CO481/CS467/PHYS467 Assignment 5

Due Friday night April 05, 2019, 10:00pm, Q3 has an extension until Sunday April 07, 3:00am

Instruction: Please submit your solutions to Crowdmark by the due date and time. Take special care to place the answer to each question in the right place. **Turn everything in before Friday 10pm, but you can resubmit Q3 until Sunday 3am.**

Question 1. The 5-qubit QECC [6 marks]

Recall that the 5-qubit QECC has four generators for the stabilizer group:

$$\begin{aligned} G_1 &= X \otimes Z \otimes Z \otimes X \otimes I \\ G_2 &= I \otimes X \otimes Z \otimes Z \otimes X \\ G_3 &= X \otimes I \otimes X \otimes Z \otimes Z \\ G_4 &= Z \otimes X \otimes I \otimes X \otimes Z \end{aligned}$$

(a)[3 marks] List the 16 possible 0- or 1-qubit Pauli errors for this code. For each of these errors, write down the $\pm$ outcome resulting from measuring each of the 4 generators. You can provide the answers in a table, similar to the one we have started in class.

(b)[3 marks] Show that $H^{\otimes 5}$ is *not* a logical operation for this code.

Question 2. Encoded R gate on 7-qubit code [7 marks]

We define the R gate as the 2×2 unitary (up to a phase) satisfying the following commutation relations:

$$RXR^\dagger = iXZ, \quad RZR^\dagger = Z.$$

Side remark: if we consider Z as a $\pi/2$ rotation, T as a $\pi/8$ rotation, R is a $\pi/4$ rotation, all along the z-axis, and up to a phase, $R = \sqrt{Z} = T^2$. R is in the Clifford group. We choose to specify R using commutation relation and not bother with the irrelevant overall phase.

Recall from class that the 7-qubit Steane code has stabilizer group generated by

$$\begin{aligned} G_1 &= I \otimes I \otimes I \otimes X \otimes X \otimes X \otimes X \\ G_2 &= I \otimes X \otimes X \otimes I \otimes I \otimes X \otimes X \\ G_3 &= X \otimes I \otimes X \otimes I \otimes X \otimes I \otimes X \\ G_4 &= I \otimes I \otimes I \otimes Z \otimes Z \otimes Z \otimes Z \\ G_5 &= I \otimes Z \otimes Z \otimes I \otimes I \otimes Z \otimes Z \\ G_6 &= Z \otimes I \otimes Z \otimes I \otimes Z \otimes I \otimes Z \end{aligned}$$

with $X_L = X^{\otimes 7}$, $Z_L = Z^{\otimes 7}$. In this question you will show that $U = R^{\otimes 7} Z^{\otimes 7}$ effects a transversal, encoded, R gate on the 7-qubit code.

(a)[4 marks] For each G_i, $i = 1, \cdots, 6$, write down $UG_iU^\dagger$ as a product of the above generators (thus U is an encoded operation on the 7-bit code). Because of the symmetry in U and similarities in the generators, it suffices to show your work/reasoning for $UG_1U^\dagger$ and $UG_4U^\dagger$, and state the answers for the rest.

(b)[3 marks] Show that $UX_LU^\dagger = iX_LZ_L$ and $UZ_LU^\dagger = Z_L$ (thus showing U is an encoded R gate).

Question 3. A magic multi-purpose 4-qubit code [13 marks + 4 marks bonus]

Consider a stabilizer code C whose stabilizer group S is generated by

$$\begin{aligned} G_1 &= X \otimes X \otimes X \otimes X \\ G_2 &= Z \otimes Z \otimes Z \otimes Z \end{aligned}$$

C encodes 2 qubits into 4 qubits.

(a) [3 marks] Explain why the following 4 matrices are encoded operations. Explain why we can choose them as the encoded Pauli X and Z operators on the two encoded qubits.

$$\begin{aligned} X_{1L} &= X \otimes X \otimes I \otimes I \\ Z_{1L} &= I \otimes Z \otimes Z \otimes I \\ X_{2L} &= I \otimes X \otimes X \otimes I \\ Z_{2L} &= I \otimes I \otimes Z \otimes Z \end{aligned}$$

(b) [2 marks] Show that $H \otimes H \otimes H \otimes H$ is an encoded operation.

(c) [4 marks] (bonus) What encoded operation does $H^{\otimes 4}$ perform? (Hint: check commutation relation with the encoded X and Z's, and recall that each element of the stabilizer group is an encoded identity operator.)

(d) [2 marks] Find the codewords $|00_L\rangle$, $|01_L\rangle$, $|10_L\rangle$, $|11_L\rangle$ using the stabilizer generators and the encoded Pauli operators X_{1L}, Z_{1L}, X_{2L}, Z_{2L} given in part (a).

(e) Show that an erasure on any of the 4 qubits can be corrected. By symmetry, it suffices ot show that erasure on the fist qubit can be corrected.

(i) [2 marks] Suppose one of I, X, Y, Z happens to the first qubit. What are the outcomes if G_1 and G_2 are measured?

(ii) [2 marks] State a method to correct the erasure on the first qubit. Detail explanation is not needed.

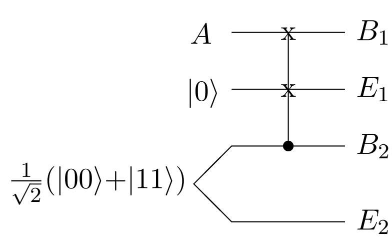
(f) [2 marks] Instead of correcting an erasure error, the same code C can be used to detect a single *unknown* Pauli error. Explain how. (Here, you want to show that there are measurements that distinguish the no error case from the case with any single-qubit Pauli error.)

(g) [0 marks] Demoting a question to a remark:

Consider a new QECC C' obtained by adding Z_{2L} to the list of stabilizer generator. This encodes 1 qubit in 4, and correct 1 amplitude damping error without satisfying the QECC condition!

Question 4. QECC for the 50-50 erasure channel? [4 marks]

Consider the quantum operation $\mathcal{E}$ which erases a qubit input with probability 50%. It is called the 50-50 erasure channel. We described it in class, and it has an alternative description:



In the above, A is the input qubit system and B_1B_2 are the output systems. $E_1B_2E_2$ are qubit systems initialized in a fixed pure state $|a\rangle_{E_1B_2E_2} = |0\rangle_{E_1} \otimes \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)_{B_2E_2}$. Then, a unitary U is applied, which conditioned on B_2 being in the state $|1\rangle$, swaps B_1 and E_1. Finally, performing a partial trace of E_1E_2 gives the output in systems B_1B_2. The information whether a system is erased or not can be found in B_2. In other words, $\mathcal{E}(\rho) = \mathrm{tr}_{E_1E_2}\, U\,(\rho_A \otimes |a\rangle\langle a|_{E_1B_2E_2})\, U^\dagger$.

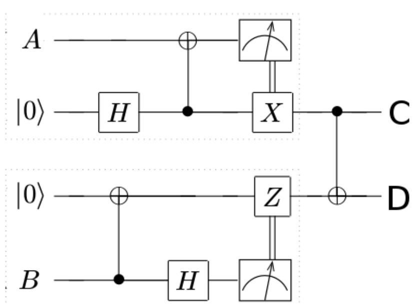
Let n be any positive integer. Explain why there is no QECC that encodes 1 qubit into n qubits such that the encoded qubit can be recovered with very high probability after the noise process $\mathcal{E}^{\otimes n}$. (Hint: you can use the fact that we cannot clone a qubit with very high probability, and find a contradiction if such a QECC exists.)

Question 5. Deriving a remote CNOT using 1-bit teleportation [0 marks]

PRACTICE QUESTION, DO NOT TURN IN.

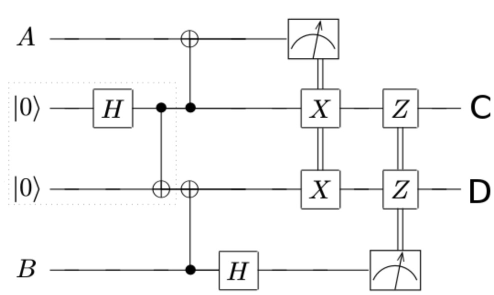
Suppose two remote parties Alice and Bob wish to perform a CNOT on two qubits in systems AB, where the control qubit A is held by Alice, and the target qubit B is held by Bob. We will derive a method for them to do so by using one maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, one classical bit of communication from Alice to Bob, and one classical bit of communication from Bob to Alice.

First consider the following circuit. A vertical *double line* connecting a measurement box (in the computational basis) to a unitary U means that "conditioned on the measurement outcome being 1, perform U, otherwise do nothing".



(a) [2 marks] Explain why the above circuit performs a CNOT on the two incoming qubits in systems AB, and leaves the output in systems CD.

(b) [4 marks] Show that the following circuit implements the same transformation as the previous circuit.



(c) [4 marks] Extract from the circuit in part (b) a protocol for Alice and Bob to apply a CNOT on AB using one maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, one classical bit of communication from Alice to Bob, and one classical bit of communication from Bob to Alice.