CO481/CS467/PHYS467 Assignment 5

Due Wed April 02, 2025, 08:30am

Instruction: Please submit your solutions to Crowdmark by the due date and time. Take special care to place the answer to each question in the right place.

Question 1. The 5-qubit QECC [6 marks]

Recall that the 5-qubit QECC has four generators for the stabilizer group:

$$\begin{aligned} G_1 &= X \otimes Z \otimes Z \otimes X \otimes I \\ G_2 &= I \otimes X \otimes Z \otimes Z \otimes X \\ G_3 &= X \otimes I \otimes X \otimes Z \otimes Z \\ G_4 &= Z \otimes X \otimes I \otimes X \otimes Z \end{aligned}$$

(a)[3 marks] List the 16 possible 0- or 1-qubit Pauli errors for this code. For each of these errors, write down the $\pm$ outcome resulting from measuring each of the 4 generators. You can provide the answers in a table, similar to the one we have started in class.

(b)[3 marks] Show that $H^{\otimes 5}$ is *not* a logical operation for this code.

Question 2. Encoded R gate on 7-qubit code [7 marks]

We define the R gate as the 2×2 unitary (up to a phase) satisfying the following commutation relations:

$$RXR^\dagger = iXZ, \quad RZR^\dagger = Z.$$

Side remark: if we consider Z as a $\pi/2$ rotation, T as a $\pi/8$ rotation, R is a $\pi/4$ rotation, all along the z-axis, and up to a phase, $R = \sqrt{Z} = T^2$. R is in the Clifford group. We choose to specify R using commutation relation and not bother with the irrelevant overall phase.

Recall from class that the 7-qubit Steane code has stabilizer group generated by

$$\begin{aligned} G_1 &= I \otimes I \otimes I \otimes X \otimes X \otimes X \otimes X \\ G_2 &= I \otimes X \otimes X \otimes I \otimes I \otimes X \otimes X \\ G_3 &= X \otimes I \otimes X \otimes I \otimes X \otimes I \otimes X \\ G_4 &= I \otimes I \otimes I \otimes Z \otimes Z \otimes Z \otimes Z \\ G_5 &= I \otimes Z \otimes Z \otimes I \otimes I \otimes Z \otimes Z \\ G_6 &= Z \otimes I \otimes Z \otimes I \otimes Z \otimes I \otimes Z \end{aligned}$$

with $X_L = X^{\otimes 7}$, $Z_L = Z^{\otimes 7}$. In this question you will show that $U = R^{\otimes 7} Z^{\otimes 7}$ effects a transversal, encoded, R gate on the 7-qubit code.

(a)[4 marks] Show that U is an encoded operation on the 7-bit code.

As a reminder, you need to show that for each G_i, $i = 1, \cdots, 6$, $UG_iU^\dagger$ is a product of the above generators.

Because of the symmetry in U and similarities in the generators, it suffices to show your work/reasoning for $UG_1U^\dagger$ and $UG_4U^\dagger$, and state the answers for the rest.

(b)[3 marks] Show that U is an encoded R gate (by showing $UX_LU^\dagger = iX_LZ_L$ and $UZ_LU^\dagger = Z_L$).

Question 3. A multi-purpose 4-qubit code [13 marks]

Consider a stabilizer code C whose stabilizer group S is generated by

$$\begin{aligned} G_1 &= X \otimes X \otimes X \otimes X \\ G_2 &= Z \otimes Z \otimes Z \otimes Z \end{aligned}$$

C encodes 2 qubits into 4 qubits.

(a) [1 mark] Explain why the following 4 matrices are encoded operations.

$$\begin{aligned} X_{1L} &= X \otimes X \otimes I \otimes I \\ Z_{1L} &= I \otimes Z \otimes Z \otimes I \\ X_{2L} &= I \otimes X \otimes X \otimes I \\ Z_{2L} &= I \otimes I \otimes Z \otimes Z \end{aligned}$$

(b) [2 marks] Explain the commutation relations between the above that enable us to choose them as the encoded Pauli X and Z operators on the two encoded qubits.

(c) [1 mark] Show that $H \otimes H \otimes H \otimes H$ is an encoded operation.

(d) [3 marks] What encoded operation does $H^{\otimes 4}$ perform? (Hint: check commutation relation with the encoded X and Z's, and recall that each element of the stabilizer group is an encoded identity operator.)

(e) [2 marks] Find the codewords $|00_L\rangle$, $|01_L\rangle$, $|10_L\rangle$, $|11_L\rangle$ using the stabilizer generators and the encoded Pauli operators X_{1L}, Z_{1L}, X_{2L}, Z_{2L} given in part (a).

(f) [3 mark] Explain how to correct an erasure on any of the 4 qubits. By symmetry, it suffices to correct an erasure on the fist qubit.

(g) [1 marks] Instead of correcting an erasure error, the same code C can be used to detect a single *unknown* Pauli error. State clearly what measurement distinguishes the no error case from the case with any single-qubit Pauli error.

Additional remark: Consider a new QECC C' obtained by adding Z_{2L} to the list of stabilizer generator. This encodes 1 qubit in 4, and correct 1 amplitude damping error without satisfying the QECC condition!