3. Summary of quantum mechanics

(a) Linear algebra and Dirac notation
(Self-study + test, NC 2.1, KLM 2.1-2.6, 2.8)

(b) Axioms of quantum mechanics
(NC 2.2.1-2.2.5, 2.2.7, KLM 3.1-3.4)

(c) Composite systems, entanglement,
operations on 1 out of 2 systems, locality of QM
(NC 2.2.8-2.2.9)

Axioms of quantum mechanics

1. Space postulate
2. State postulate
3. Composite systems
4. Evolution
5. Measurements

Two equivalent formalisms: pure vs mixed states.
We use the simpler pure state formalism on parts 2-7.

Like Go, QM can be complex with simple rules.

1. Space postulate

A finite physical system S is associated with a complex Hilbert space H with finite dimension, say, d.

1. Space postulate

A finite physical system S is associated with a complex Hilbert space H with finite dimension, say, d.

Side note: this is a common description in QM texts. On finite dimensions, this is equivalent to complex Eucliden space with the usual norm (& inner product).

1. Space postulate

A finite physical system S is associated with a complex Hilbert space H with finite dimension, say, d.

2. State postulate

The state of the system S is given by a unit vector in the associated Hilbert space H.

Example: when d=2, the state, a 2-dim complex unit vector, is called a "qubit" (a quantum bit)

coined by Schumacher

In vector form: $\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$ where $a_0, a_1 \in \mathbb{C}$,

$|a_0|^2 + |a_1|^2 = 1$.

Example: when d=2, the state, a 2-dim complex unit vector, is called a "qubit" (a quantum bit)

coined by Schumacher

In vector form: $\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$ where $a_0, a_1 \in \mathbb{C}$,

$|a_0|^2 + |a_1|^2 = 1$.

In Dirac notation, vectors are written as "kets":

(Computational) basis vectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow |0\rangle, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightarrow |1\rangle$$

So: $$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = a_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= a_0 |0\rangle + a_1 |1\rangle = |\psi\rangle$$

A general d-dim complex vector is given by:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \quad \text{where} \quad \forall i \;\; a_i \in \mathbb{C}, \quad \sum_{i=1}^{d} |a_i|^2 = 1 .$$

In Dirac notation, the ket is written as $\sum_{i=1}^{d} a_i |i\rangle$,

where the i-th basis vector

$$\text{i-th position} \rightarrow \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \text{is written as } |i\rangle .$$

The ket $|\Psi\rangle$, as a vector, has a dual written as $\langle\Psi|$

which is called the "bra".

For finite dimensional Hilbert spaces, the dual of a vector v can be taken as the conjugate-transpose of v.

The ket $|\psi\rangle$, as a vector, has a dual written as $\langle\psi|$ which is called the "bra".

For finite dimensional Hilbert spaces, the dual of a vector v can be taken as the conjugate-transpose of v.

$$|\psi\rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix}$$

the dual is $\langle\psi| = \left[a_1^* a_2^* \ldots a_d^* \right]$

a_i^* $\leftarrow$ complex conjugate

The ket $|\psi\rangle$, as a vector, has a dual written as $\langle\psi|$ which is called the "bra".

For finite dimensional Hilbert spaces, the dual of a vector v can be taken as the conjugate-transpose of v.

$$|\psi\rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix}$$

the dual is $\langle\psi| = [\, a_1^* \, a_2^* \, \ldots \, a_d^* \,]$

$$= \sum_{i=1}^{d} a_i^* \langle i| \, ..$$

dual of $|i\rangle$

$$= [\, 0 \ldots 1 \ldots 0 \,]$$

= transpose of $|i\rangle$

special, only
for real vectors

$$= \sum_{i=1}^{d} a_i |i\rangle$$

Linear algebra in the bra-ket notation

1. Inner product

Let $\{|e_i\rangle\}_{i=1}^{d}$ be ANY basis.

(a) the inner product of $|e_i\rangle, |e_j\rangle$ is

$$\langle e_i | e_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Kronecker delta-function
which we call "delta-function"
in this course.
(Elsewhere delta-function may
refer to the "Dirac delta-function"
which is not a function, and will
not be used in this course.)

Linear algebra in the bra-ket notation

1. Inner product

Let $\{|e_i\rangle\}_{i=1}^{d}$ be ANY basis.

(a) the inner product of $|e_i\rangle, |e_j\rangle$ is

$$\langle e_i|e_j\rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

(b) For $|\psi\rangle = \sum_{i=1}^{d} a_i |e_i\rangle$, $|\phi\rangle = \sum_{j=1}^{d} b_j |e_j\rangle$

their inner product is

$$\langle\psi|\phi\rangle = \sum_{i=1}^{d} a_i^* \langle e_i| \cdot \sum_{j=1}^{d} b_j |e_j\rangle = \sum_{i=1}^{d} a_i^* b_i$$

the "bra-ket"

δ_{ij} so set i=j and obtain a single sum

2. Outer product

For $|\psi\rangle = \sum_{i=1}^{d} a_i |i\rangle$, $|\phi\rangle = \sum_{j=1}^{d} b_j |j\rangle$

their outer-product is

$$|\psi\rangle\langle\phi| = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \times \begin{bmatrix} b_1^* & b_2^* & \cdots & b_d^* \end{bmatrix}$$

matrix multiplication

$$= \begin{pmatrix} a_1 b_1^* & a_1 b_2^* & \cdots & a_1 b_d^* \\ a_2 b_1^* & a_2 b_2^* & \cdots & a_2 b_d^* \\ \vdots & & & \\ a_d b_1^* & a_d b_2^* & \cdots & a_d b_d^* \end{pmatrix}$$

matrix representation

2. Outer product

For $|\psi\rangle = \sum_{i=1}^{d} a_i |i\rangle$, $|\phi\rangle = \sum_{j=1}^{d} b_j |j\rangle$

their outer-product in Dirac notation is

$$|\psi\rangle\langle\phi| = \sum_{i=1}^{d} a_i |i\rangle \sum_{j=1}^{d} b_j^* \langle j|$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{d} \underbrace{a_i b_j^*}_{\text{(i,j) entry of the matrix } |\psi\rangle\langle\phi|} |i\rangle\langle j|$$

2. Outer product

For $|\psi\rangle = \sum_{i=1}^{d} a_i |i\rangle$, $|\phi\rangle = \sum_{j=1}^{d} b_j |j\rangle$

their outer-product in Dirac notation is

$$|\psi\rangle\langle\phi| = \sum_{i=1}^{d} a_i |i\rangle \sum_{j=1}^{d} b_j^* \langle j|$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{d} \underbrace{a_i b_j^*}_{\text{(i,j) entry of the matrix } |\psi\rangle\langle\phi|} |i\rangle\langle j|$$

In general, any matrix M can be written as:

$$M = \sum_{i=1}^{r}\sum_{j=1}^{c} M_{ij} |i\rangle\langle j|$$

where M has r rows and c columns.

In general, any matrix M can be written as:

$$M = \sum_{i=1}^{r} \sum_{j=1}^{c} M_{ij} \, |i\rangle\langle j|$$

where M has r rows and c columns.

Exercise:

$$M = \sum_{i=1}^{r} \sum_{j=1}^{c} M_{ij} \, |i\rangle\langle j| \,, \quad N = \sum_{k=1}^{c} \sum_{\ell=1}^{c'} N_{k\ell} \, |k\rangle\langle \ell|$$

show that the product of M and N is given by

$$MN = \sum_{i=1}^{r} \sum_{\ell=1}^{c'} \left(\sum_{j=1}^{c} M_{ij} N_{j\ell} \right) |i\rangle\langle \ell|$$

using the delta function for the inner product in bra-ket notations.

2. Outer product

For $|\psi\rangle = \sum_{i=1}^{d} a_i |i\rangle$, $|\phi\rangle = \sum_{j=1}^{d} b_j |j\rangle$

their outer-product $|\psi\rangle\langle\phi|$

is a rank 1, dxd matrix taking $|\phi\rangle$ to $|\psi\rangle$

and any state orthogonal to $|\phi\rangle$ to 0 .

3. Projectors

Let K be a c-dim subspace of H, with basis $\{|f_i\rangle\}_{i=1}^{c}$.

Then, the projector onto K can be written as

$$\Pi_K = \sum_{i=1}^{c} |f_i\rangle\langle f_i| .$$

(Projectors are crucial when we discuss measurements.)

Axioms of quantum mechanics

1. Space postulate
2. State postulate
   Dirac / bra-ket notation for states,
   inner product, outer product, and projectors
3. Composite systems
4. Evolution
5. Measurements

3. Composite system postulate

Consider 2 systems S, T with respective associated Hilbert spaces H and K.

The bipartite system ST is associated with the Hilbert space which is the tensor product of H and K.

3. Composite system postulate

Consider 2 systems S, T with respective associated Hilbert spaces H and K.

The bipartite system ST is associated with the Hilbert space which is the tensor product of H and K.

If H has c dimensions and basis $\{|1\rangle, |2\rangle, \ldots, |c\rangle\}$,

& K has d dimensions and basis $\{|1\rangle, |2\rangle, \ldots, |d\rangle\}$.

Then $H \otimes K$ has cd dimensions and basis

$$\left\{ \begin{array}{llll} |1\rangle \otimes |1\rangle, & |1\rangle \otimes |2\rangle, & \ldots & |1\rangle \otimes |d\rangle \\ |2\rangle \otimes |1\rangle, & |2\rangle \otimes |2\rangle, & \ldots & |2\rangle \otimes |d\rangle \\ & \vdots & & \\ |c\rangle \otimes |1\rangle, & |c\rangle \otimes |2\rangle, & \ldots & |c\rangle \otimes |d\rangle \end{array} \right\}.$$

What states are possible on a composite system?

(a) product states, which are of the form

$$|\Psi\rangle \otimes |\Phi\rangle \in H \otimes K$$

where $|\Psi\rangle \in H$ and $|\Phi\rangle \in K$.

What states are possible on a composite system?

(a) product states, which are of the form

$$|\psi\rangle \otimes |\phi\rangle \in H \otimes K$$

where $|\psi\rangle \in H$ and $|\phi\rangle \in K$.

(b) entangled states, which cannot be written as product states.

e.g., $\frac{1}{\sqrt{2}} \underbrace{|00\rangle}_{|0\rangle \otimes |0\rangle} + \frac{1}{\sqrt{2}} \underbrace{|11\rangle}_{|1\rangle \otimes |1\rangle} \in \mathbb{C}^2 \otimes \mathbb{C}^2$

Proof idea: by contradiction. If it is a tensor product, say, of $a|0\rangle + b|1\rangle$ and $c|0\rangle + d|1\rangle$.

Derive a contradiction concerning a,b,c,d.

Axioms of quantum mechanics

1. Space postulate
2. State postulate

   Dirac / bra-ket notation for states,
   inner product, outer product, and projectors
3. Composite systems

   Product and entangled states
4. Evolution
5. Measurements

4. Evolution postulate:

The time evolution of states in a closed quantum system is described by a unitary operator.

4. Evolution postulate:

The time evolution of states in a closed quantum system is described by a unitary operator.

Notation: we use the "dagger" to denote the conjugate transpose (aka the adjoint).

e.g., U is unitary iff $UU^\dagger = U^\dagger U = I$.

4. Evolution postulate:

The time evolution of states in a closed quantum system is described by a unitary operator.

Notation: we use the "dagger" to denote the conjugate transpose (aka the adjoint).

e.g., U is unitary iff $UU^\dagger = U^\dagger U = I$.

Crucial fact: a unitary matrix takes a unit vector to another unit vector. So, a legitimate quantum state is evolved to another legitimate quantum state.

Proof: exercise.

Examples of unitary evolution:

For d = 2, consider the Pauli matrices:

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$\sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$\sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$

$$\sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

NB: 1. No need to write the 0 entries.
2. Read from R->L.

e.g., $\sigma_x (a|0\rangle + b|1\rangle) = a|1\rangle + b|0\rangle$ (NOT gate)

The Hadamard matrix, H = $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. (Fourier trsf)

For d = 4, with basis $\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \}$

the unitary $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ takes $|a\,b\rangle$ to $|a\; a\oplus b\rangle$

and is called the CNOT.

For d = 8, with basis $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle,$
$|100\rangle, |101\rangle, |110\rangle, |111\rangle\}$

the unitary
$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & & \text{zeros} & & \\ 0 & 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \\ & & & & 1 & 0 & 0 & 0 \\ & \text{zeros} & & & 0 & 1 & 0 & 0 \\ & & & & 0 & 0 & 0 & 1 \\ & & & & 0 & 0 & 1 & 0 \end{array}\right)$$
called the TOFFOLI

takes $|a\,b\,c\rangle$ to $|a\,b\,ab \oplus c\rangle$.

For d = 8, with basis $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$

the unitary $\begin{pmatrix} \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} & \text{zeros} \\ \text{zeros} & \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix} \end{pmatrix}$ called the TOFFOLI

takes $|a\,b\,c\rangle$ to $|a\,b\,ab \oplus c\rangle$.

note: AND(a,b) = ab

Preview: action of these gates similar to classical setting; just that our unit vector is in the 2-norm.

In general, a matrix U is unitary

iff $U = e^{-iHr}$ for some hermitian matrix H and real number r.

PS A unitary is the most general transformation effecting a change of basis.

In general, a matrix U is unitary

iff $U = e^{-iHr}$ for some hermitian matrix H and real number r.

Question: suppose a system is evolved under U, and then under V, what is the combine evolution?

Answer: VU. Note that the product of two unitary matrices is still unitary, so, composing two evolutions indeed gives a valid evolution.

In physics, the evolution postulate is given by Schroedinger's equation:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H(t)\,|\psi(t)\rangle$$

where t is time, and the hermitian matrix H(t) is called the "Hamiltonian" at time t.

If H(t) = H is time independent, then,

$$|\psi(t)\rangle = e^{-iHt}\,|\psi(0)\rangle .$$

In physics, the evolution postulate is given by Schroedinger's equation:

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle$$

where t is time, and the hermitian matrix H(t) is called the "Hamiltonian" at time t.

$\hbar = 1$

If H(t) = H is time independent, then,

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle.$$

important, not just a choice since
we cannot travel back in time

In physics, the evolution postulate is given by Schroedinger's equation:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

where t is time, and the hermitian matrix H(t) is called the "Hamiltonian" at time t.

If H(t) = H is time independent, then,

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle .$$

The physical theory or the experimental setup determines H(t) and the evolution.

In contrast, in quantum information processing, we focus on the abstract unitary evolution.

Exercise: For any hamiltonian H, any unitary U, any t, (a) show that

$$U e^{-iHt} U^{\dagger} = e^{-iKt}$$

where K is another hamiltonian.

(b) What is K in terms of U and H?

(c) What is the physical interpretation of the answers in (a) and (b)?

Hint: use poser series decomposition for e^{-iHt}.

We will see that this is a very useful result in quantum computation, so, you will derive it in the test.

Axioms of quantum mechanics

✓ 1. Space postulate
✓ 2. State postulate
    Dirac / bra-ket notation for states,
    inner product, outer product, and projectors
✓ 3. Composite systems
    Product and entangled states
✓ 4. Evolution
    Linear, unitary
5. Measurements

5. Measurement postulate

Consider a d-dimensonal Hilbert space H.

Consider an arbitrary basis B = $\{ |e_i\rangle \}_{i=1}^{d}$ for H.

A complete von Neumann measurement on H along the basis B does the following.

5. Measurement postulate

Consider a d-dimensonal Hilbert space H.

Consider an arbitrary basis B = $\{ |e_i\rangle \}_{i=1}^{d}$ for H.

A complete von Neumann measurement on H along the basis B does the following.

If the pre-measurement state is $|\psi\rangle = \sum_{i=1}^{d} a_i |e_i\rangle$

then the measurement outputs:

(1) a measurement outcome i with probability $|a_i|^2$.

(2) a postmeasurement state $|e_i\rangle$ if the outcome is i.

Note that the state being a unit vector gives a proper probability distribution on the outcome.

Example: d=5, $B = \{ |1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle \}$

$$|\psi\rangle = \sqrt{\frac{1}{8}}\,|1\rangle + \frac{i}{\sqrt{2}}\,|3\rangle + \sqrt{\frac{3}{8}}\,|4\rangle$$

The complete measurement along B has outcome

"1" with prob 1/8,

"2" with prob 0,

"3" with prob 1/2,

"4" with prob 3/8,

"5" with prob 0.

Example: d=2,

$$B = \{ |+\rangle, |-\rangle \} \text{ where } |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle),$$

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

What are the probs to obtain the outcomes + & - ?

Example: d=2,

$B = \{ |+\rangle, |-\rangle \}$ where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$,

$|\psi\rangle = a|0\rangle + b|1\rangle$

What are the probs to obtain the outcomes + & - ?

Valuable trick in QM: express info in a useful basis.

We want to rewrite $|\psi\rangle = a'|+\rangle + b'|-\rangle$.

Example: d=2,

$$B = \{ |+\rangle, |-\rangle \} \text{ where } |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle),$$

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

What are the probs to obtain the outcomes + & - ?

We want to rewrite $|\psi\rangle = a'|+\rangle + b'|-\rangle$.

From $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

we have $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$, $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$.

i.e., express the original basis in terms of the basis
we are measuring along

Example: d=2,

$$B = \{ |+\rangle, |-\rangle \} \text{ where } |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle),$$

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

What are the probs to obtain the outcomes + & - ?

We want to rewrite $|\psi\rangle = a'|+\rangle + b'|-\rangle$.

From $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

we have $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$, $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$.

Answer: $|\psi\rangle = a|0\rangle + b|1\rangle$

$$= a\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) + b\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

$$= \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle$$

$\therefore$, Prob("+") = $\frac{1}{2}|a+b|^2$, Prob("-") = $\frac{1}{2}|a-b|^2$.

5. Measurement postulate (from a few pages ago)

Consider a d-dimensonal Hilbert space H.

Consider an arbitrary basis B = $\{ |e_i\rangle \}_{i=1}^{d}$ for H.

A complete von Neumann measurement on H along the basis B does the following.

If the pre-measurement state is $|\psi\rangle = \sum_{i=1}^{d} a_i |e_i\rangle$

then the measurement outputs:

(1) a measurement outcome i with probability $|a_i|^2$.

(2) a postmeasurement state $|e_i\rangle$ if the outcome is i.

What are the most general measurements in QM?

Incomplete measurements!

The most general measurement is a coarse-graining of a complete basis measurement. This is called an incomplete measurement.

Let B = $\{|e_i\rangle\}_{i=1}^{d}$, $|\psi\rangle = \sum_{i=1}^{d} a_i |e_i\rangle$ as before.

Let $S_1, S_2, \ldots, S_k$ be a partition of {1,2,...,d}.

ie $\forall j \neq l$, $S_j \cap S_l = \emptyset$, $S_1 \cup S_2 \cup \cdots S_k$ = {1,2,...,d}.

e.g. 1, {1,2,3,4,5} can be partitioned into

S1 = {1,4}, S2 = {2,5}, S3 = {3}.

The most general measurement is a coarse-graining of a complete basis measurement. This is called an incomplete measurement.

Let $B = \{|e_i\rangle\}_{i=1}^{d}$, $|\psi\rangle = \sum_{i=1}^{d} a_i |e_i\rangle$ as before.

Let $S_1, S_2, \ldots, S_k$ be a partition of {1,2,...,d}.

ie $\forall j \neq l$, $S_j \cap S_l = \emptyset$, $S_1 \cup S_2 \cup \cdots S_k$ = {1,2,...,d}.

e.g. 1, {1,2,3,4,5} can be partitioned into

S1 = {1,4}, S2 = {2,5}, S3 = {3}.

e.g. 2, S_{odd} = {1,3,5}, S_{even} = {2,4} is another partition (k=2, odd=1, even=2).

The most general measurement is a coarse-graining of a complete basis measurement. This is called an incomplete measurement.

Let B = $\{|e_i\rangle\}_{i=1}^{d}$, $|\psi\rangle = \sum_{i=1}^{d} a_i |e_i\rangle$ as before.

Let $S_1, S_2, \ldots, S_k$ be a partition of {1,2,...,d}.

ie $\forall j \neq \ell$, $S_j \cap S_\ell = \emptyset$, $S_1 \cup S_2 \cup \cdots S_k$ = {1,2,...,d}.

The partition defines a measurement with

(1) outcome j $\in$ {1,2,..,k} with prob $\sum_{i \in S_j} |a_i|^2$.

labels for
the partition

(2) a postmeasurement state $\dfrac{\sum_{i \in S_j} a_i |e_i\rangle}{\sum_{i \in S_j} |a_i|^2}$

if the outcome is j.

In other words, the outcome is "which partition" and we do not seek to distinguish the outcomes within each partition.

Crucial: the postmeasurement state remains a linear combination of basis states within the partition corr to the outcome, and rescaled.

Complete measurement is the special case of the partition: $S1=\{1\}$, $S2=\{2\}$, ... , $Sd=\{d\}$.

Example: d=5, $B = \{ |1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle \}$

$$|\psi\rangle = \sqrt{\frac{1}{8}}\,|1\rangle + \frac{i}{\sqrt{2}}\,|3\rangle + \sqrt{\frac{3}{8}}\,|4\rangle$$

Consider an incomplete measurement with 2 outcomes where :

$$S_{odd} = \{1, 3, 5\}, \quad S_{even} = \{2, 4\}$$

With prob 5/8, outcome = "odd",

postmeasurement state is $\left(\sqrt{\frac{1}{8}}\,|1\rangle + \frac{i}{\sqrt{2}}\,|3\rangle \right) \Big/ \sqrt{\frac{5}{8}}$.

With prob 3/8, outcome = "even",

postmeasurement state is $|4\rangle$.

3. Summary of quantum mechanics

(a) Linear algebra and Dirac notation
(Self-study + test, NC 2.1, KLM 2.1-2.6, 2.8)

(b) Axioms of quantum mechanics
(NC 2.2.1-2.2.5, 2.2.7, KLM 3.1-3.4)

mostly done!

(c) Composite systems, entanglement,
operations on 1 out of 2 systems, locality of QM
(NC 2.2.8-2.2.9)

Axioms of quantum mechanics

✓ 1. Space postulate

✓ 2. State postulate

    Dirac / bra-ket notation for states,
    inner product, outer product, and projectors

✓ 3. Composite systems

    Product and entangled states

✓ 4. Evolution

    Linear, unitary

5. Measurements

    Incomplete measurement along a basis, defined
    by a partition of the labels of the basis vectors.

Exercise:

Let d = 3.

Consider the incomplete measurement along the basis:

$B = \{ |+\rangle, |-\rangle, |2\rangle \}$ where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$,

with partition S1 = {+,2}, S2 = {-}.

If the pre-measurement state is $|\psi\rangle = a|0\rangle + b|1\rangle + c|2\rangle$,

what is the probability to obtain the outcome "1"?

(a) $|b|^2$ (b) $|a|^2 + |b|^2$ (c) $\left|\frac{a+b}{\sqrt{2}}\right|^2$

(d) $\left|\frac{a+b}{\sqrt{2}}\right|^2 + |c|^2$ (e) $\left|\frac{a-b}{\sqrt{2}}\right|^2 + |c|^2$

Answer:

Let d = 3.
Consider the incomplete measurement along the basis:
$B = \{ |+\rangle, |-\rangle, |2\rangle \}$ where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$,
with partition S1 = {+,2}, S2 = {-}.

If the pre-measurement state is $|\psi\rangle = a|0\rangle + b|1\rangle + c|2\rangle$,
what is the probability to obtain the outcome "1"?

(a) $|b|^2$ (b) $|a|^2 + |b|^2$ (c) $\left|\frac{a+b}{\sqrt{2}}\right|^2$

(d) $\left|\frac{a+b}{\sqrt{2}}\right|^2 + |c|^2$ (e) $\left|\frac{a-b}{\sqrt{2}}\right|^2 + |c|^2$

since $|\psi\rangle = a|0\rangle + b|1\rangle + c|2\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle + c|2\rangle$

Postmeasurement state is $\left(\frac{a+b}{\sqrt{2}}|+\rangle + c|2\rangle\right) \Big/ \sqrt{\left|\frac{a+b}{\sqrt{2}}\right|^2 + |c|^2}$.

Alternative way to specific an incomplete measurement

Previous e.g.: d=5, $B = \{|1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle\}$

$S_{odd} = \{1, 3, 5\}$, $S_{even} = \{2, 4\}$

Instead of the partition, define 2 projectors:

$P_{odd} = |1\rangle\langle 1| + |3\rangle\langle 3| + |5\rangle\langle 5|$, $P_{even} = |2\rangle\langle 2| + |4\rangle\langle 4|$.

Alternative way to specific an incomplete measurement

Previous e.g.: d=5, $\mathcal{B} = \{|1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle\}$

$$S_{odd} = \{1, 3, 5\}, \quad S_{even} = \{2, 4\}$$

Instead of the partition, define 2 projectors:

$$P_{odd} = |1\rangle\langle 1| + |3\rangle\langle 3| + |5\rangle\langle 5|, \quad P_{even} = |2\rangle\langle 2| + |4\rangle\langle 4|.$$

For premeas state: $|\psi\rangle = \sqrt{\frac{1}{8}}\,|1\rangle + \frac{i}{\sqrt{2}}\,|3\rangle + \sqrt{\frac{3}{8}}\,|4\rangle$

$$P_{odd}|\psi\rangle = \sqrt{\frac{1}{8}}\,|1\rangle + \frac{i}{\sqrt{2}}\,|3\rangle$$

Prob(outcome = "odd") = $\left\| P_{odd}|\psi\rangle \right\|_2^2 = \frac{5}{8}$.

Postmeas state is $\dfrac{P_{odd}|\psi\rangle}{\left\| P_{odd}|\psi\rangle \right\|_2} = \left(\sqrt{\frac{1}{8}}\,|1\rangle + \frac{i}{\sqrt{2}}\,|3\rangle\right) \Big/ \sqrt{\frac{5}{8}}$.

$$P_{even}|\psi\rangle = \sqrt{\frac{3}{8}}|\psi\rangle$$

Prob(outcome = "even") = $\left\| P_{even}|\psi\rangle \right\|_2^2 = \frac{3}{8}$.

Postmeas state is $\dfrac{P_{even}|\psi\rangle}{\left\| P_{even}|\psi\rangle \right\|_2} = |\psi\rangle$.

Initial specification $\longrightarrow$ Alternative specification

Initial specification

$B = \{|e_i\rangle\}_{i=1}^{d}$

Let $S_1, S_2, \ldots, S_K$ be a partition of {1,2,...,d}.

$$|\psi\rangle = \sum_{i=1}^{d} a_i |e_i\rangle$$

(1) outcome = j $\in$ {1,..,k}

with prob $\sum_{i \in S_j} |a_i|^2$.

(2) corresponding postmeas state

$$= \frac{\sum_{i \in S_j} a_i |e_i\rangle}{\sum_{i \in S_j} |a_i|^2}.$$

Alternative specification

Projectors $P_1, P_2, \ldots, P_K$

$\forall_j \; P_j = \sum_{i \in S_j} |e_i\rangle\langle e_i| \quad \left(\sum_{j=1}^{K} P_j = I.\right)$

$$|\psi\rangle = \sum_{i=1}^{d} a_i |e_i\rangle$$

(1) outcome = j $\in$ {1,..,k}

with prob $\| P_j |\psi\rangle \|_2^2$.

(2) corresponding postmeas state

$$= \frac{P_j |\psi\rangle}{\| P_j |\psi\rangle \|_2}.$$

First equivalent way to specify a measurement:

(I) specifying a measurement using projectors

Consider a d-dimensonal Hilbert space H.

The most general measurement on H can be specified by a set of projectors acting on H, $\{P_j\}_{j=1}^{K}$, such that

$$\sum_j P_j = I.$$

First equivalent way to specify a measurement:

(I) specifying a measurement using projectors

Consider a d-dimensonal Hilbert space H.

The most general measurement on H can be specified

by a set of projectors acting on H, $\{P_j\}_{j=1}^{K}$, such that

$$\sum_j P_j = I.$$

If the pre-measurement state is $|\psi\rangle$

then the measurement outputs:

(1) measurement outcome j with prob $\| P_j |\psi\rangle \|_2^2$.

(2) a postmeasurement state $\dfrac{P_j|\psi\rangle}{\| P_j|\psi\rangle \|_2}$

if the outcome is j.

Euclidean
2-norm

Reminder:

P is a projector

$\iff$ (i) P is hermitian, and
(ii) eigenvalues of P are either 0 or 1

$\iff$ P is normal. $P = P^2$.

Reminder:

P is a projector

$\Longleftrightarrow$ (i) P is hermitian, and
(ii) eigenvalues of P are either 0 or 1

$\Longleftrightarrow$ P is normal. $P = P^2$.

Exercise: Show that

(1) If a list of projectors $P_1, \ldots, P_K$ acting on H sum to I, then the projectors are mutually orthogonal.
(2) For each j, the support of P_j is a subspace of H.
(3) Let $\{|e_{i,j}\rangle\}$ be a basis for the support of P_j.

Note $P_j = \sum_i |e_{i,j}\rangle\langle e_{i,j}|$.

Let $S_j = \{(i,j)\}_{i=1}^{\dim(\mathrm{supp}(P_j))}$.

Then, the S_j's are disjoint with a total of d elements.

This turns the second specification back to the first.

Exercise:

Let d = 3.

Consider the incomplete measurement along the basis:

$\mathcal{B} = \{ |+\rangle, |-\rangle, |2\rangle \}$ where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$,

with partition S1 = {+,2}, S2 = {-}.

Which of the following is an equivalent specification of the above measurement?

(a) $P_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $P_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(c) $P_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Second equivalent way to specify a measurement:

(II) Specifying a measurement using an observable.

Notation: an observable is a hermitian matrix
acting on a Hilbert Space.

For an observable M, suppose $\lambda_1, \ldots, \lambda_K$ are the distinct eigenvalues. Let P_j be the projector onto the eigenspace corresponding to λ_j. Then $\sum_{j=1}^{K} P_j = I$ and $\{P_j\}$ defines a measurement.

Second equivalent way to specify a measurement:

(II) Specifying a measurement using an observable.

Notation: an observable is a hermitian matrix acting on a Hilbert Space.

For an observable M, suppose $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues. Let P_j be the projector onto the eigenspace corresponding to λ_j. Then $\sum_{j=1}^{k} P_j = I$ and $\{P_j\}$ defines a measurement.

Conversely, starting from a set of projectors $P_1, \ldots, P_k$, let $M = \sum_{j=1}^{k} r_j P_j$ where the r_j's are distinct real numbers.

The observable M specifies the same measurement as the projectors $P_1, \ldots, P_k$.

Exercise:

Recall the Pauli matrices σ_x, σ_z.

(1) Show that each of σ_x, σ_z has eigenvalues 1, -1.

Find the corresponding eigenvectors.

(2) What are the eigenvalues of $\sigma_x \otimes \sigma_z$?

What is the multiplicity of each eigenvalue?

(3) What are the projectors for the measurement specified by $\sigma_x \otimes \sigma_z$?

(4) Describe a basis, and a partition of the labels of the basis that gives the same measurement.

Discuss the exercise?

Why overall phase of the state vector is irrelevant:

1. The probabilities of the outcomes of a measurement is INDEPENDENT of an overall phase in the state vector.

2. The overall phase can carry over from the pre-measurement state to the post-measurement state but it still will not be observed in later measurements.

Relative phase is crucial, and must be carried over through all steps, e.g., from pre-measurement to post-measurement state.

3. Summary of quantum mechanics

(a) Linear algebra and Dirac notation
(Self-study + test, NC 2.1, KLM 2.1-2.6, 2.8)

(b) Axioms of quantum mechanics
(NC 2.2.1-2.2.5, 2.2.7, KLM 3.1-3.4)

(c) Composite systems, entanglement,
operations on 1 out of 2 systems, locality of QM
(NC 2.2.8-2.2.9)