3. Summary of quantum mechanics

(a) Linear algebra and Dirac notation
(Self-study + test, NC 2.1, KLM 2.1-2.6, 2.8)

(b) Axioms of quantum mechanics
(NC 2.2.1-2.2.5, 2.2.7, KLM 3.1-3.4)

(c) Composite systems, entanglement,
operations on 1 out of 2 systems, locality of QM
(NC 2.2.8-2.2.9)

States on composite systems

Consider two systems S1, S2, with dimensions d1, d2.

How do we describe the most general state on S1 S2?

Method 1: express in tensor product basis

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Take ANY basis of S1 $\{|e_1\rangle, |e_2\rangle, \dots, |e_{d_1}\rangle\}$
and ANY basis of S2 $\{|f_1\rangle, |f_2\rangle, \dots, |f_{d_2}\rangle\}$.

Then, ANY state on S1 S2 can be expressed as:

$|\psi\rangle = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} c_{ij} \, |e_i\rangle \otimes |f_j\rangle$ where $c_{ij} \in \mathbb{C}$, $\sum_{i=1}^{d_1}\sum_{j=1}^{d_2} |c_{ij}|^2 = 1$.

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e.g. $|\psi\rangle = \sqrt{\frac{1}{10}}\,|0\rangle|+\rangle + \sqrt{\frac{2}{10}}\,|0\rangle|-\rangle + i\sqrt{\frac{3}{10}}\,|1\rangle|+\rangle + \sqrt{\frac{4}{10}}\,|1\rangle|-\rangle$

Features: we can choose both basis, but there are
d1 d2 terms in the expression.

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$$= \sum_{i=1}^{d_1} |e_i\rangle \otimes \sum_{j=1}^{d_2} c_{ij} \, |f_j\rangle = \sum_{i=1}^{d_1} |e_i\rangle \otimes |\eta_i\rangle \; \alpha_i$$

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where $\forall i \quad |\eta_i\rangle = \dfrac{\sum_{j=1}^{d_2} c_{ij}\, |f_j\rangle}{\sqrt{\sum_{j=1}^{d_2} |c_{ij}|^2}}$ is a unit vector

$$\alpha_i = \sqrt{\sum_{j=1}^{d_2} |c_{ij}|^2}, \quad \sum_i |\alpha_i|^2 = 1.$$

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$$|\psi\rangle = \sum_{i=1}^{d_1} |e_i\rangle \otimes |\eta_i\rangle \alpha_i$$

where $\forall i$ $|\eta_i\rangle$ is a unit vector, $\sum_i |\alpha_i|^2 = 1$.

Features: only d1 terms, we choose the basis $\{|e_i\rangle\}$

no control what $|\eta_i\rangle$'s are (not even orthonormal).

Note basis for S2 totally disappears.

NB we can choose a basis for S2 and derive the state on S1 instead.

e.g., choose basis for S2 to be $\{|+\rangle, |-\rangle\}$

$$|\psi\rangle = \sqrt{\frac{1}{10}}|0\rangle|+\rangle + \sqrt{\frac{2}{10}}|0\rangle|-\rangle + i\sqrt{\frac{3}{10}}|1\rangle|+\rangle + \sqrt{\frac{4}{10}}|1\rangle|-\rangle$$

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$$= \left(\sqrt{\tfrac{1}{10}}|0\rangle + i\sqrt{\tfrac{3}{10}}|1\rangle\right)|+\rangle + \left(\sqrt{\tfrac{2}{10}}|0\rangle + \sqrt{\tfrac{4}{10}}|1\rangle\right)|-\rangle$$

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$$= \sqrt{\tfrac{4}{10}}\,|\eta_+\rangle|+\rangle + \sqrt{\tfrac{6}{10}}\,|\eta_-\rangle|-\rangle$$

where $|\eta_+\rangle = \dfrac{\left(\sqrt{\tfrac{1}{10}}|0\rangle + i\sqrt{\tfrac{3}{10}}|1\rangle\right)}{\sqrt{\tfrac{4}{10}}} = \tfrac{1}{2}|0\rangle + i\sqrt{\tfrac{3}{2}}|1\rangle,$

$$|\eta_-\rangle = \frac{\left(\sqrt{\tfrac{2}{10}}|0\rangle + \sqrt{\tfrac{4}{10}}|1\rangle\right)}{\sqrt{\tfrac{6}{10}}} = \sqrt{\tfrac{1}{3}}|0\rangle + \sqrt{\tfrac{2}{3}}|1\rangle.$$

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where $d = \min(d_1, d_2)$, $c_i \in \mathbb{C}$, $\sum_{i=1}^{d} |c_i|^2 = 1$,

$\{|e_1\rangle, |e_2\rangle, \dots, |e_d\rangle\}$ is an orthonormal set of states on S1

$\{|f_1\rangle, |f_2\rangle, \dots, |f_d\rangle\}$ is an orthonormal set of states on S2

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Features: only d terms, BOTH $\{|e_i\rangle\}, \{|f_i\rangle\}$ are o.n.

(like bases) but we CANNOT choose either.

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Proof will be defered to week 7-8. We will not need this representation until then.

Summary:

Method 1: double sum,
expressed in a basis in each system,
can choose both bases.

Method 2: single sum,
expressed in a basis in one system,
can choose that basis.

Method 3: single sum,
expressed in a basis in each system,
cannot choose either basis.

States on composite systems

What about 3 or more systems, say, r systems?

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A state on S1 S2 ... Sr can be expressed as a state on S1 and S2 ... Sr (the latter as a single system), then, recursively, each state on S2 ... Sr can be expressed as a state on S2 and S3 ... Sr and so on.

States on composite systems

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Each analysis is facilitated by some representation.

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e.g., In the Schmidt decomposition, we will see that the state is entangled (not a product state) if and only if there are at least two nonzero c_i's.

e.g., second representation is particularly useful when considering operation on one out of two systems that we will see next.

Recall:

In the linear algebraic formalism for evolving a classical circuit, we apply the matrix representation of the gate to the vector residing on the input register of the gate, tensor with the identity operator on the other registers.

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Why such formalism?

Because that's how we evolve quantum mechanical systems and quantum circuits.

The only difference is that, in classical circuits, the state vector is a probability distribution over the classical bit values, so, the vector has non-negative entries, and is a unit vector in 1-norm. In QM, the state vector is a unit vector in 2-norm.

need definitions?

Operation on one out of two quantum systems

Consider two systems S1, S2 with d1, d2 dims.

If the initial state is $|\psi\rangle$, a unitary U is applied to S1, then the final state is $U \otimes I \, |\psi\rangle$.

e.g., d1 = d2 = 2.

$$|\psi\rangle = \sqrt{\frac{1}{10}}|0\rangle|0\rangle + \sqrt{\frac{2}{10}}|0\rangle|1\rangle + \sqrt{\frac{3}{10}}|1\rangle|0\rangle + \sqrt{\frac{4}{10}}|1\rangle|1\rangle$$

If $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is applied to S2, what is the final state?

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If $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is applied to S2, what is the final state?

Method 1: in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$$|\psi\rangle = \begin{pmatrix} \sqrt{\frac{1}{10}} \\ \sqrt{\frac{2}{10}} \\ \sqrt{\frac{3}{10}} \\ \sqrt{\frac{4}{10}} \end{pmatrix}, \quad I \otimes X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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Final state = $I \otimes X |\psi\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{10}} \\ \sqrt{\frac{2}{10}} \\ \sqrt{\frac{3}{10}} \\ \sqrt{\frac{4}{10}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{10}} \\ \sqrt{\frac{1}{10}} \\ \sqrt{\frac{4}{10}} \\ \sqrt{\frac{3}{10}} \end{pmatrix}$.

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e.g., $|\psi\rangle = \sqrt{\frac{1}{10}}|0\rangle|0\rangle + \sqrt{\frac{2}{10}}|0\rangle|1\rangle + \sqrt{\frac{3}{10}}|1\rangle|0\rangle + \sqrt{\frac{4}{10}}|1\rangle|1\rangle$

If $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is applied to S2, what is the final state?

Method 2:

Final state = $I \otimes X |\psi\rangle$

$= I \otimes X \left(\sqrt{\frac{1}{10}}|0\rangle|0\rangle + \sqrt{\frac{2}{10}}|0\rangle|1\rangle + \sqrt{\frac{3}{10}}|1\rangle|0\rangle + \sqrt{\frac{4}{10}}|1\rangle|1\rangle \right)$

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$= I \otimes X \sqrt{\frac{1}{10}}|0\rangle|0\rangle + I \otimes X \sqrt{\frac{2}{10}}|0\rangle|1\rangle$

$+ I \otimes X \sqrt{\frac{3}{10}}|1\rangle|0\rangle + I \otimes X \sqrt{\frac{4}{10}}|1\rangle|1\rangle$

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If $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is applied to S2, what is the final state?

Method 2:

Final state = $I \otimes X \, |\psi\rangle$

$$= I \otimes X \left(\sqrt{\tfrac{1}{10}}|0\rangle|0\rangle + \sqrt{\tfrac{2}{10}}|0\rangle|1\rangle + \sqrt{\tfrac{3}{10}}|1\rangle|0\rangle + \sqrt{\tfrac{4}{10}}|1\rangle|1\rangle \right)$$

$$= I \otimes X \sqrt{\tfrac{1}{10}}|0\rangle|0\rangle + I \otimes X \sqrt{\tfrac{2}{10}}|0\rangle|1\rangle$$

$$+ I \otimes X \sqrt{\tfrac{3}{10}}|1\rangle|0\rangle + I \otimes X \sqrt{\tfrac{4}{10}}|1\rangle|1\rangle$$

$$= \sqrt{\tfrac{1}{10}}|0\rangle|1\rangle + \sqrt{\tfrac{2}{10}}|0\rangle|0\rangle$$

$$+ \sqrt{\tfrac{3}{10}}|1\rangle|1\rangle + \sqrt{\tfrac{4}{10}}|1\rangle|0\rangle .$$

Ex. Check that the 2 methods give the same answers.

Quick question:

e.g., $|\psi\rangle = \sqrt{\frac{1}{10}}|0\rangle|0\rangle + \sqrt{\frac{2}{10}}|0\rangle|1\rangle + \sqrt{\frac{3}{10}}|1\rangle|0\rangle + \sqrt{\frac{4}{10}}|1\rangle|1\rangle$

If $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is applied to the first qubit, what is the final state?

(a) $\sqrt{\frac{1}{10}}|0\rangle|0\rangle + \sqrt{\frac{2}{10}}|0\rangle|1\rangle + \sqrt{\frac{3}{10}}|1\rangle|0\rangle - \sqrt{\frac{4}{10}}|1\rangle|1\rangle$

(b) $\sqrt{\frac{1}{10}}|0\rangle|0\rangle + \sqrt{\frac{2}{10}}|0\rangle|1\rangle - \sqrt{\frac{3}{10}}|1\rangle|0\rangle - \sqrt{\frac{4}{10}}|1\rangle|1\rangle$

(c) $\sqrt{\frac{1}{10}}|0\rangle|0\rangle - \sqrt{\frac{2}{10}}|0\rangle|1\rangle + \sqrt{\frac{3}{10}}|1\rangle|0\rangle - \sqrt{\frac{4}{10}}|1\rangle|1\rangle$

(d) $-\sqrt{\frac{1}{10}}|0\rangle|0\rangle + \sqrt{\frac{2}{10}}|0\rangle|1\rangle + \sqrt{\frac{3}{10}}|1\rangle|0\rangle + \sqrt{\frac{4}{10}}|1\rangle|1\rangle$

Measuring one out of two quantum systems

Vast majority of this course is based on measuring one of two quantum systems ...

e.g., $|\psi\rangle = \sqrt{\frac{1}{10}}|0\rangle|0\rangle + \sqrt{\frac{2}{10}}|0\rangle|1\rangle + \sqrt{\frac{3}{10}}|1\rangle|0\rangle + \sqrt{\frac{4}{10}}|1\rangle|1\rangle$

What happens if we measure the second qubit along the computational basis?

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What happens if we measure the second qubit along the computational basis?

note shorthand

Answer: in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

this is an incomplete measurement distinguishing

$S_0 = \{|00\rangle, |10\rangle\}$ and $S_1 = \{|01\rangle, |11\rangle\}$.

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$S_0 = \{|00\rangle, |10\rangle\}$ and $S_1 = \{|01\rangle, |11\rangle\}$.

Probability to get "0": 1/10 + 3/10 = 0.4

Postmeasurement state: $\left(\sqrt{\frac{1}{10}}|0\rangle + \sqrt{\frac{3}{10}}|1\rangle\right) \otimes |0\rangle \Big/ \sqrt{0.4}$

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Postmeasurement state: $\left(\sqrt{\frac{1}{10}}|0\rangle + \sqrt{\frac{3}{10}}|1\rangle\right) \otimes |0\rangle / \sqrt{0.4}$

Probability to get "1": 2/10 + 4/10 = 0.6

Postmeasurement state: $\left(\sqrt{\frac{2}{10}}|0\rangle + \sqrt{\frac{4}{10}}|1\rangle\right) \otimes |1\rangle / \sqrt{0.6}$.

Exercise:

This is an incomplete measurement distinguishing

$S_0 = \{|00\rangle, |10\rangle\}$ and $S_1 = \{|01\rangle, |11\rangle\}$.

Show that the measurement can also be described by the projectors:

$$P_0 = I \otimes |0\rangle\langle 0|, \quad P_1 = I \otimes |1\rangle\langle 1|.$$

Find the probability of each outcome and the post-measurement states using the projectors.

Measuring one out of two quantum systems

e.g., $|\psi\rangle = \sqrt{\frac{1}{10}}|0\rangle|0\rangle + \sqrt{\frac{2}{10}}|0\rangle|1\rangle + \sqrt{\frac{3}{10}}|1\rangle|0\rangle + \sqrt{\frac{4}{10}}|1\rangle|1\rangle$

What happens if we measure the second qubit along the $\{|+\rangle, |-\rangle\}$ basis, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$?

Measuring one out of two quantum systems

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Answer: in the basis $\{|0+\rangle, |0-\rangle, |1+\rangle, |1-\rangle\}$

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$S_+ = \{|0+\rangle, |1+\rangle\}$ and $S_- = \{|0-\rangle, |1-\rangle\}$.

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this is an incomplete measurement distinguishing

$S_+ = \{|0+\rangle, |1+\rangle\}$ and $S_- = \{|0-\rangle, |1-\rangle\}$.

The pre-measurement state is given to us in a basis different from the measurement basis.

Method 1: express the pre-measurement state in the measurement basis.

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Recall: $|0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$, $|1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \quad + \quad \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$|\psi\rangle = \sqrt{\tfrac{1}{10}}\,|0\rangle|0\rangle + \sqrt{\tfrac{2}{10}}\,|0\rangle|1\rangle + \sqrt{\tfrac{3}{10}}\,|1\rangle|0\rangle + \sqrt{\tfrac{4}{10}}\,|1\rangle|1\rangle$$

Method 1: express the pre-measurement state in the measurement basis.

Note that: $|0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$, $|1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \quad + \quad \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$|\psi\rangle = \sqrt{\tfrac{1}{10}}\,|0\rangle|0\rangle + \sqrt{\tfrac{2}{10}}\,|0\rangle|1\rangle + \sqrt{\tfrac{3}{10}}\,|1\rangle|0\rangle + \sqrt{\tfrac{4}{10}}\,|1\rangle|1\rangle$$

$$\begin{aligned}
&= \sqrt{\tfrac{1}{10}}\,|0\rangle \otimes \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \\
&\quad + \sqrt{\tfrac{2}{10}}\,|0\rangle \otimes \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right) \\
&\quad + \sqrt{\tfrac{3}{10}}\,|1\rangle \otimes \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \\
&\quad + \sqrt{\tfrac{4}{10}}\,|1\rangle \otimes \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)
\end{aligned}$$

Method 1: express the pre-measurement state in the measurement basis.

Note that: $$|0\rangle = \frac{1}{\sqrt{2}}\Big(\underbrace{|+\rangle}_{\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)} + \underbrace{|-\rangle}_{\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)}\Big), \quad |1\rangle = \frac{1}{\sqrt{2}}\big(|+\rangle - |-\rangle\big)$$

$$|\psi\rangle = \sqrt{\tfrac{1}{10}}\,|0\rangle|0\rangle + \sqrt{\tfrac{2}{10}}\,|0\rangle|1\rangle + \sqrt{\tfrac{3}{10}}\,|1\rangle|0\rangle + \sqrt{\tfrac{4}{10}}\,|1\rangle|1\rangle$$

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\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2}}\left[\left(\sqrt{\tfrac{1}{10}} + \sqrt{\tfrac{2}{10}}\right)|0\rangle + \left(\sqrt{\tfrac{3}{10}} + \sqrt{\tfrac{4}{10}}\right)|1\rangle\right] \otimes |+\rangle \\
&\quad + \frac{1}{\sqrt{2}}\left[\left(\sqrt{\tfrac{1}{10}} - \sqrt{\tfrac{2}{10}}\right)|0\rangle + \left(\sqrt{\tfrac{3}{10}} - \sqrt{\tfrac{4}{10}}\right)|1\rangle\right] \otimes |-\rangle
\end{aligned}$$

$$= \left[0.5398 \,|0\rangle + 0.8345 \,|1\rangle \right] \otimes |+\rangle$$

$$+ \left[-0.0599 \,|0\rangle + -0.0926 \,|1\rangle \right] \otimes |-\rangle$$

$$= \left[0.5398 \, |0\rangle + 0.8345 \, |1\rangle \right] \otimes |+\rangle$$

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$$= \sqrt{0.5398^2 + 0.8345^2} \; \frac{\left[0.5398 \, |0\rangle + 0.8345 \, |1\rangle \right]}{\sqrt{0.5398^2 + 0.8345^2}} \otimes |+\rangle$$

$$+ \underbrace{\sqrt{0.0599^2 + 0.0926^2}}_{\text{square of this is the prob to get "-"}} \; \underbrace{\frac{\left[-0.0599 \, |0\rangle + -0.0926 \, |1\rangle \right]}{\sqrt{0.0599^2 + 0.0926^2}}}_{\text{unit vector, tensored with } |-\rangle \text{ is the postmeas state if "-"}} \otimes |-\rangle$$

Similarly for the "+" outcome.

Note what we just did was to express the state

$$|\psi\rangle = \sqrt{\tfrac{1}{10}}|0\rangle|0\rangle + \sqrt{\tfrac{2}{10}}|0\rangle|1\rangle + \sqrt{\tfrac{3}{10}}|1\rangle|0\rangle + \sqrt{\tfrac{4}{10}}|1\rangle|1\rangle$$

in the composite system (of two qubits) using method 2, where we choose $\{|+\rangle, |-\rangle\}$ as the basis for the 2nd qubit.

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Measuring one out of two quantum systems

e.g., $|\psi\rangle = \sqrt{\frac{1}{10}}|0\rangle|0\rangle + \sqrt{\frac{2}{10}}|0\rangle|1\rangle + \sqrt{\frac{3}{10}}|1\rangle|0\rangle + \sqrt{\frac{4}{10}}|1\rangle|1\rangle$

What happens if we measure the second qubit along the $\{|+\rangle, |-\rangle\}$ basis, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$?

Answer: in the basis $\{|0+\rangle, |0-\rangle, |1+\rangle, |1-\rangle\}$

this is an incomplete measurement distinguishing

$S_+ = \{|0+\rangle, |1+\rangle\}$ and $S_- = \{|0-\rangle, |1-\rangle\}$.

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Method 2: the meas is described by the projectors

$P_+ = I \otimes |+\rangle\langle+|$, $P_- = I \otimes |-\rangle\langle-|$.

Ex: find the prob of each outcome $\| P_+|\psi\rangle \|^2$, $\| P_-|\psi\rangle \|^2$
and the post-measurement states $\frac{P_+|\psi\rangle}{\| P_+|\psi\rangle \|}$, $\frac{P_-|\psi\rangle}{\| P_-|\psi\rangle \|}$.

Generalizing the above example ...

Theorem: for two systems S1, S2, of d1, d2 dims,

if S1 is measured along the basis $\{|e_i\rangle\}$

then, the measurement has outcome "i" with prob $|\alpha_i|^2$

with corresponding postmeasurement state $|e_i\rangle \otimes |\eta_i\rangle$

for an initial state expressible in the form:

$$|\psi\rangle = \sum_{i=1}^{d_1} |e_i\rangle \otimes |\eta_i\rangle \, \alpha_i$$

basis being measured

unit vectors, need not be mutually orthonormal

3. Summary of quantum mechanics

✓ (a) Linear algebra and Dirac notation
(Self-study + test, NC 2.1, KLM 2.1-2.6, 2.8)

✓ (b) Axioms of quantum mechanics
(NC 2.2.1-2.2.5, 2.2.7, KLM 3.1-3.4)

→ (c) Composite systems, entanglement,
operations on 1 out of 2 systems, locality of QM
(NC 2.2.8-2.2.9)

Locality of quantum mechanics

Suppose Alice and Bob each holds one quantum system, and they share a joint initial state.

When, Alice measures her system, the GLOBAL post-measurement state can depend on her measurement outcome.

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Alice measures along the $\{|0\rangle, |1\rangle\}$ basis.

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Better not!

Alice knows her measurement outcome and Bob's state. BUT Bob DOESN'T.

Bob doesn't even know if Alice has made the meas or if she ever would. Whether Alice has measured or not, Bob only knows his state is $|0\rangle$ with prob 1/2, and $|1\rangle$ with prob 1/2.

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Without physically transmitting data, Alice's and Bob's independent actions cannot affect one another, else we run into a logical inconsistency.

Mathematically, their operations commute.

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Mathematically, their operations commute.

In A1, you will prove a special case, wherein, for any initial 2-qubit state, for any measurement Alice can apply, and any measurement Bob can apply, Alice's output distribution is independent of Bob's measurement, and vice versa.