# 5. Quantum circuits

(a) Quantum circuit model (KLM 4.1, NC 1.3.4) ✓

(b) Quantum gates (NC 4.2-4.3, KLM 4.2) ✓

(c) Continuous universal set of quantum gates (reading) (NC 4.5.1-4.5.2, KLM 4.3)

(d) Quantum gate approximations (NC Box 4.1, KLM 4.3)

(e) Finite universal set of q. gates (NC 4.5.3, KLM 4.3)

(f) Efficiency & Kitaev-Solovay thm (NC App 3, KLM 4.4)

(g) Quantum circuits for measurements (reading/defer) (KLM 4.5*)

(h) Hardness of approximating most unitaries (reading) (NC 4.5.6)

The unitaries acting on a Hilbert space form a continuous set. Is a continuous set of gates needed for universality (e.g., CNOT+all 1-qubit gates)?

Idea: approximating any unitary to arbitrary accuracy is good enough.

<u>How to measure the quality of approximate unitaries?</u>

Depends on the goal !

Here: replace one gate (that we want) by another (that we can apply) in a circuit without affecting the correctness of the "computation."

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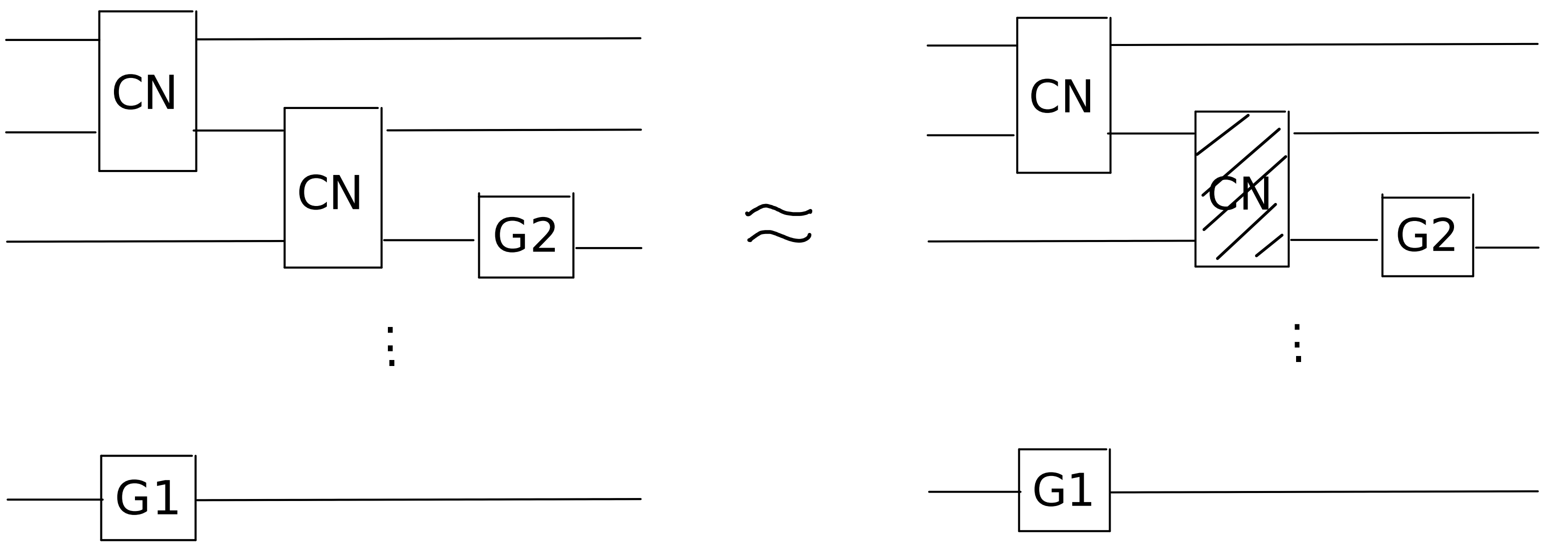
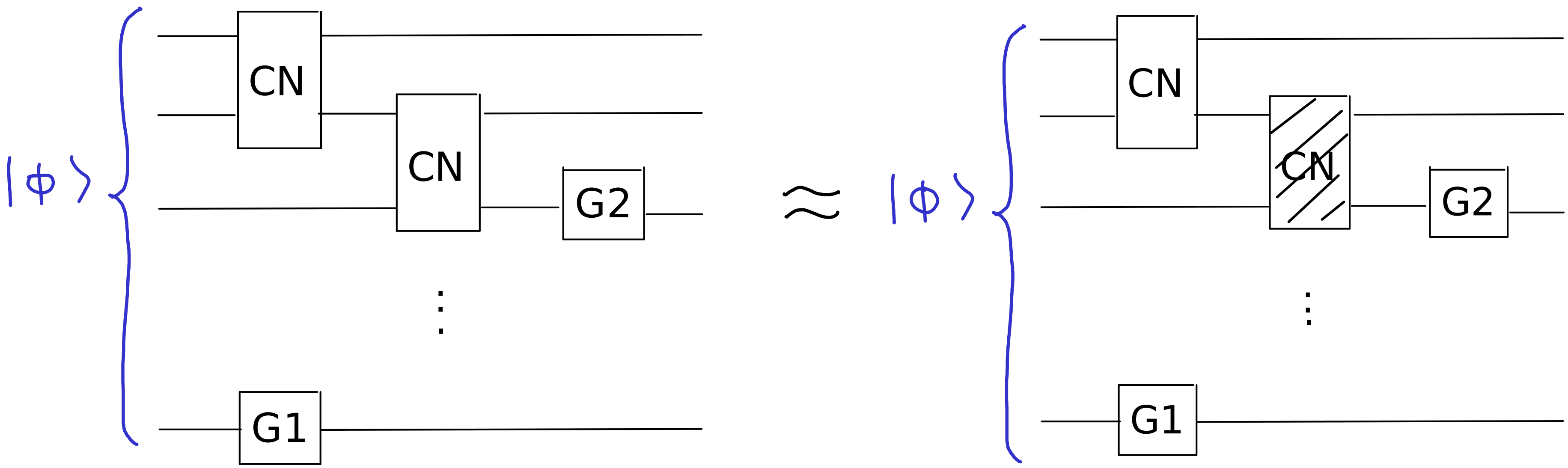


## How to measure the quality of approximate unitaries?

If, for all possible input $|\phi\rangle$ , the outputs from the two circuits are "similar", our computation is not too affected and the approximation is good.



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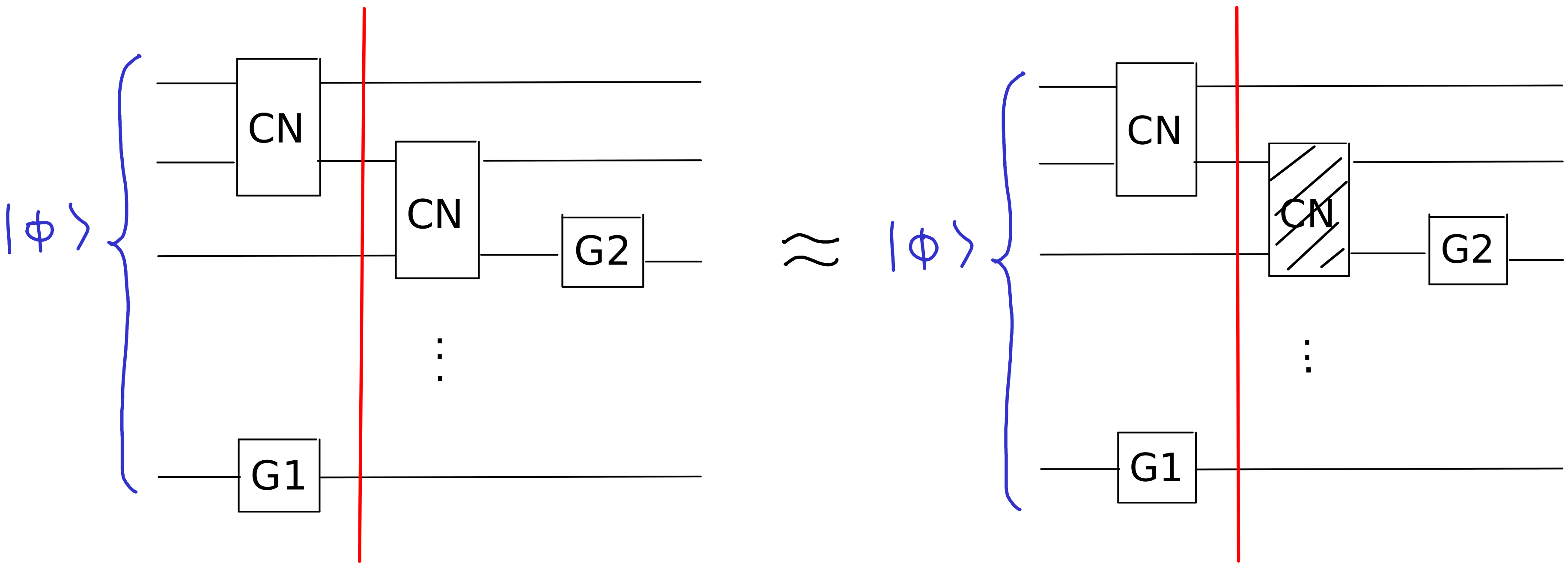


Up to red stage: identical states in both circuits

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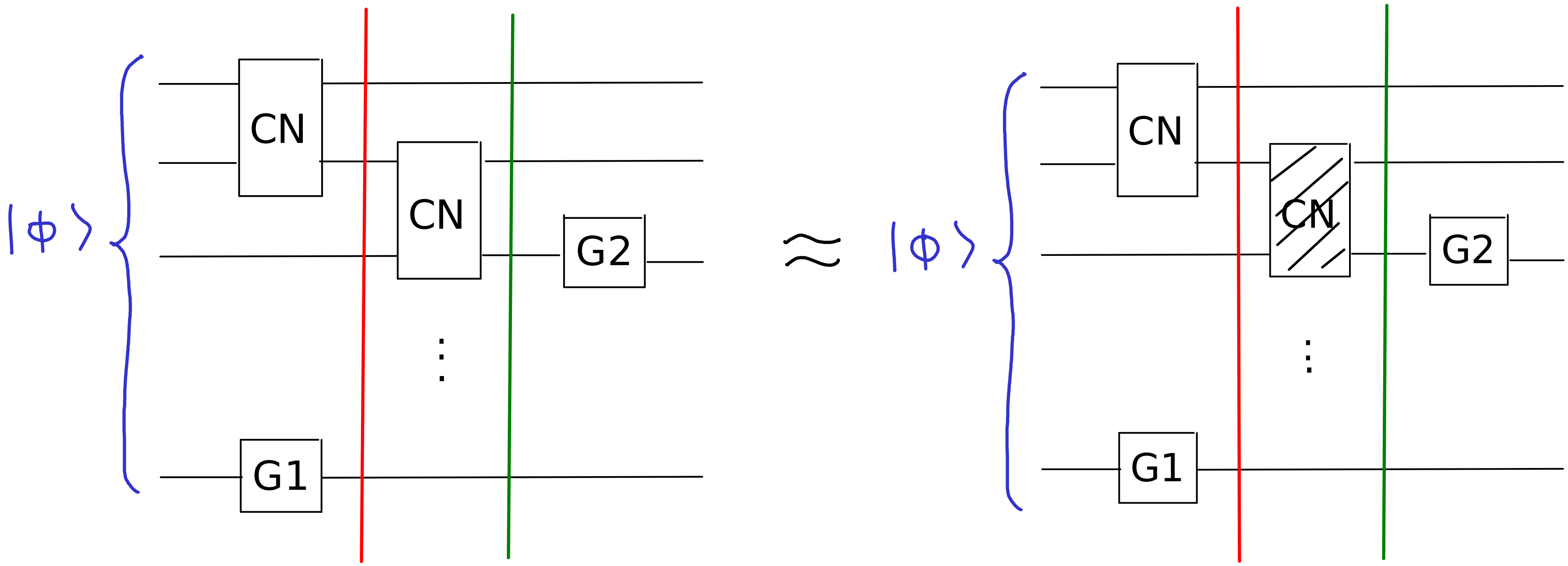


Up to red stage: identical states in both circuits

Green stage onwards, identical computation

Suffices if the states at the green stage are similar.

# How to measure the quality of approximate unitaries?

Definition: the error of approximating U by V is

$$E^*(U,V) = \max_{|\psi\rangle_{RS}} \left\| (I \otimes U)|\psi\rangle - (I \otimes V)|\psi\rangle \right\|$$

where U, V act on system S, and system R is arbitrary.

($|\psi\rangle_{RS}$ unit vector.)

Euclidean norm

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For our circuit:
S = qubit(s) acted on by the gate to be approximate
R = all other qubits

$|\psi\rangle_{RS}$ the worst case state right before the gate.

## Indistinguisability:

Exercise: for two unit vectors $|a\rangle, |b\rangle$

$$\| |a\rangle - |b\rangle \| = \sqrt{2}\sqrt{1 - \mathrm{Re}\langle a|b\rangle}$$

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If each of $|\psi_1\rangle, |\psi_2\rangle$ is chosen with probability 1/2,

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$$\leq \frac{1}{2} + \frac{1}{2}\frac{1}{\sqrt{2}}\| |\psi_1\rangle - |\psi_2\rangle \|\sqrt{2}$$

So, $I\otimes U|\psi\rangle$, $I\otimes V|\psi\rangle$ can be distinguished with prob

$$\leq \frac{1}{2} + \frac{1}{2}E^*(U, V)$$

so no one can tell if U or V has been applied if E* small

# How to evaluate this error?

$$E^*(U,V) = \max_{|\psi\rangle_{RS}} \| I \otimes U |\psi\rangle - I \otimes V |\psi\rangle \|$$

R is arbitrary

i.e., We max over all possible R.
Non-trivial: we can limit dim(R) to dim(S) without affecting the value of the optimization (deferring the proof which needs more about the Schmidt decomposition).

Note the difference from NC etc.
R needed to compose approximations.

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NB. for fixed R, the optimization is compact so $|\psi\rangle$ exists.

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(3) Let $\{|e_i\rangle\}_{i=1}^{r}$ be a basis for R (dim r). Express

$$|\psi\rangle_{RS} = \sum_{i=1}^{r} \alpha_i |e_i\rangle \otimes |\eta_i\rangle$$

where $\alpha_i \geq 0$, $\sum_{i=1}^{r} \alpha_i^2 = 1$, $|\eta_i\rangle$ are unit vectors on S.

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$$= \sum_{i=1}^{r} \alpha_i^2 \| (U-V)|\eta_i\rangle \|^2 \leq \underbrace{\sum_{i=1}^{r} \alpha_i^2}_{=1} E(U,V)^2 .$$

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Suppose a gate U1 on S1 is approximated by V1, and then a gate U2 on S2 is approximate by V2.

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$$E\left( (U_{2\,S_2} \otimes I_{R_2})(U_{1\,S_1} \otimes I_{R_1}) \,,\, (V_{2\,S_2} \otimes I_{R_2})(V_{1\,S_1} \otimes I_{R_1}) \right)$$

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see NC for proof

$$= E(U_2, V_2) + E(U_1, V_1).$$

So, error of composition is subadditive, & without *.

Recursively, for a circuit with m gates each with error e', the overall circuit error is at most e'm.
To have overall error e, it suffices to implement each gate with error at most e/m.

Definition: A set of gates G is universal for quantum computation, if for any positive integer n, any n-qubit unitary U, and any e > 0, there are $V_1, V_2, \ldots, V_k$ in G

s.t. $E(U, V_k \ldots V_2 V_1) \leq e$ .

suppressing the $I \otimes$ parts

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Theorem: {H, T, CNOT} is universal for QC, where

T is called the $\frac{\pi}{8}$-gate

$T = R_z\left(\frac{\pi}{4}\right)$ .

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Proof: see NC 4.5.2-4.5.3 or PMATH 343.

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Ideas:

(1) HTH = $R_x\left(\frac{\pi}{4}\right)$

irrational multiple of $\pi$ !!

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(3) $\forall \xi, \forall \varepsilon, \exists r$ s.t. $E(U^r, R_{\hat{n}}(\xi)) \leq \varepsilon$

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Any single qubit gate is a composition of some sequence of $R_{\hat{n}}$ and $R_{\hat{m}}$.

It is CRUCIAL that the universal gate set is discrete !

Quantum computation is discrete, not analog.
This is how noise can be handled.

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Solovay-Kitaev Theorem

Given any universal set of 1-qubit gates G, whose inverses can be implemented exactly, any 1-qubit gate can be approximated with error $< \varepsilon$

using poly( log( $\frac{1}{\varepsilon}$ ) ) gates.

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Remark:
Most universal sets of gates are very efficient in approximating single qubit gates.

Proof idea: optional reading in NC Appendix 3,
or PMATH 343.

## Consequence of the Solovay-Kitaev Theorem

Suppose circuit C has m CNOT and 1-qubit gates.

There is a circuit C' with m' = m poly( log( m/e ) ) gates from {CNOT, H, T} approximating C with error at most e.

i.e., circuit complexity is largely preserved.

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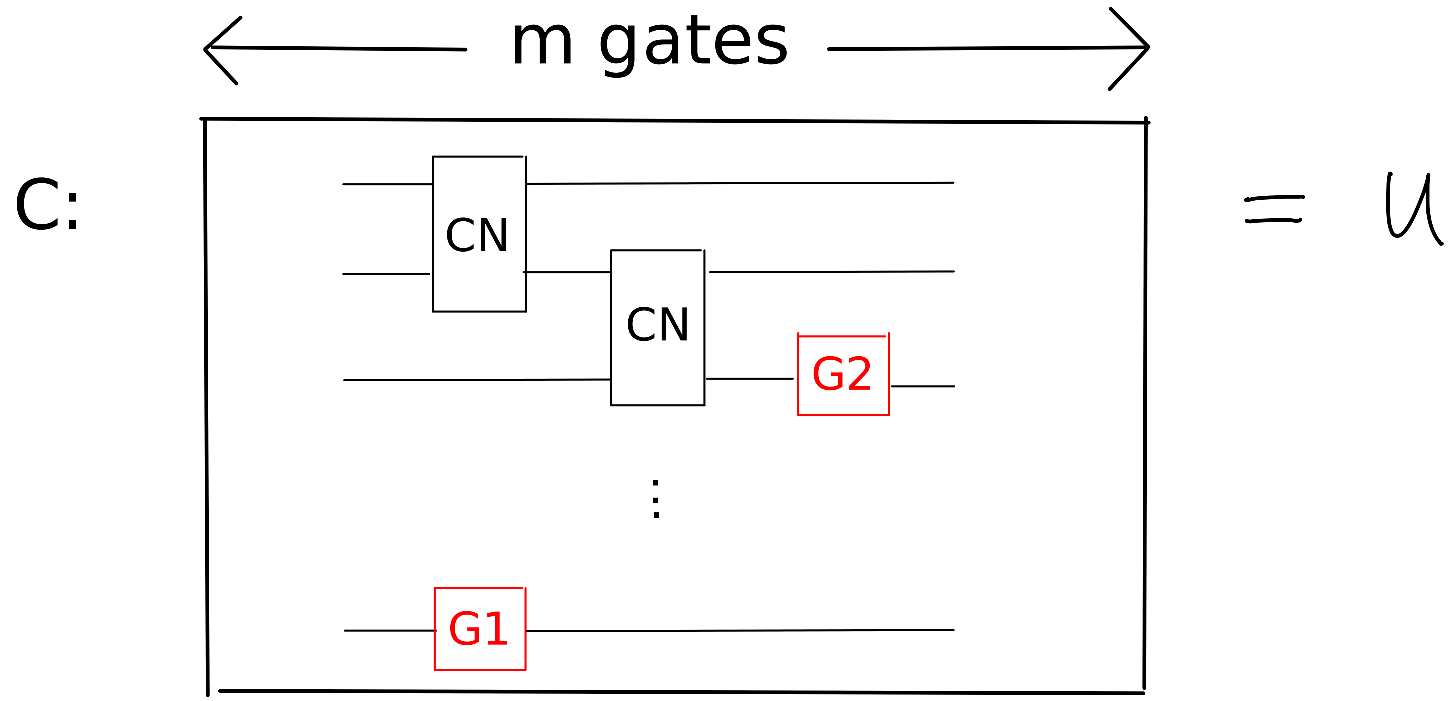
Idea:

Suffices to approx each 1-qubit gate in C with error <= e/m, which takes poly(log( m/e )) H & T gates by SK-thm. So, total # gates m'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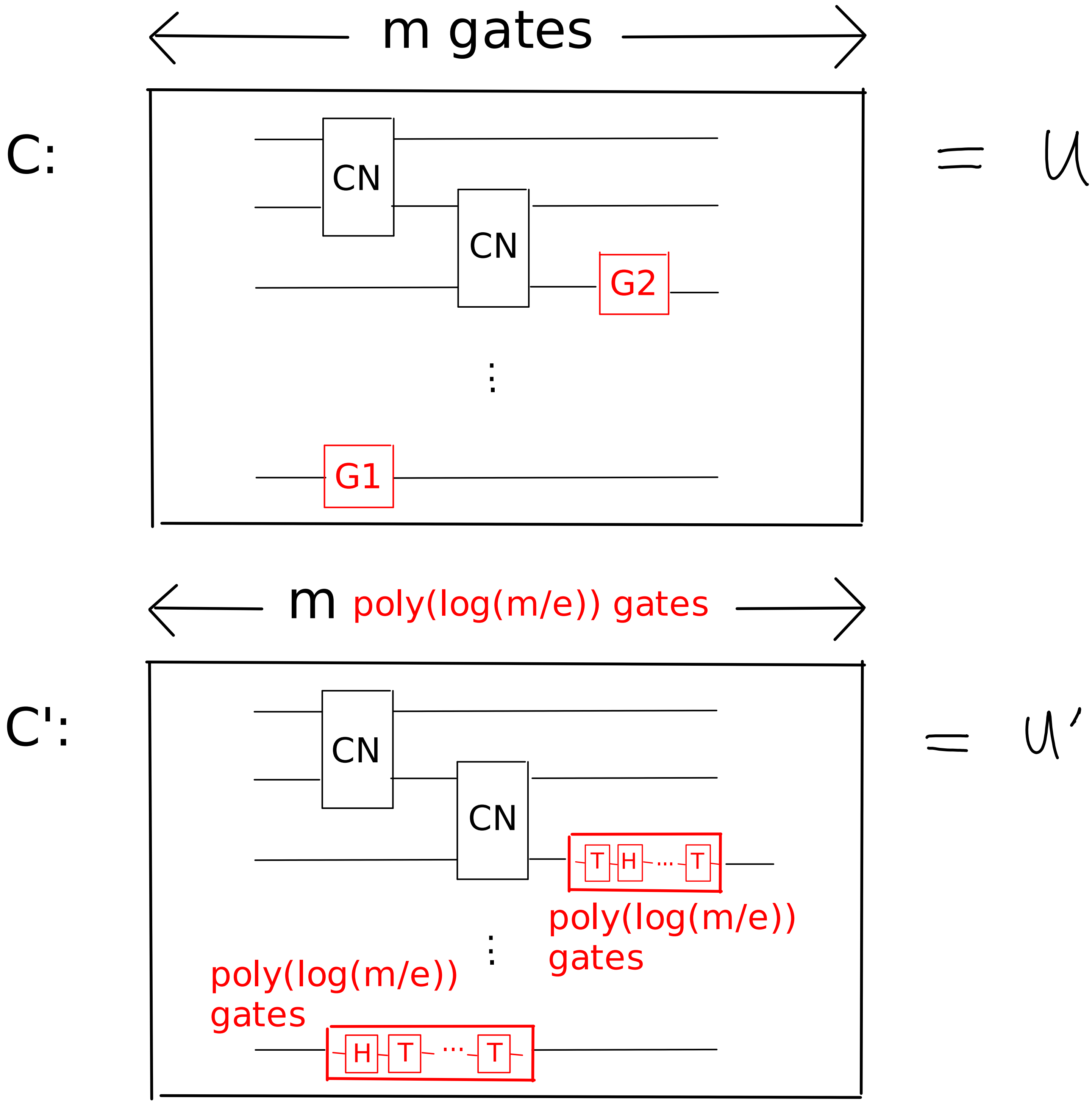
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In short, most classical and quantum computations requires a circuit of exponentially many gates ...

There are too many different computations, but too few gates in the universal gate set.

Polynomial-sized circuits and computations are rare!