Syllabus:

When QM meets information theory (wk 1-3)

QM & immediate consequences for info processing

Noiseless Q computation of classical problems (wk 4-7)

Q circuits, universality, Q algorithms

* Q computation in the presence of noise (wk 8-10)

(8) Modelling noise: mixed state QM & Q operations

(9) Quantum error correction

(10) Reliable Q computation using noisy components

Q computation in the presence of adversary (wk 11-12)

Q cryptography

8. Modelling noise: mixed state QM & Q operations

States:

(b) States on composite systems (NC 8.3.1, 2.5, KLM 3.5.2)

(a) Noisy quantum data (NC 2.4, KLM 3.5.1)

Evolution:

(c) Most general (noisy) quantum dynamics

(d) Characterizations (NC 8.2, KLM 3.5.3)

(e) Important examples (NC 8.3)

Measurements (reading exercise):

(f) POVM measurements (NC 2.2.6, KLM A8)

(g) Trace distance, indistinguishability,
Helstrom-Holevo theorem (NC 9.2, KLM A8)

(a) Noisy quantum data on S alone

(b) States on composite systems R&S

(b)

Pure state
QM on 2
systems RS

partial trace
of R

Density matrix on S
(revised measurement)

(a) Noisy quantum data on S alone

(b) States on composite systems R&S

(b)

Pure state
QM on 2
systems RS

partial trace
of R

(a)

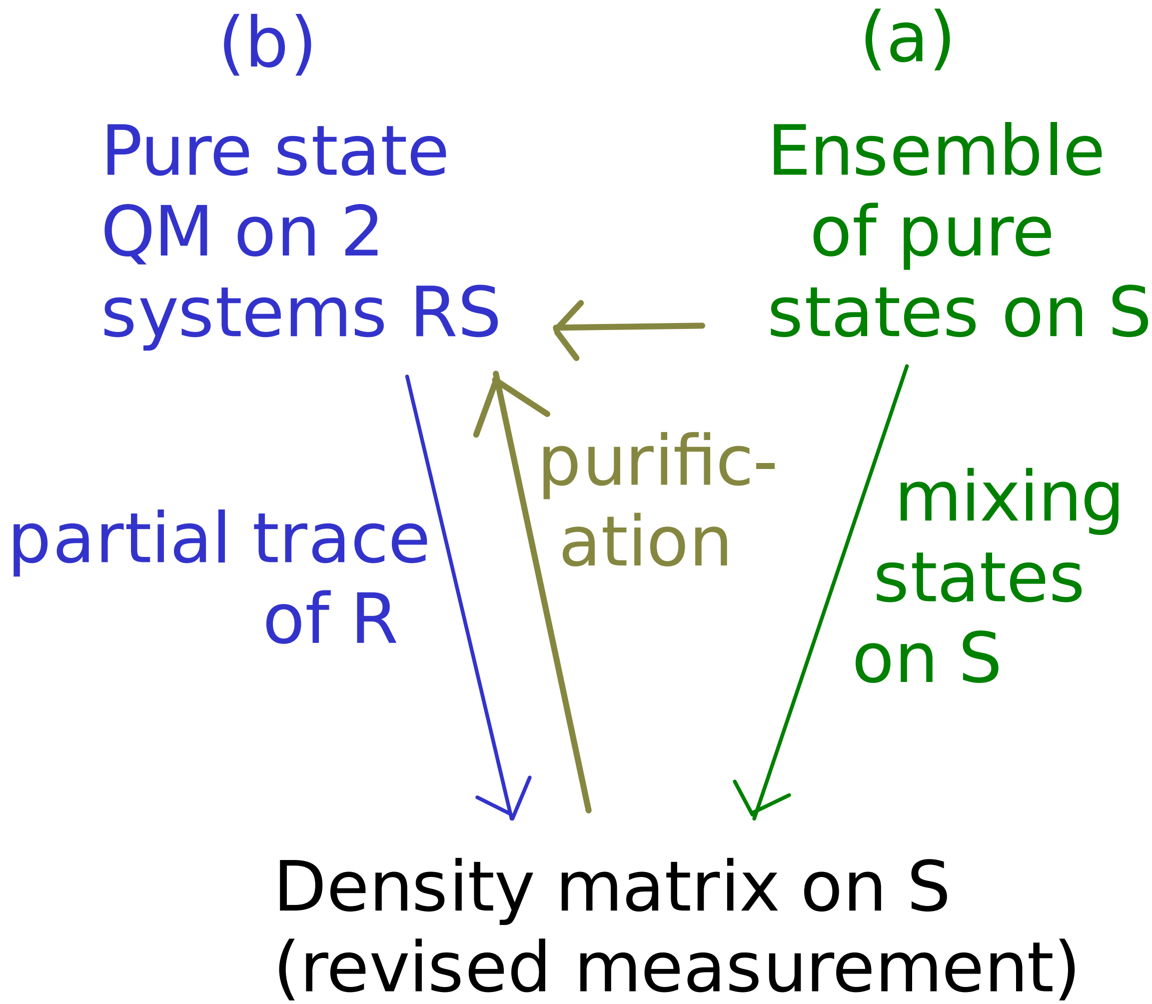
Ensemble
of pure
states on S

mixing
states
on S

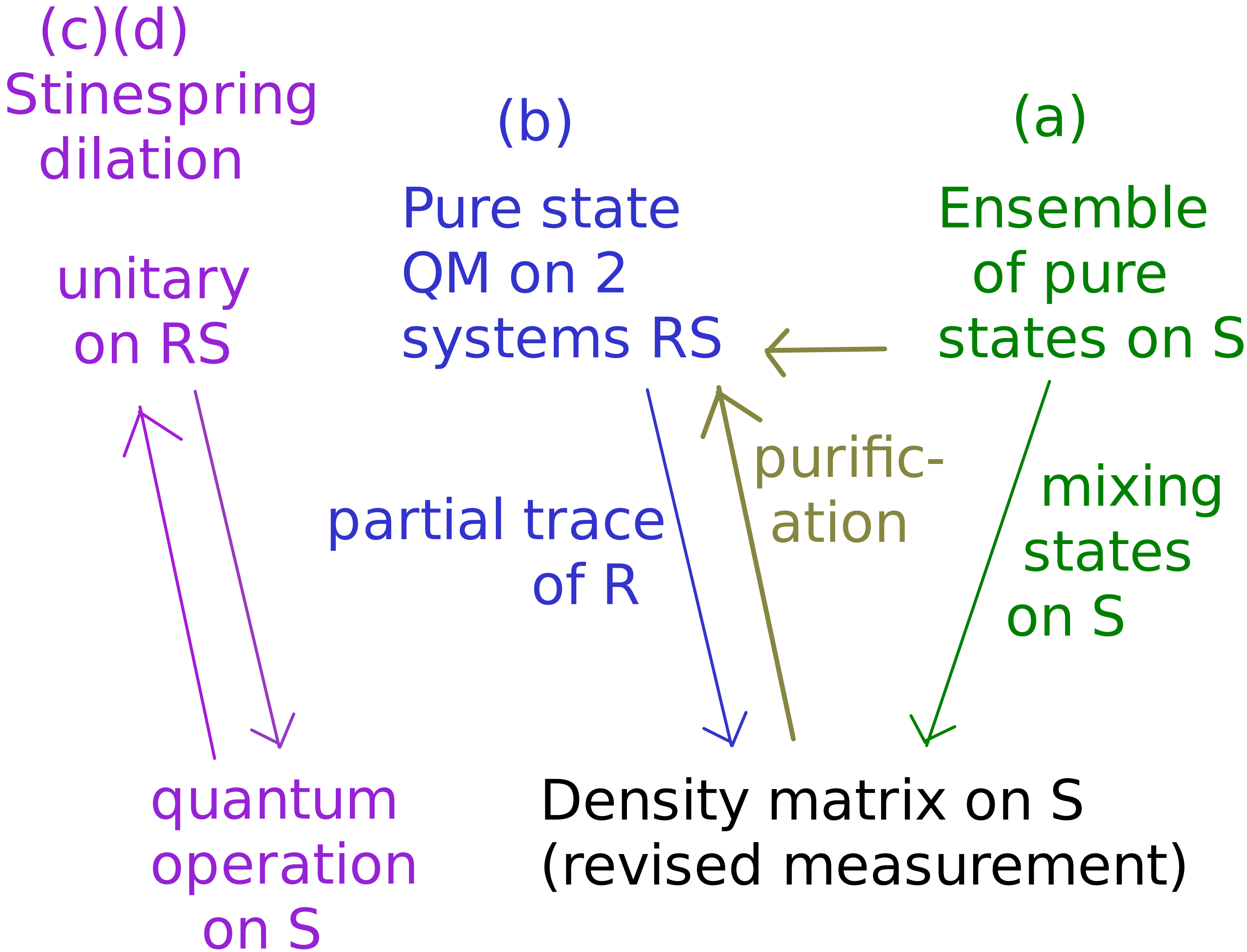
Density matrix on S
(revised measurement)

(a) Noisy quantum data on S alone

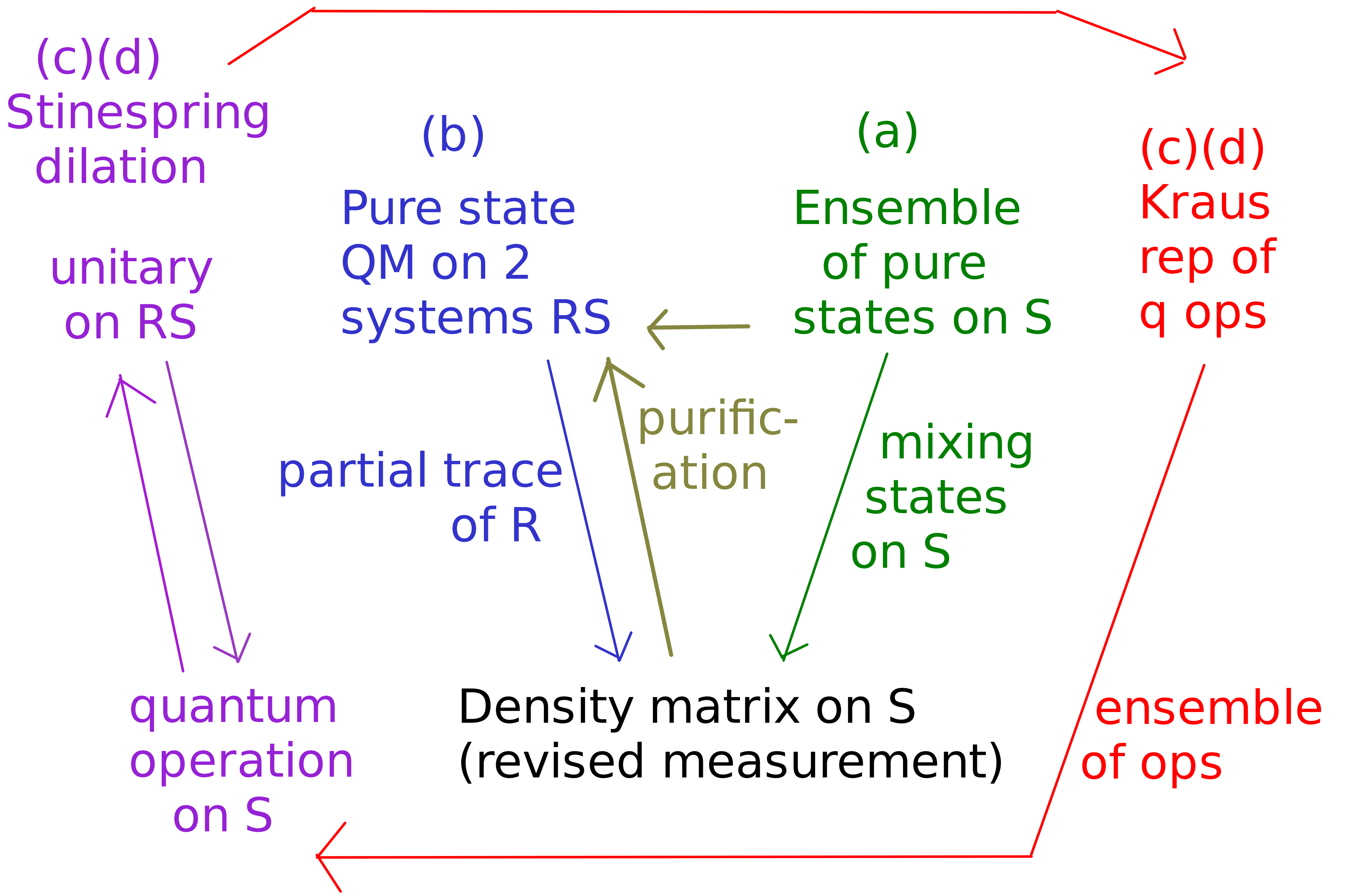
(b) States on composite systems R&S



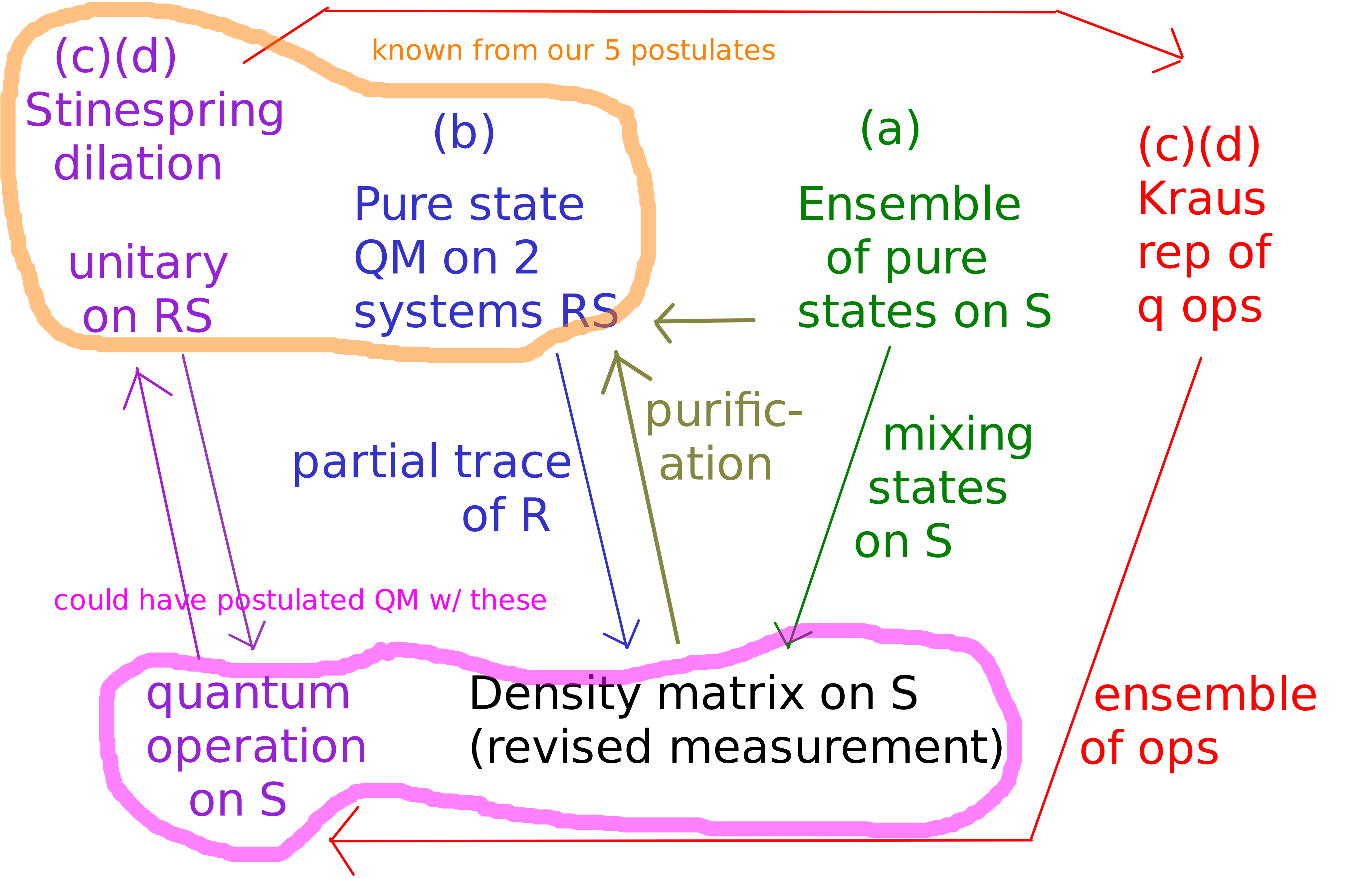
(c)(d) Most general (noisy) quantum dynamics



(c)(d) Most general (noisy) quantum dynamics



(c)(d) Most general (noisy) quantum dynamics



either way, once proved, we use all of the above methods

(b)

Pure state
QM on 2
systems RS

partial trace
of R

Density matrix on S
(revised measurement)

Motivation:

Given a possibly entangled pure state on systems RS, but we operate (unitaries and measurements) on S.

QM on S (without including R)?

Motivation:

Given a possibly entangled pure state on systems RS, but we operate (unitaries and measurements) on S.

QM on S (without including R)?

Example: $|\psi\rangle = a|00\rangle + b|11\rangle$ on RS, a,b real.

For any measurement on S given by projectors $\{P_k\}$

pr(k) = $\| I \otimes P_k |\psi\rangle \|^2$

Motivation:

Given a possibly entangled pure state on systems RS, but we operate (unitaries and measurements) on S.

QM on S (without including R)?

Example: $|\psi\rangle = a|00\rangle + b|11\rangle$ on RS, a,b real.

For any measurement on S given by projectors $\{P_k\}$

pr(k) = $\| I \otimes P_k |\psi\rangle \|^2$

$$I \otimes P_k |\psi\rangle = a|0\rangle \otimes P_k|0\rangle + b|1\rangle \otimes P_k|1\rangle$$

Motivation:

Given a possibly entangled pure state on systems RS, but we operate (unitaries and measurements) on S.

QM on S (without including R)?

Example: $|\psi\rangle = a|00\rangle + b|11\rangle$ on RS, a,b real.

For any measurement on S given by projectors $\{P_k\}$

pr(k) = $\| I \otimes P_k |\psi\rangle \|^2$

$$I \otimes P_k |\psi\rangle = a|0\rangle \otimes P_k|0\rangle + b|1\rangle \otimes P_k|1\rangle$$

$$\| I \otimes P_k |\psi\rangle \|^2 = a^2 \|P_k|0\rangle\|^2 + b^2 \|P_k|1\rangle\|^2$$ (exercise)

Example: $|\psi\rangle = a|00\rangle + b|11\rangle$ on RS, a,b real.

For any measurement on S given by projectors $\{P_k\}$

pr(k) = $\| I \otimes P_k |\psi\rangle \|^2$

$$I \otimes P_k |\psi\rangle = a|0\rangle \otimes P_k|0\rangle + b|1\rangle \otimes P_k|1\rangle$$

$$\| I \otimes P_k |\psi\rangle \|^2 = a^2 \|P_k|0\rangle\|^2 + b^2 \|P_k|1\rangle\|^2$$ (exercise)

Lemma: for any column vector v, $\|v\|^2 = \mathrm{tr}\, v v^\dagger$.

Proof:

Example: $|\psi\rangle = a|00\rangle + b|11\rangle$ on RS, a,b real.

For any measurement on S given by projectors $\{P_k\}$

pr(k) = $\| I \otimes P_k |\psi\rangle \|^2$

$$I \otimes P_k |\psi\rangle = a|0\rangle \otimes P_k|0\rangle + b|1\rangle \otimes P_k|1\rangle$$

$$\| I \otimes P_k |\psi\rangle \|^2 = a^2 \|P_k|0\rangle\|^2 + b^2 \|P_k|1\rangle\|^2$$ (exercise)

Lemma: for any column vector v, $\|v\|^2 = \mathrm{tr}\, v v^\dagger$.

Proof: $v = \begin{pmatrix} v_1 \\ v_2 \\ \\ v_d \end{pmatrix}$, $v v^\dagger = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} \begin{bmatrix} v_1^* & v_2^* & \cdots & v_d^* \end{bmatrix} = \begin{pmatrix} v_1 v_1^* & \cdots & v_1 v_d^* \\ v_2 v_1^* & & \\ \vdots & & \\ v_d v_1^* & & v_d v_d^* \end{pmatrix}$

Example: $|\psi\rangle = a|00\rangle + b|11\rangle$ on RS, a,b real.

For any measurement on S given by projectors $\{P_k\}$

pr(k) = $\| I \otimes P_k |\psi\rangle \|^2$

$$I \otimes P_k |\psi\rangle = a|0\rangle \otimes P_k|0\rangle + b|1\rangle \otimes P_k|1\rangle$$

$$\| I \otimes P_k |\psi\rangle \|^2 = a^2 \|P_k|0\rangle\|^2 + b^2 \|P_k|1\rangle\|^2$$ (exercise)

Lemma: for any column vector v, $\|v\|^2 = \operatorname{tr} v v^\dagger$.

Proof: $v = \begin{pmatrix} v_1 \\ v_2 \\ \\ v_d \end{pmatrix}$, $v v^\dagger = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} \begin{bmatrix} v_1^* & v_2^* & \cdots & v_d^* \end{bmatrix} = \begin{pmatrix} v_1 v_1^* & \cdots & v_1 v_d^* \\ v_2 v_1^* & & \\ \vdots & & \\ v_d v_1^* & & v_d v_d^* \end{pmatrix}$

$$\operatorname{tr} v v^\dagger = v_1 v_1^* + \cdots + v_d v_d^* = \|v\|^2$$

Example: $|\psi\rangle = a|00\rangle + b|11\rangle$ on RS, a,b real.

For any measurement on S given by projectors $\{P_k\}$

pr(k) = $\| I \otimes P_k |\psi\rangle \|^2$

$$I \otimes P_k |\psi\rangle = a|0\rangle \otimes P_k|0\rangle + b|1\rangle \otimes P_k|1\rangle$$

$$\| I \otimes P_k |\psi\rangle \|^2 = a^2 \|P_k|0\rangle\|^2 + b^2 \|P_k|1\rangle\|^2$$ (exercise)

$$= a^2 \operatorname{tr} P_k|0\rangle\langle 0|P_k + b^2 \operatorname{tr} P_k|1\rangle\langle 1|P_k$$

<u>Example:</u> $|\Psi\rangle = a|00\rangle + b|11\rangle$ on RS, a,b real.

For any measurement on S given by projectors $\{P_k\}$

pr(k) = $\| I \otimes P_k |\Psi\rangle \|^2$

$$I \otimes P_k |\Psi\rangle = a|0\rangle \otimes P_k|0\rangle + b|1\rangle \otimes P_k|1\rangle$$

$$\| I \otimes P_k |\Psi\rangle \|^2 = a^2 \|P_k|0\rangle\|^2 + b^2 \|P_k|1\rangle\|^2$$ (exercise)

$$= a^2 \, \mathrm{tr}\, P_k|0\rangle\langle 0|P_k + b^2 \, \mathrm{tr}\, P_k|1\rangle\langle 1|P_k$$

$$= a^2 \, \mathrm{tr}\, P_k|0\rangle\langle 0| + b^2 \, \mathrm{tr}\, P_k|1\rangle\langle 1|$$

(cyclic property of trace and $P_k^2 = P_k$)

Example: $|\psi\rangle = a|00\rangle + b|11\rangle$ on RS, a,b real.

For any measurement on S given by projectors $\{P_k\}$

pr(k) = $\| I \otimes P_k |\psi\rangle \|^2$

$$I \otimes P_k |\psi\rangle = a|0\rangle \otimes P_k|0\rangle + b|1\rangle \otimes P_k|1\rangle$$

$$\| I \otimes P_k |\psi\rangle \|^2 = a^2 \|P_k|0\rangle\|^2 + b^2 \|P_k|1\rangle\|^2$$ (exercise)

$$= a^2 \operatorname{tr} P_k|0\rangle\langle 0|P_k + b^2 \operatorname{tr} P_k|1\rangle\langle 1|P_k$$

$$= a^2 \operatorname{tr} P_k|0\rangle\langle 0| + b^2 \operatorname{tr} P_k|1\rangle\langle 1|$$

(cyclic property of trace and $P_k^2 = P_k$)

$$= \operatorname{tr} P_k \left(a^2|0\rangle\langle 0| + b^2|1\rangle\langle 1| \right)$$ (tr linear)

Example: $|\psi\rangle = a|00\rangle + b|11\rangle$ on RS, a,b real.

For any measurement on S given by projectors $\{P_k\}$

pr(k) = $\| I \otimes P_k |\psi\rangle \|^2$

$$I \otimes P_k |\psi\rangle = a|0\rangle \otimes P_k|0\rangle + b|1\rangle \otimes P_k|1\rangle$$

$$\| I \otimes P_k |\psi\rangle \|^2 = a^2 \|P_k|0\rangle\|^2 + b^2 \|P_k|1\rangle\|^2$$ (exercise)

$$= a^2 \, \mathrm{tr}\, P_k|0\rangle\langle 0|P_k + b^2 \, \mathrm{tr}\, P_k|1\rangle\langle 1|P_k$$

$$= a^2 \, \mathrm{tr}\, P_k|0\rangle\langle 0| + b^2 \, \mathrm{tr}\, P_k|1\rangle\langle 1|$$

(cyclic property of trace and $P_k^2 = P_k$)

$$= \mathrm{tr}\, P_k \left(a^2|0\rangle\langle 0| + b^2|1\rangle\langle 1| \right)$$ (tr linear)

$$= \mathrm{tr}\, P_k \left(v_0 v_0^\dagger + v_1 v_1^\dagger \right)$$

where
$$v_0 = \langle 0| \otimes I \, |\psi\rangle = \langle 0| \otimes I \left(a|00\rangle + b|11\rangle \right) = a|0\rangle_S$$
$$v_1 = \langle 1| \otimes I \, |\psi\rangle = \langle 1| \otimes I \left(a|00\rangle + b|11\rangle \right) = b|1\rangle_S$$

<u>Example:</u> $|\psi\rangle = a|00\rangle + b|11\rangle$ on RS, a,b real.

For any measurement on S given by projectors $\{P_k\}$

pr(k) = $\| I \otimes P_k |\psi\rangle \|^2$

$$I \otimes P_k |\psi\rangle = a|0\rangle \otimes P_k|0\rangle + b|1\rangle \otimes P_k|1\rangle$$

$$\| I \otimes P_k |\psi\rangle \|^2$$

revised formula for prob of meas

$$= \mathrm{tr}\, P_k \overbrace{\left(a^2 |0\rangle\langle 0| + b^2 |1\rangle\langle 1| \right)}^{\text{revised way to rep quantum data on S}} \quad \text{(tr linear)}$$

$$= \mathrm{tr}\, P_k \left(v_0 v_0^{\dagger} + v_1 v_1^{\dagger} \right)$$

where

$$v_0 = \langle 0| \otimes I \, |\psi\rangle = \langle 0| \otimes I \left(a|00\rangle + b|11\rangle \right) = a|0\rangle_S$$

$$v_1 = \langle 1| \otimes I \, |\psi\rangle = \langle 1| \otimes I \left(a|00\rangle + b|11\rangle \right) = b|1\rangle_S$$

General: $|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle$ on RS.

real ortho-normal unit vector on S

(derived in topic03-02.pdf p5-8)

General: $|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle$ on RS.

real ortho-normal unit vector on S

For any measurement on S given by projectors $\{P_k\}$

$$I \otimes P_k |\psi\rangle = \sum_i a_i |i\rangle \otimes P_k |\eta_i\rangle$$

General: $|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle$ on RS.

real ortho-normal unit vector on S

For any measurement on S given by projectors $\{P_k\}$

$$I \otimes P_k |\psi\rangle = \sum_i a_i |i\rangle \otimes P_k |\eta_i\rangle$$

$$\text{pr}(k) = \| I \otimes P_k |\psi\rangle \|^2 = \sum_i a_i^2 \| P_k |\eta_i\rangle \|^2$$

General: $|\Psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle$ on RS.

real ortho-normal unit vector on S

For any measurement on S given by projectors $\{P_k\}$

$$I \otimes P_k |\Psi\rangle = \sum_i a_i |i\rangle \otimes P_k |\eta_i\rangle$$

$$\text{pr}(k) = \| I \otimes P_k |\Psi\rangle \|^2 = \sum_i a_i^2 \| P_k |\eta_i\rangle \|^2$$

$$= \sum_i a_i^2 \ \mathrm{tr}\, P_k |\eta_i\rangle\langle\eta_i| P_k$$

$$= \sum_i a_i^2 \ \mathrm{tr}\, P_k |\eta_i\rangle\langle\eta_i|$$

General: $|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle$ on RS.

real ortho-normal unit vector on S

For any measurement on S given by projectors $\{P_k\}$

$$I \otimes P_k |\psi\rangle = \sum_i a_i |i\rangle \otimes P_k |\eta_i\rangle$$

$$\text{pr}(k) = \| I \otimes P_k |\psi\rangle \|^2 = \sum_i a_i^2 \| P_k |\eta_i\rangle \|^2$$

$$= \sum_i a_i^2 \ \mathrm{tr}\, P_k |\eta_i\rangle\langle\eta_i| P_k$$

$$= \sum_i a_i^2 \ \mathrm{tr}\, P_k |\eta_i\rangle\langle\eta_i|$$

$$= \mathrm{tr}\, P_k \left(\sum_i a_i^2 |\eta_i\rangle\langle\eta_i| \right)$$ where $a_i |\eta_i\rangle_S = \langle i| \otimes I |\psi\rangle$.

General: $|\Psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle$ on RS.

real ortho-normal unit vector on S

For any measurement on S given by projectors $\{P_k\}$

$$I \otimes P_k |\Psi\rangle = \sum_i a_i |i\rangle \otimes P_k |\eta_i\rangle$$

$$\text{pr}(k) = \| I \otimes P_k |\Psi\rangle \|^2 = \sum_i a_i^2 \| P_k |\eta_i\rangle \|^2$$

$$= \sum_i a_i^2 \, \text{tr}\, P_k |\eta_i\rangle\langle\eta_i| P_k$$

$$= \sum_i a_i^2 \, \text{tr}\, P_k |\eta_i\rangle\langle\eta_i|$$

revised formula for prob of meas

$$= \text{tr}\, P_k \underbrace{\left(\sum_i a_i^2 |\eta_i\rangle\langle\eta_i| \right)}_{\rho_S}$$ where $a_i |\eta_i\rangle_S = \langle i| \otimes I |\Psi\rangle$.

ρ_S : density matrix on S

dxd if d = dim(S)

trace 1, positive semidefinite

revised rep for quantum data on S

The partial trace

Revised formulation of QM:

Revised description of quantum state:

$$|\Psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle \xrightarrow{\text{1. outer product}} |\Psi\rangle\langle\Psi| \xrightarrow{\text{2. partial trace}} \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

1. outer product 2. partial trace

revised description of measurement:

$$\mathrm{pr}(k) = \| I \otimes P_k |\Psi\rangle \|^2 \longrightarrow \mathrm{pr}(k) = \mathrm{tr}\, P_k \rho_S$$

Revised formulation of QM:

Revised description of quantum state:

$$|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle \longrightarrow |\psi\rangle\langle\psi| \longrightarrow \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

1. outer product 2. partial trace

revised description of measurement:

$$\mathrm{pr}(k) = \| I \otimes P_k |\psi\rangle \|^2 \longrightarrow \mathrm{pr}(k) = \mathrm{tr}\, P_k \rho_S$$

Define partial trace (describing a state on S from a state on RS) so postmeasurement states & dynamics also makes sense.

The partial trace

Recall the trace of a matrix M is the sum of all the diagonal elements. In the Dirac notation:

$d \times d$

$$\mathrm{tr}\, M = \mathrm{tr}\left(M \sum_{i=1}^{d} |i\rangle\langle i|\right) = \sum_{i=1}^{d} \langle i|M|i\rangle$$

insert identity

$\{|i\rangle\}$ basis

tr is cyclic and linear

- in terms of a basis
- can use any basis

basis independent

The partial trace

Recall the trace of a matrix M is the sum of all the diagonal elements. In the Dirac notation:

$d \times d$

$$\mathrm{tr}\, M = \mathrm{tr}\Big(M \sum_{i=1}^{d} |i\rangle\langle i|\Big) = \sum_{i=1}^{d} \langle i|M|i\rangle$$

insert identity

$\{|i\rangle\}$ basis

tr is cyclic and linear

d dim, basis $\{|i\rangle\}$

Definition: the partial trace of system B, denoted tr_B, is defined on matrices acting on systems AB as

d_A dim

$$\mathrm{tr}_B\, M = \underbrace{\sum_{i=1}^{d} (\underset{A}{I} \otimes \underset{B}{\langle i|})\, M\, (I \otimes |i\rangle)}_{d_A \times d_A}$$

$d_A d \times d_A d$

i.e., trace B and do nothing on A

Question:

real

Suppose $|\psi\rangle_{RS} = a|00\rangle + b|1+\rangle$ and we apply partial trace to R, what is the density matrix on S?

(a) $\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$ (b) $\begin{bmatrix} a^2+\frac{b^2}{2} & \frac{b^2}{2} \\ \frac{b^2}{2} & \frac{b^2}{2} \end{bmatrix}$ (c) $\begin{bmatrix} a^2 & \frac{ab}{\sqrt{2}} \\ \frac{ab}{\sqrt{2}} & b^2 \end{bmatrix}$

Please do not scroll down ...

Useful from previous slides:

$$|\psi\rangle = \sum_i a_i |i\rangle|\eta_i\rangle \xrightarrow{\text{1. outer product}} |\psi\rangle\langle\psi| \xrightarrow{\text{2. partial trace}} \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

$$\mathrm{tr}_B M = \sum_{i=1}^{d} (\underbrace{I}_{A} \otimes \underbrace{\langle i|}_{B})\, \underbrace{M}_{d_A d \times d_A d}\, (I \otimes |i\rangle)$$

Question:

real

Suppose $|\psi\rangle_{RS} = a|00\rangle + b|1+\rangle$ and we apply partial trace to R, what is the density matrix on S?

(a) $\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$ (b) $\begin{bmatrix} a^2+\frac{b^2}{2} & \frac{b^2}{2} \\ \frac{b^2}{2} & \frac{b^2}{2} \end{bmatrix}$ (c) $\begin{bmatrix} a^2 & \frac{ab}{\sqrt{2}} \\ \frac{ab}{\sqrt{2}} & b^2 \end{bmatrix}$

Answer is (b).

Question:

real

Suppose $|\psi\rangle_{RS} = a|00\rangle + b|1+\rangle$ and we apply partial trace to R, what is the density matrix on S?

(a) $\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$ (b) $\begin{bmatrix} a^2+\frac{b^2}{2} & \frac{b^2}{2} \\ \frac{b^2}{2} & \frac{b^2}{2} \end{bmatrix}$ (c) $\begin{bmatrix} a^2 & \frac{ab}{\sqrt{2}} \\ \frac{ab}{\sqrt{2}} & b^2 \end{bmatrix}$

Answer is (b).

Exercise: show that, if we apply the partial trace to S instead, the density matrix, now on R, is (c)! Hint: rewrite the joint state on RS using method 2 with the basis $\{|0\rangle, |1\rangle\}$ for S.

Question:

real

Suppose $|\psi\rangle_{RS} = a|00\rangle + b|1+\rangle$ and we apply partial trace to R, what is the density matrix on S?

(a) $\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$ (b) $\begin{bmatrix} a^2+\frac{b^2}{2} & \frac{b^2}{2} \\ \frac{b^2}{2} & \frac{b^2}{2} \end{bmatrix}$ (c) $\begin{bmatrix} a^2 & \frac{ab}{\sqrt{2}} \\ \frac{ab}{\sqrt{2}} & b^2 \end{bmatrix}$

Answer is (b).

Exercise: show that, if we apply the partial trace to S instead, the density matrix, now on R, is (c)! Hint: rewrite the joint state on RS using method 2 with the basis $\{|0\rangle, |1\rangle\}$ for S.

NB: partial tracing R is different from partial tracing S! Outcomes even have different dims if dim(R) $\neq$ dim(S).

Exercise: show that partial tracing R or S from any of the 4 Bell states give density matrices equal to I/2 !

$$|\Phi_0\rangle_{RS} = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad |\Phi_z\rangle_{RS} = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\Phi_x\rangle_{RS} = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \qquad |\Phi_y\rangle_{RS} = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Exercise: show that partial tracing R or S from any of the 4 Bell states give density matrices equal to I/2 !

$$|\Phi_0\rangle_{RS} = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad |\Phi_z\rangle_{RS} = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\Phi_x\rangle_{RS} = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \qquad |\Phi_y\rangle_{RS} = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

So, the partial trace is a many-to-one mapping, and is mathematically irreversible.

Crucial in cryptography (later in the course) to learn when two states have the same partial trace, and if you see system S, what do you know about RS jointly.

We have defined the partial trace in the dirac notation.

Next we derive how it looks in matrix representation.

The partial trace (example for 2 qubits)

$$I \otimes \langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{pmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \end{bmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$I \otimes \langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{pmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The partial trace (example for 2 qubits)

$$I \otimes \langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} [1 \;\; 0] & [0 \;\; 0] \\ [0 \;\; 0] & [1 \;\; 0] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$I \otimes \langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} [0 \;\; 1] & [0 \;\; 0] \\ [0 \;\; 0] & [0 \;\; 1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(I \otimes \langle 0|)\, M\, (I \otimes |0\rangle) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{bmatrix}$$

The partial trace (example for 2 qubits)

$$I \otimes \langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} [1 \;\; 0] & [0 \;\; 0] \\ [0 \;\; 0] & [1 \;\; 0] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$I \otimes \langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} [0 \;\; 1] & [0 \;\; 0] \\ [0 \;\; 0] & [0 \;\; 1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(I \otimes \langle 0|)\, M\, (I \otimes |0\rangle) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{bmatrix}$$

$$(I \otimes \langle 1|)\, M\, (I \otimes |1\rangle) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m_{22} & m_{24} \\ m_{42} & m_{44} \end{bmatrix}$$

The partial trace (example for 2 qubits)

$$I \otimes \langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} [1 \ 0] & [0 \ 0] \\ [0 \ 0] & [1 \ 0] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$I \otimes \langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} [0 \ 1] & [0 \ 0] \\ [0 \ 0] & [0 \ 1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(I \otimes \langle 0|)\, M \,(I \otimes |0\rangle) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{bmatrix}$$

$$(I \otimes \langle 1|)\, M \,(I \otimes |1\rangle) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m_{22} & m_{24} \\ m_{42} & m_{44} \end{bmatrix}$$

$$\operatorname{tr}_B M = \sum_{i=1}^{d} (I \otimes \langle i|)\, M \,(I \otimes |i\rangle) = \begin{bmatrix} m_{11} + m_{22} & m_{13} + m_{24} \\ m_{31} + m_{42} & m_{33} + m_{44} \end{bmatrix}$$

$$\mathrm{tr}_B M = \begin{pmatrix} \mathrm{tr}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} & \mathrm{tr}\begin{bmatrix} m_{13} & m_{14} \\ m_{23} & m_{24} \end{bmatrix} \\ \mathrm{tr}\begin{bmatrix} m_{31} & m_{32} \\ m_{41} & m_{42} \end{bmatrix} & \mathrm{tr}\begin{bmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{bmatrix} \end{pmatrix} \xleftarrow{\text{tracing each block}} \left(\begin{array}{cc|cc} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ \hline m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{array}\right) = M$$

$$\mathrm{tr}_B M = \begin{pmatrix} \mathrm{tr}\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} & \mathrm{tr}\begin{pmatrix} m_{13} & m_{14} \\ m_{23} & m_{24} \end{pmatrix} \\ \mathrm{tr}\begin{pmatrix} m_{31} & m_{32} \\ m_{41} & m_{42} \end{pmatrix} & \mathrm{tr}\begin{pmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{pmatrix} \end{pmatrix} \xleftarrow{\text{tracing each block}} \left(\begin{array}{cc|cc} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ \hline m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{array}\right) = M$$

consistent with the idea entries in each block rep states in B while the block labels corr to states in A

$$\begin{array}{cc} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \left(\begin{array}{cc|cc} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ \hline m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{array}\right) \end{array}$$

$$\operatorname{tr}_B M = \begin{pmatrix} \operatorname{tr}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} & \operatorname{tr}\begin{bmatrix} m_{13} & m_{14} \\ m_{23} & m_{24} \end{bmatrix} \\ \operatorname{tr}\begin{bmatrix} m_{31} & m_{32} \\ m_{41} & m_{42} \end{bmatrix} & \operatorname{tr}\begin{bmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{bmatrix} \end{pmatrix} \xleftarrow{\text{tracing each block}} \left(\begin{array}{cc|cc} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ \hline m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{array}\right) = M$$

summing diagonal blocks

Exercise:

$$\operatorname{tr}_A M = \operatorname{tr}_A \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{bmatrix}$$

Example: A, B are 3- and 2-dim respectively. (M: 6x6)

$$M = \left[\begin{array}{c|c|c} M_{11} & M_{12} & M_{13} \\ \hline M_{21} & M_{22} & M_{23} \\ \hline M_{31} & M_{32} & M_{33} \end{array}\right]$$

Each M_{ij} is a 2x2 matrix.

Example: A, B are 3- and 2-dim respectively. (M: 6x6)

$$M = \left[\begin{array}{c|c|c} M_{11} & M_{12} & M_{13} \\ \hline M_{21} & M_{22} & M_{23} \\ \hline M_{31} & M_{32} & M_{33} \end{array}\right]$$

Each M_{ij} is a 2x2 matrix.

$$\mathrm{tr}_A M = M_{11} + M_{22} + M_{33}$$

(note, the reduced matrix on B is 2x2)

Example: A, B are 3- and 2-dim respectively. (M: 6x6)

$$M = \left[\begin{array}{c|c|c} M_{11} & M_{12} & M_{13} \\ \hline M_{21} & M_{22} & M_{23} \\ \hline M_{31} & M_{32} & M_{33} \end{array}\right]$$

Each M_{ij} is a 2x2 matrix.

$$\mathrm{tr}_A M = M_{11} + M_{22} + M_{33}$$

(note, the reduced matrix on B is 2x2)

$$\mathrm{tr}_B M = \left[\begin{array}{c|c|c} \mathrm{tr}\, M_{11} & \mathrm{tr}\, M_{12} & \mathrm{tr}\, M_{13} \\ \hline \mathrm{tr}\, M_{21} & \mathrm{tr}\, M_{22} & \mathrm{tr}\, M_{23} \\ \hline \mathrm{tr}\, M_{31} & \mathrm{tr}\, M_{32} & \mathrm{tr}\, M_{33} \end{array}\right]$$

(note, the reduced matrix on A is 3x3)

Remark:

The trace of an r-dim system is a linear map from r x r matrices to real numbers.

Remark:

The trace of an r-dim system is a linear map from r x r matrices to real numbers.

The partial trace of an r-dim system is a linear map from rs x rs matrices to s x s matrices where the trace is applied to R, and the identity map on S.

Remark:

The trace of an r-dim system is a linear map from r x r matrices to real numbers.

The partial trace of an r-dim system is a linear map from rs x rs matrices to s x s matrices where the trace is applied to R, and the identity map on S.

It acts on tensor product matrices as:

$$\mathrm{tr}_R\, M_R \otimes M_S = (\mathrm{tr}\, M_R) \cdot M_S$$

scalar   scalar product

and extends to any rs x rs matrix (by linearity).

Revised formulation of QM:

Revised description of quantum state:

$$|\Psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle \xrightarrow{\text{1. outer product}} |\Psi\rangle\langle\Psi| \xrightarrow{\text{2. partial trace}} \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

Checking the new definition of partial trace on $|\Psi\rangle\langle\Psi|$:

$$\mathrm{tr}_R |\Psi\rangle\langle\Psi| = \sum_{i=1}^{\dim R} (\langle i| \otimes I)\, |\Psi\rangle\langle\Psi|\, (|i\rangle \otimes I)$$

Revised formulation of QM:

Revised description of quantum state:

$$|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle \xrightarrow{\text{1. outer product}} |\psi\rangle\langle\psi| \xrightarrow{\text{2. partial trace}} \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

1. outer product 2. partial trace

Checking the new definition of partial trace on $|\psi\rangle\langle\psi|$:

$$\mathrm{tr}_R |\psi\rangle\langle\psi| = \sum_{i=1}^{\dim R} \left(\langle i| \otimes I\right) |\psi\rangle\langle\psi| \left(|i\rangle \otimes I\right)$$

$$\left(\langle i| \otimes I\right) |\psi\rangle = \left(\langle i| \otimes I\right) \sum_j a_j |j\rangle |\eta_j\rangle$$

Revised formulation of QM:

Revised description of quantum state:

$$|\Psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle \xrightarrow{\text{1. outer product}} |\Psi\rangle\langle\Psi| \xrightarrow{\text{2. partial trace}} \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

Checking the new definition of partial trace on $|\Psi\rangle\langle\Psi|$:

$$\mathrm{tr}_R |\Psi\rangle\langle\Psi| = \sum_{i=1}^{\dim R} \left(\langle i| \otimes I\right) |\Psi\rangle\langle\Psi| \left(|i\rangle \otimes I\right)$$

$$\left(\langle i| \otimes I\right)|\Psi\rangle = \left(\langle i| \otimes I\right) \sum_j a_j |j\rangle |\eta_j\rangle$$

$$= \sum_j a_j \langle i|j\rangle \otimes |\eta_j\rangle = a_i |\eta_i\rangle$$

Revised formulation of QM:

Revised description of quantum state:

$$|\psi\rangle = \sum_i a_i |i\rangle|\eta_i\rangle \longrightarrow |\psi\rangle\langle\psi| \longrightarrow \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

1. outer product 2. partial trace

Checking the new definition of partial trace on $|\psi\rangle\langle\psi|$:

$$\mathrm{tr}_R |\psi\rangle\langle\psi| = \sum_{i=1}^{\dim R} (\langle i| \otimes I)|\psi\rangle\langle\psi|(|i\rangle \otimes I)$$

$$(\langle i| \otimes I)|\psi\rangle = (\langle i| \otimes I)\sum_j a_j |j\rangle|\eta_j\rangle$$

$$= \sum_j a_j \langle i|j\rangle \otimes |\eta_j\rangle = a_i |\eta_i\rangle$$

So $\mathrm{tr}_R |\psi\rangle\langle\psi| = \sum_{i=1}^{\dim R} a_i^2 |\eta_i\rangle\langle\eta_i|$ same as wanted above.

Revised formulation of QM:

Revised description of quantum state:

$$|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle \longrightarrow |\psi\rangle\langle\psi| \longrightarrow \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

1. outer product 2. partial trace

revised description of measurement:

$$\mathrm{pr}(k) = \| I \otimes P_k |\psi\rangle \|^2 \longrightarrow \mathrm{tr}\, P_k \rho_S$$

Revised formulation of QM:

Revised description of quantum state:

$$|\psi\rangle = \sum_i a_i |i\rangle|\eta_i\rangle \longrightarrow |\psi\rangle\langle\psi| \longrightarrow \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

1. outer product 2. partial trace

revised description of measurement:

pr(k) = $\| I \otimes P_k |\psi\rangle \|^2 \longrightarrow \mathrm{tr}\, P_k \rho_S$

post-meas state: $|\psi_k\rangle = \dfrac{I \otimes P_k |\psi\rangle}{\| I \otimes P_k |\psi\rangle \|} = \dfrac{\sum_i a_i |i\rangle \otimes P_k |\eta_i\rangle}{\| I \otimes P_k |\psi\rangle \|}$

Revised formulation of QM:

Revised description of quantum state:

$$|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle \longrightarrow |\psi\rangle\langle\psi| \longrightarrow \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

1. outer product   2. partial trace

revised description of measurement:

$$\mathrm{pr}(k) = \| I \otimes P_k |\psi\rangle \|^2 \longrightarrow \mathrm{tr}\, P_k \rho_S$$

post-meas state: $|\psi_k\rangle = \dfrac{I \otimes P_k |\psi\rangle}{\| I \otimes P_k |\psi\rangle \|} = \dfrac{\sum_i a_i |i\rangle \otimes P_k |\eta_i\rangle}{\| I \otimes P_k |\psi\rangle \|}$

$$\mathrm{tr}_R |\psi_k\rangle\langle\psi_k| = \sum_{i=1}^{\dim R} (\langle i| \otimes I)\, |\psi_k\rangle\langle\psi_k|\, (|i\rangle \otimes I)$$

$$(\langle i| \otimes I)\, |\psi_k\rangle$$

Revised formulation of QM:

Revised description of quantum state:

$$|\Psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle \xrightarrow{\text{1. outer product}} |\Psi\rangle\langle\Psi| \xrightarrow{\text{2. partial trace}} \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

1. outer product 2. partial trace

revised description of measurement:

$$\mathrm{pr}(k) = \| I \otimes P_k |\Psi\rangle \|^2 \longrightarrow \mathrm{tr}\, P_k \rho_S$$

post-meas state: $$|\Psi_k\rangle = \frac{I \otimes P_k |\Psi\rangle}{\| I \otimes P_k |\Psi\rangle \|} = \frac{\sum_i a_i |i\rangle \otimes P_k |\eta_i\rangle}{\| I \otimes P_k |\Psi\rangle \|}$$

$$\mathrm{tr}_R |\Psi_k\rangle\langle\Psi_k| = \sum_{i=1}^{\dim R} (\langle i| \otimes I)\; |\Psi_k\rangle\langle\Psi_k|\; (|i\rangle \otimes I)$$

$$(\langle i| \otimes I)\, |\Psi_k\rangle = (\langle i| \otimes I)\, \frac{\sum_j a_j |j\rangle \otimes P_k |\eta_j\rangle}{\| I \otimes P_k |\Psi\rangle \|} = \frac{a_i\, P_k |\eta_i\rangle}{\| I \otimes P_k |\Psi\rangle \|}$$

$$\mathrm{tr}_R\,|\psi_k\rangle\langle\psi_k| = \sum_{i=1}^{\dim R} \frac{a_i\, P_k\,|\eta_i\rangle}{\|\, I\otimes P_k\,|\psi\rangle\,\|}\;\frac{a_i\,\langle\eta_i|\,P_k}{\|\, I\otimes P_k\,|\psi\rangle\,\|}$$

$$\mathrm{tr}_R\, |\psi_k\rangle\langle\psi_k| = \sum_{i=1}^{\dim R} \frac{a_i\, P_k |\eta_i\rangle}{\| I \otimes P_k |\psi\rangle \|} \frac{a_i \langle\eta_i| P_k}{\| I \otimes P_k |\psi\rangle \|}$$

$$= \sum_{i=1}^{\dim R} \frac{a_i^2\, P_k |\eta_i\rangle\langle\eta_i| P_k}{\mathrm{tr}\, P_k\, \rho_S}$$

$$\mathrm{tr}_R |\Psi_K\rangle\langle\Psi_K| = \sum_{i=1}^{\dim R} \frac{a_i\, P_K |\eta_i\rangle}{\| I \otimes P_K |\Psi\rangle \|} \frac{a_i \langle\eta_i| P_K}{\| I \otimes P_K |\Psi\rangle \|}$$

$$= \sum_{i=1}^{\dim R} \frac{a_i^2\, P_K |\eta_i\rangle\langle\eta_i| P_K}{\mathrm{tr}\, P_K \rho_S}$$

$$= P_K \frac{\sum_{i=1}^{\dim R} a_i^2 |\eta_i\rangle\langle\eta_i|}{\mathrm{tr}\, P_K \rho_S} P_K$$

$$\mathrm{tr}_R |\psi_k\rangle\langle\psi_k| = \sum_{i=1}^{\dim R} \frac{a_i P_k |\eta_i\rangle}{\| I \otimes P_k |\psi\rangle \|} \frac{a_i \langle\eta_i| P_k}{\| I \otimes P_k |\psi\rangle \|}$$

$$= \sum_{i=1}^{\dim R} \frac{a_i^2 P_k |\eta_i\rangle\langle\eta_i| P_k}{\mathrm{tr}\, P_k \rho_S}$$

$$= P_k \frac{\sum_{i=1}^{\dim R} a_i^2 |\eta_i\rangle\langle\eta_i|}{\mathrm{tr}\, P_k \rho_S} P_k$$

$$= \frac{P_k \rho_S P_k}{\mathrm{tr}\, P_k \rho_S}$$

revised description of post-measurement state

Revised formulation of QM:

Revised description of evolution by unitary U on S:

$$|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle$$

$$I \otimes U |\psi\rangle = \sum_i a_i |i\rangle \, U|\eta_i\rangle$$

Revised formulation of QM:

Revised description of evolution by unitary U on S:

$$|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle$$

$$I \otimes U |\psi\rangle = \sum_i a_i |i\rangle U |\eta_i\rangle$$

$$(\langle i| \otimes I)\, I \otimes U |\psi\rangle = a_i U |\eta_i\rangle$$ similar to applying P_k

Revised formulation of QM:

Revised description of evolution by unitary U on S:

$$|\Psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle$$

$$I \otimes U |\Psi\rangle = \sum_i a_i |i\rangle \, U |\eta_i\rangle$$

$$(\langle i| \otimes I) \; I \otimes U |\Psi\rangle = a_i \, U |\eta_i\rangle$$ similar to applying P_k

---

$$\mathrm{tr}_R \; I \otimes U \; |\Psi\rangle\langle\Psi| \; I \otimes U^{\dagger}$$

$$= \sum_{i=1}^{\dim R} (\langle i| \otimes I) \; I \otimes U \; |\Psi\rangle\langle\Psi| \; I \otimes U^{\dagger} \; (|i\rangle \otimes I)$$

Revised formulation of QM:

Revised description of evolution by unitary U on S:

$$|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle$$

$$I \otimes U |\psi\rangle = \sum_i a_i |i\rangle \, U |\eta_i\rangle$$

$$(\langle i| \otimes I) \; I \otimes U |\psi\rangle = a_i \, U |\eta_i\rangle$$ similar to applying P_k

---

$$\mathrm{tr}_R \; I \otimes U \; |\psi\rangle\langle\psi| \; I \otimes U^\dagger$$

$$= \sum_{i=1}^{\dim R} (\langle i| \otimes I) \; I \otimes U \; |\psi\rangle\langle\psi| \; I \otimes U^\dagger \; (|i\rangle \otimes I)$$

$$= \sum_{i=1}^{\dim R} a_i^2 \, U |\eta_i\rangle\langle\eta_i| U^\dagger \;\; = \; U \rho_S U^\dagger$$

So, $\rho_S \longrightarrow U \rho_S U^\dagger$.

Revised formulation of QM:

Revised description of quantum state:

$$|\Psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle \longrightarrow |\Psi\rangle\langle\Psi| \longrightarrow \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

Revised formulation of QM:

Revised description of quantum state:

$$|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle \longrightarrow |\psi\rangle\langle\psi| \longrightarrow \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

Revised description of measurement:

$$\mathrm{pr}(k) = \| I \otimes P_k |\psi\rangle \|^2 \longrightarrow \mathrm{pr}(k) = \mathrm{tr}\, P_k \rho_S$$

post-meas state: $|\psi_k\rangle = \dfrac{I \otimes P_k |\psi\rangle}{\| I \otimes P_k |\psi\rangle \|} \longrightarrow \dfrac{P_k \rho_S P_k}{\mathrm{tr}\, P_k \rho_S}$

Revised formulation of QM:

Revised description of quantum state:

$$|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle \longrightarrow |\psi\rangle\langle\psi| \longrightarrow \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

Revised description of measurement:

$$\text{pr}(k) = \| I \otimes P_k |\psi\rangle \|^2 \longrightarrow \text{pr}(k) = \operatorname{tr} P_k \rho_S$$

post-meas state: $$|\psi_k\rangle = \frac{I \otimes P_k |\psi\rangle}{\| I \otimes P_k |\psi\rangle \|} \longrightarrow \frac{P_k \rho_S P_k}{\operatorname{tr} P_k \rho_S}$$

Revised description of evolution by unitary U on S:

$$|\psi\rangle \longrightarrow I \otimes U |\psi\rangle \longrightarrow \rho_S \rightarrow U \rho_S U^\dagger$$

Revised formulation of QM:

Revised description of quantum state:

$$|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle \longrightarrow |\psi\rangle\langle\psi| \longrightarrow \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

Revised description of measurement:

$$\mathrm{pr}(k) = \| I \otimes P_k |\psi\rangle \|^2 \longrightarrow \mathrm{pr}(k) = \mathrm{tr}\, P_k \rho_S$$

post-meas state: $|\psi_k\rangle = \dfrac{I \otimes P_k |\psi\rangle}{\| I \otimes P_k |\psi\rangle \|} \longrightarrow \dfrac{P_k \rho_S P_k}{\mathrm{tr}\, P_k \rho_S}$

Revised description of evolution by unitary U on S:

$$|\psi\rangle \longrightarrow I \otimes U |\psi\rangle \qquad \longrightarrow \qquad \rho_S \rightarrow U \rho_S U^\dagger$$

Revised rules coincide with alternative interpretation of density matrix as ensemble of quantum states (later).

We took our 5 postulates, applied to composite system RS and derived QM as viewed from S.

We now deduce some properties for the states and evolution/measurement.

We can instead start with these properties and formulate all QM from scratch based on them ...

Properties of the density matrix and partial trace:

1. For any $|\psi\rangle$ on RS, for any unitary U on R,

$$\mathrm{tr}_R \, U_R \otimes I_S \, |\psi\rangle\langle\psi| \, U_R^\dagger \otimes I_S = \mathrm{tr}_R \, |\psi\rangle\langle\psi|$$

Properties of the density matrix and partial trace:

1. For any $|\psi\rangle$ on RS, for any unitary U on R,

$$\mathrm{tr}_R\, U_R \otimes I_S\, |\psi\rangle\langle\psi|\, U_R^\dagger \otimes I_S = \mathrm{tr}_R\, |\psi\rangle\langle\psi|$$

Remarks:

(a) This restates the non-signalling principle in the presence of entanglement: if 1 doesn't hold, a party holding R can affect the density matrix of S and it measurement statistics, thereby partially communicating whether U is applied or not.

Properties of the density matrix and partial trace:

1. For any $|\psi\rangle$ on RS, for any unitary U on R,

$$\mathrm{tr}_R\, U_R \otimes I_S\, |\psi\rangle\langle\psi|\, U_R^\dagger \otimes I_S = \mathrm{tr}_R\, |\psi\rangle\langle\psi|$$

Remarks:

(a) This restates the non-signalling principle in the presence of entanglement: if 1 doesn't hold, a party holding R can affect the density matrix of S and it measurement statistics, thereby partially communicating whether U is applied or not.

(b) Or, the above can be proved directly as an alternative proof to the non-signalling principle.

Properties of the density matrix and partial trace:

1. For any $|\psi\rangle$ on RS, for any unitary U on R,

$$\mathrm{tr}_R\, U_R \otimes I_S\, |\psi\rangle\langle\psi|\, U_R^{\dagger} \otimes I_S \;=\; \mathrm{tr}_R\, |\psi\rangle\langle\psi|$$

Proof: Let r, s be dim of R, S. $I_S \otimes U_R = \underbrace{\begin{array}{c} \begin{array}{cccc} \leftarrow r \rightarrow & & & \end{array} \\ \left(\begin{array}{c|c|c|c} U & 0 & 0 & \\ \hline 0 & U & & \\ \hline 0 & & & \\ \vdots & & & U \end{array}\right) \end{array}}_{\leftarrow s \text{ blocks} \rightarrow}$

note the switch
of the ordering

Properties of the density matrix and partial trace:

1. For any $|\psi\rangle$ on RS, for any unitary U on R,

$$\mathrm{tr}_R\, U_R \otimes I_S\, |\psi\rangle\langle\psi|\, U_R^{\dagger} \otimes I_S = \mathrm{tr}_R\, |\psi\rangle\langle\psi|$$

Proof: Let r, s be dim of R, S. $I_S \otimes U_R \simeq \begin{array}{c} \overset{\leftarrow r \rightarrow}{} \\ \left(\begin{array}{c|c|c|c} U & 0 & 0 & \\ \hline 0 & U & & \\ \hline 0 & & & \\ \vdots & & & U \end{array}\right) \\ \leftarrow s \text{ blocks} \rightarrow \end{array}$

$M = \left(\begin{array}{c|c|c|c} M_{11} & M_{12} & M_{13} \cdots & \\ \hline M_{21} & M_{22} & & \\ \hline M_{31} & & & \\ \vdots & & & M_{ss} \end{array}\right)$, M_{ij} : r-by-r matrices.

Properties of the density matrix and partial trace:

1. For any $|\psi\rangle$ on RS, for any unitary U on R,

$$\mathrm{tr}_R\, U_R \otimes I_S\, |\psi\rangle\langle\psi|\, U_R^\dagger \otimes I_S = \mathrm{tr}_R\, |\psi\rangle\langle\psi|$$

Proof: Let r, s be dim of R, S. $I_S \otimes U_R = \begin{array}{c} \begin{array}{c} \leftarrow r \rightarrow \end{array} \\ \begin{pmatrix} U & 0 & 0 & \\ 0 & U & & \\ 0 & & & \\ \vdots & & & U \end{pmatrix} \\ \leftarrow s \text{ blocks} \longrightarrow \end{array}$

$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \cdots & \\ M_{21} & M_{22} & & \\ M_{31} & & & \\ \vdots & & & M_{ss} \end{pmatrix}$, M_{ij} : r-by-r matrices.

$$(I_S \otimes U_R)\, M\, (I_S \otimes U_R^\dagger) = \begin{pmatrix} U & 0 & 0 & \\ 0 & U & & \\ 0 & & & \\ \vdots & & & U \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} & M_{13} \cdots & \\ M_{21} & M_{22} & & \\ M_{31} & & & \\ \vdots & & & M_{ss} \end{pmatrix} \begin{pmatrix} U^\dagger & 0 & 0 & \\ 0 & U^\dagger & & \\ 0 & & & \\ \vdots & & & U^\dagger \end{pmatrix}$$

Properties of the density matrix and partial trace:

1. For any $|\psi\rangle$ on RS, for any unitary U on R,

$$\mathrm{tr}_R \; U_R \otimes I_S \, |\psi\rangle\langle\psi| \, U_R^\dagger \otimes I_S = \mathrm{tr}_R \, |\psi\rangle\langle\psi|$$

Proof: Let r, s be dim of R, S. $I_S \otimes U_R \doteq \begin{array}{c} \overset{\leftarrow r \rightarrow}{} \\ \left(\begin{array}{c|c|c|c} U & 0 & 0 & \\ \hline 0 & U & & \\ \hline 0 & & & \\ \vdots & & & U \end{array}\right) \\ \longleftarrow s \text{ blocks} \longrightarrow \end{array}$

$$M = \left(\begin{array}{c|c|c|c} M_{11} & M_{12} & M_{13} \cdots & \\ \hline M_{21} & M_{22} & & \\ \hline M_{31} & & & \\ \vdots & & & M_{ss} \end{array}\right), \quad M_{ij}$$

: r-by-r matrices.

$$(I_S \otimes U_R)\, M \,(I_S \otimes U_R^\dagger) \doteq \left(\begin{array}{c|c|c|c} U & 0 & 0 & \\ \hline 0 & U & & \\ \hline 0 & & & \\ \vdots & & & U \end{array}\right) \left(\begin{array}{c|c|c|c} M_{11} & M_{12} & M_{13} \cdots & \\ \hline M_{21} & M_{22} & & \\ \hline M_{31} & & & \\ \vdots & & & M_{ss} \end{array}\right) \left(\begin{array}{c|c|c|c} U^\dagger & 0 & 0 & \\ \hline 0 & U^\dagger & & \\ \hline 0 & & & \\ \vdots & & & U^\dagger \end{array}\right)$$

$$\doteq \left(\begin{array}{c|c|c|c} UM_{11}U^\dagger & UM_{12}U^\dagger & UM_{13}U^\dagger & \cdots \\ \hline UM_{21}U^\dagger & UM_{22}U^\dagger & UM_{23}U^\dagger & \cdots \\ \hline UM_{31}U^\dagger & \cdots & & \\ \vdots & & & UM_{ss}U^\dagger \end{array}\right)$$

(exercise)

$$\mathrm{tr}_R \left(I_S \otimes U_R\right) M \left(I_S \otimes U_R^\dagger\right) = \mathrm{tr}_R \begin{pmatrix} \begin{array}{c|c|c|c} UM_{11}U^\dagger & UM_{12}U^\dagger & UM_{13}U^\dagger & \cdots \\ \hline UM_{21}U^\dagger & UM_{22}U^\dagger & UM_{23}U^\dagger & \cdots \\ \hline UM_{31}U^\dagger & \ddots & & \\ \vdots & & & \\ & & & UM_{ss}U^\dagger \end{array} \end{pmatrix}$$

$$\operatorname{tr}_R \left(I_S \otimes U_R\right) M \left(I_S \otimes U_R^\dagger\right) = \operatorname{tr}_R \left(\begin{array}{c|c|c|c} UM_{11}U^\dagger & UM_{12}U^\dagger & UM_{13}U^\dagger & \cdots \\ \hline UM_{21}U^\dagger & UM_{22}U^\dagger & UM_{23}U^\dagger & \cdots \\ \hline UM_{31}U^\dagger & \ddots & & \\ \vdots & & & UM_{ss}U^\dagger \end{array}\right)$$

$$= \left(\begin{array}{c|c|c|c|c} \operatorname{tr} UM_{11}U^\dagger & \operatorname{tr} UM_{12}U^\dagger & \operatorname{tr} UM_{13}U^\dagger & \cdots & \\ \hline \operatorname{tr} UM_{21}U^\dagger & \operatorname{tr} UM_{22}U^\dagger & \operatorname{tr} UM_{23}U^\dagger & \cdots & \\ \hline \operatorname{tr} UM_{31}U^\dagger & \ddots & & & \\ \vdots & & & & \operatorname{tr} UM_{ss}U^\dagger \end{array}\right)$$

$$\mathrm{tr}_R\,(I_S \otimes U_R)\,M\,(I_S \otimes U_R^\dagger) = \mathrm{tr}_R \left[\begin{array}{c|c|c|c} UM_{11}U^\dagger & UM_{12}U^\dagger & UM_{13}U^\dagger & \cdots \\ \hline UM_{21}U^\dagger & UM_{22}U^\dagger & UM_{23}U^\dagger & \cdots \\ \hline UM_{31}U^\dagger & \ddots & & \\ \vdots & & & UM_{ss}U^\dagger \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c|c} \mathrm{tr}\,UM_{11}U^\dagger & \mathrm{tr}\,UM_{12}U^\dagger & \mathrm{tr}\,UM_{13}U^\dagger & \cdots & \\ \hline \mathrm{tr}\,UM_{21}U^\dagger & \mathrm{tr}\,UM_{22}U^\dagger & \mathrm{tr}\,UM_{23}U^\dagger & \cdots & \\ \hline \mathrm{tr}\,UM_{31}U^\dagger & \ddots & & & \\ \vdots & & & & \mathrm{tr}\,UM_{ss}U^\dagger \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c|c} \mathrm{tr}\,M_{11} & \mathrm{tr}\,M_{12} & \mathrm{tr}\,M_{13} & \cdots & \\ \hline \mathrm{tr}\,M_{21} & \mathrm{tr}\,M_{22} & \mathrm{tr}\,M_{23} & \cdots & \\ \hline \mathrm{tr}\,M_{31} & \ddots & & & \\ \vdots & & & & \mathrm{tr}\,M_{ss} \end{array}\right] = \mathrm{tr}_R\,M.$$

Properties of the density matrix and partial trace:

1. For any $|\psi\rangle$ on RS, for any unitary U on R,

$$tr_R \, U_R \otimes I_S \, |\psi\rangle\langle\psi| \, U_R^\dagger \otimes I_S = tr_R \, |\psi\rangle\langle\psi|$$

2. Corollary: the partial trace of R can be taken along any basis of R (just as the trace is basis indep).

Proof (reading exercise for W25):

Properties of the density matrix and partial trace:

1. For any $|\psi\rangle$ on RS, for any unitary U on R,

$$\mathrm{tr}_R\, U_R \otimes I_S\, |\psi\rangle\langle\psi|\, U_R^\dagger \otimes I_S = \mathrm{tr}_R\, |\psi\rangle\langle\psi|$$

2. Corollary: the partial trace of R can be taken along any basis of R (just as the trace is basis indep).

Proof: $\mathrm{tr}_R\, M = \mathrm{tr}_R\, U_R \otimes I_S\, M\, U_R^\dagger \otimes I_S$

$$= \sum_{i=1}^{r} \left(\langle i|U_R \otimes I\right) M \left(U_R^\dagger |i\rangle \otimes I\right)$$

Properties of the density matrix and partial trace:

1. For any $|\psi\rangle$ on RS, for any unitary U on R,

$$\mathrm{tr}_R\, U_R \otimes I_S\, |\psi\rangle\langle\psi|\, U_R^\dagger \otimes I_S = \mathrm{tr}_R\, |\psi\rangle\langle\psi|$$

2. Corollary: the partial trace of R can be taken along any basis of R (just as the trace is basis indep).

Proof: $\mathrm{tr}_R\, M = \mathrm{tr}_R\, U_R \otimes I_S\, M\, U_R^\dagger \otimes I_S$

$$= \sum_{i=1}^{r} \left(\langle i| U_R \otimes I\right) M \left(U_R^\dagger |i\rangle \otimes I\right)$$

and any basis of R can be written as $\{U_R^\dagger |i\rangle\}_i$ for some U_R.

<u>Properties of the density matrix and partial trace</u>:

1. For any $|\psi\rangle$ on RS, for any unitary U on R,

$$\mathrm{tr}_R \, U_R \otimes I_S \, |\psi\rangle\langle\psi| \, U_R^\dagger \otimes I_S = \mathrm{tr}_R \, |\psi\rangle\langle\psi|$$

2. Corollary: the partial trace of R can be taken along any basis of R (just as the trace is basis indep).

3. Partial tracing a system R has physical meaning of not accessing R. One example is losing access to R irreversibly (disgarded or corrupted by noise).

Properties of the density matrix and partial trace:

4. The partial trace is:
   (a) linear
   (b) trace preserving
   (c) completely positive (i.e., applied to 1 out of 2 systems, it takes positive semidefinite (PSD) matrices to PSD matrices)

hermitian with non-negative eigenvalues

Properties of the density matrix and partial trace:

4. The partial trace is:
   (a) linear
   (b) trace preserving
   (c) completely positive (i.e., applied to 1 out of 2 systems, it takes positive semidefinite (PSD) matrices to PSD matrices)

Proof: (a) immediate from definition

Properties of the density matrix and partial trace:

4. The partial trace is:
   (a) linear
   (b) trace preserving
   (c) completely positive (i.e., applied to 1 out of 2 systems, it takes positive semidefinite (PSD) matrices to PSD matrices)

Proof: (a) immediate from definition

(b) $\mathrm{tr}_R M = \sum_{i=1}^{r} (I \otimes \langle i|) M_{SR} (I \otimes |i\rangle)$

$$\mathrm{tr}(\mathrm{tr}_R M) = \sum_{j=1}^{s} \langle j| \mathrm{tr}_R M |j\rangle$$

Properties of the density matrix and partial trace:

4. The partial trace is:
   (a) linear
   (b) trace preserving
   (c) completely positive (i.e., applied to 1 out of 2 systems, it takes positive semidefinite (PSD) matrices to PSD matrices)

Proof: (a) immediate from definition

(b) $\mathrm{tr}_R M = \sum_{i=1}^{r} (I \otimes \langle i|) M_{SR} (I \otimes |i\rangle)$

$$\mathrm{tr}(\mathrm{tr}_R M) = \sum_{j=1}^{s} \langle j| \mathrm{tr}_R M |j\rangle$$

$$= \sum_{j=1}^{s} \langle j| \sum_{i=1}^{r} (I \otimes \langle i|) M_{SR} (I \otimes |i\rangle) |j\rangle$$

Properties of the density matrix and partial trace:

4. The partial trace is:

   (a) linear

   (b) trace preserving

   (c) completely positive (i.e., applied to 1 out of 2 systems, it takes positive semidefinite (PSD) matrices to PSD matrices)

Proof: (a) immediate from definition

(b) $$\mathrm{tr}_R M = \sum_{i=1}^{r} (I \otimes \langle i|)\, M_{SR}\, (I \otimes |i\rangle)$$

$$\mathrm{tr}(\mathrm{tr}_R M) = \sum_{j=1}^{s} \langle j|\, \mathrm{tr}_R M\, |j\rangle$$

matrices on S

$$= \sum_{j=1}^{s} \langle j| \underbrace{\sum_{i=1}^{r} (I \otimes \langle i|)\, M_{SR}\, (I \otimes |i\rangle)}_{\text{matrix on S}} \underbrace{|j\rangle}_{\text{vector on S}}$$

Properties of the density matrix and partial trace:

4. The partial trace is:
   (a) linear
   (b) trace preserving
   (c) completely positive (i.e., applied to 1 out of 2 systems, it takes positive semidefinite (PSD) matrices to PSD matrices)

Proof: (a) immediate from definition

(b) $\mathrm{tr}_R M = \sum_{i=1}^{r} (I \otimes \langle i|) M_{SR} (I \otimes |i\rangle)$

matrices on S

$$\mathrm{tr}(\mathrm{tr}_R M) = \sum_{j=1}^{s} \langle j| \mathrm{tr}_R M |j\rangle$$

$$= \sum_{j=1}^{s} \langle j| \underbrace{\sum_{i=1}^{r} (I \otimes \langle i|) M_{SR} (I \otimes |i\rangle)}_{\text{matrix on S}} |j\rangle$$

vector on S

$$= \sum_{j=1}^{s} \sum_{i=1}^{r} (\underset{S}{\langle j|} \otimes \underset{R}{\langle i|}) M_{SR} (|j\rangle \otimes |i\rangle) = \mathrm{tr}\, M$$

Properties of the density matrix and partial trace:

4. The partial trace is:
    (a) linear
    (b) trace preserving
    (c) completely positive (i.e., applied to 1 out of 2 systems, it takes positive semidefinite (PSD) matrices to PSD matrices)

Proof: (c) if M is PSD, M = $\sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$ where $\lambda_j > 0$

on RS

Properties of the density matrix and partial trace:

4. The partial trace is:
   (a) linear
   (b) trace preserving
   (c) completely positive (i.e., applied to 1 out of 2 systems, it takes positive semidefinite (PSD) matrices to PSD matrices)

Proof: (c) if M is PSD, M = $\sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$ where $\lambda_j > 0$

For each j, $tr_R |\psi_j\rangle\langle\psi_j| = \sum_{i=1}^{r} (I \otimes \langle i|) |\psi_j\rangle\langle\psi_j|_{SR} (I \otimes |i\rangle)$

$= \sum_{i=1}^{r} v_{ij} v_{ij}^{\dagger}$ where $v_{ij} = (I \otimes \langle i|) |\psi_j\rangle$

Properties of the density matrix and partial trace:

4. The partial trace is:
   (a) linear
   (b) trace preserving
   (c) completely positive (i.e., applied to 1 out of 2 systems, it takes positive semidefinite (PSD) matrices to PSD matrices)

Proof: (c) if M is PSD, M = $\sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$ where $\lambda_j > 0$

For each j, $tr_R |\psi_j\rangle\langle\psi_j| = \sum_{i=1}^{r} (I \otimes \langle i|) |\psi_j\rangle\langle\psi_j|_{SR} (I \otimes |i\rangle)$

$$= \sum_{i=1}^{r} v_{ij} v_{ij}^{\dagger}$$ where $v_{ij} = (I \otimes \langle i|) |\psi_j\rangle$

$$tr_R M = tr_R \sum_j \lambda_j |\psi_j\rangle\langle\psi_j| = \sum_j \lambda_j \, tr_R |\psi_j\rangle\langle\psi_j|$$

$$= \sum_j \sum_{i=1}^{r} \lambda_j \underbrace{v_{ij} v_{ij}^{\dagger}}_{\text{PSD}} \geq 0$$

Properties of the density matrix and partial trace:

4. The partial trace is:
   (a) linear
   (b) trace preserving
   (c) completely positive (i.e., applied to 1 out of 2 systems, it takes positive semidefinite (PSD) matrices to PSD matrices)

5. Corollary: $\rho_S = \mathrm{tr}_R |\psi\rangle\langle\psi|$ is trace 1 and PSD.

Properties of the density matrix and partial trace:

4. The partial trace is:
   (a) linear
   (b) trace preserving
   (c) completely positive (i.e., applied to 1 out of 2 systems, it takes positive semidefinite (PSD) matrices to PSD matrices)

5. Corollary: $\rho_S = \mathrm{tr}_R |\psi\rangle\langle\psi|$ is trace 1 and PSD.

6. Corollary: $\rho_S = \sum_k \mu_k |\psi_k\rangle\langle\psi_k|$ where $\mu_k \geq 0$, $\sum_k \mu_k = 1$

eigenvector of ρ_S

Properties of the density matrix and partial trace:

7. For $\rho_S = \sum_k \mu_k |\psi_k\rangle\langle\psi_k|$ where $\mu_k \geq 0$, $\sum_k \mu_k = 1$

   a measurement on S has same statistics as drawing k with prob μ_k, preparing $|\psi_k\rangle$, and measuring.

   Proof:

Properties of the density matrix and partial trace:

7. For $\rho_S = \sum_k \mu_k |\psi_k\rangle\langle\psi_k|$ where $\mu_k \geq 0$, $\sum_k \mu_k = 1$

a measurement on S has same statistics as drawing k with prob μ_k, preparing $|\psi_k\rangle$, and measuring.

Proof: if measurement is given by projectors $\{P_\ell\}$

$$\text{prob}(\ell) = \text{tr}\, P_\ell \rho_S$$

$$= \text{tr}\, P_\ell \sum_k \mu_k |\psi_k\rangle\langle\psi_k|$$

Properties of the density matrix and partial trace:

7. For $\rho_S = \sum_k \mu_k |\psi_k\rangle\langle\psi_k|$ where $\mu_k \geq 0$, $\sum_k \mu_k = 1$

a measurement on S has same statistics as drawing k with prob μ_k, preparing $|\psi_k\rangle$, and measuring.

Proof: if measurement is given by projectors $\{P_\ell\}$

$$\begin{aligned} \text{prob}(\ell) &= \text{tr}\, P_\ell \rho_S \\ &= \text{tr}\, P_\ell \sum_k \mu_k |\psi_k\rangle\langle\psi_k| \\ &= \sum_k \mu_k\, \text{tr}\, P_\ell |\psi_k\rangle\langle\psi_k| \\ &= \sum_k \mu_k\, \text{prob}(\ell \,|\, \text{state} = |\psi_k\rangle) \end{aligned}$$

$\square$

Properties of the density matrix and partial trace:

7. For $\rho_S = \sum_k \mu_k |\psi_k\rangle\langle\psi_k|$ where $\mu_k \geq 0$, $\sum_k \mu_k = 1$

a measurement on S has same statistics as drawing k with prob μ_k, preparing $|\psi_k\rangle$, and measuring.

Proof: if measurement is given by projectors $\{P_l\}$

$$\begin{aligned} \text{prob}(l) &= \text{tr}\, P_l \rho_S \\ &= \text{tr}\, P_l \sum_k \mu_k |\psi_k\rangle\langle\psi_k| \\ &= \sum_k \mu_k\, \text{tr}\, P_l |\psi_k\rangle\langle\psi_k| \\ &= \sum_k \mu_k\, \text{prob}(l \mid \text{state} = |\psi_k\rangle) \end{aligned}$$

$\square$

Property 7 gives a second interpretation of density matrix as a probabilistic mixture (or ensemble) of quantum states (later).

We took our 5 postulates, applied to composite system RS and derived QM as viewed from S.

We now deduce some properties for the states and evolution/measurement.

We can instead start with these properties and formulate all QM from scratch based on them ...

Alternative approach to
deriving the density matrix

(b)

Pure state
QM on 2
systems RS

partial trace
of R

(a)

Ensemble
of pure
states on S

mixing
states
on S

Density matrix on S
(revised measurement)

Mixed state quantum mechanics

From vectors to density matrices

The density matrix of a pure state $|\psi\rangle$ is $\rho = |\psi\rangle\langle\psi|$.

Definition: if a system is in the state $|\psi_0\rangle$ with prob p_0 and $|\psi_1\rangle$ with prob p_1, $p_0 + p_1 = 1$, then the state of the system is given by the density matrix

$$\rho = p_0 |\psi_0\rangle\langle\psi_0| + p_1 |\psi_1\rangle\langle\psi_1| .$$

From vectors to density matrices

The density matrix of a pure state $|\psi\rangle$ is $\rho = |\psi\rangle\langle\psi|$.

Definition: if a system is in the state $|\psi_0\rangle$ with prob p_0 and $|\psi_1\rangle$ with prob p_1, $p_0 + p_1 = 1$, then the state of the system is given by the density matrix

$$\rho = p_0 |\psi_0\rangle\langle\psi_0| + p_1 |\psi_1\rangle\langle\psi_1| .$$

Terminology: this is called a mixed state.

From vectors to density matrices

The density matrix of a pure state $|\psi\rangle$ is $\rho = |\psi\rangle\langle\psi|$.

Definition: if a system is in the state $|\psi_0\rangle$ with prob p_0 and $|\psi_1\rangle$ with prob p_1, $p_0 + p_1 = 1$, then the state of the system is given by the density matrix

$$\rho = p_0 |\psi_0\rangle\langle\psi_0| + p_1 |\psi_1\rangle\langle\psi_1| .$$

Terminology: this is called a mixed state.

The definition extends to arbitrary mixtures, over any number of states, which can also be mixed.

From vectors to density matrices

The density matrix of a pure state $|\psi\rangle$ is $\rho = |\psi\rangle\langle\psi|$.

Definition: if a system is in the state $|\psi_0\rangle$ with prob p_0 and $|\psi_1\rangle$ with prob p_1, $p_0 + p_1 = 1$, then the state of the system is given by the density matrix

$$\rho = p_0 |\psi_0\rangle\langle\psi_0| + p_1 |\psi_1\rangle\langle\psi_1| .$$

Terminology: this is called a mixed state.

The definition extends to arbitrary mixtures, over any number of states, which can also be mixed.

The prescription: "with probability p_i the state is ρ_i" is called an ensemble, or a probabilistic mixture, of states.

Example 1

If the state is $|0\rangle$ with prob 2/3, $|+\rangle$ with prob 1/3, what is the density matrix ?

Example 1

If the state is $|0\rangle$ with prob 2/3, $|+\rangle$ with prob 1/3, what is the density matrix ?

Answer: $|0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$|+\rangle\langle +| = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Example 1

If the state is $|0\rangle$ with prob 2/3, $|+\rangle$ with prob 1/3, what is the density matrix ?

Answer: $|0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$|+\rangle\langle +| = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

So the density matrix of interest is

$$\rho = \frac{2}{3}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{3}\cdot\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

Example 1 (continued)

Applying spectral decomposition to ρ

$$\rho = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0.23 & 0.97 \\ -0.97 & 0.23 \end{bmatrix} \begin{bmatrix} 0.13 & 0 \\ 0 & 0.87 \end{bmatrix} \begin{bmatrix} 0.23 & 0.97 \\ -0.97 & 0.23 \end{bmatrix}$$

$$= 0.13 \begin{bmatrix} 0.23 \\ -0.97 \end{bmatrix} \begin{bmatrix} 0.23 & -0.97 \end{bmatrix} + 0.87 \begin{bmatrix} 0.97 \\ 0.23 \end{bmatrix} \begin{bmatrix} 0.97 & 0.23 \end{bmatrix}$$

Example 1 (continued)

Applying spectral decomposition to ρ

$$\rho = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0.23 & 0.97 \\ -0.97 & 0.23 \end{bmatrix} \begin{bmatrix} 0.13 & 0 \\ 0 & 0.87 \end{bmatrix} \begin{bmatrix} 0.23 & 0.97 \\ -0.97 & 0.23 \end{bmatrix}$$

$$= 0.13 \begin{bmatrix} 0.23 \\ -0.97 \end{bmatrix} \begin{bmatrix} 0.23 & -0.97 \end{bmatrix} + 0.87 \begin{bmatrix} 0.97 \\ 0.23 \end{bmatrix} \begin{bmatrix} 0.97 & 0.23 \end{bmatrix}$$

It means that, for a different mixture, where the state is $\begin{bmatrix} 0.23 \\ -0.97 \end{bmatrix}$ with prob 0.13, $\begin{bmatrix} 0.97 \\ 0.23 \end{bmatrix}$ with prob 0.87, the density matrix is also ρ.

Example 1 (continued)

Applying spectral decomposition to ρ

$$\rho = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0.23 & 0.97 \\ -0.97 & 0.23 \end{bmatrix} \begin{bmatrix} 0.13 & 0 \\ 0 & 0.87 \end{bmatrix} \begin{bmatrix} 0.23 & 0.97 \\ -0.97 & 0.23 \end{bmatrix}$$

$$= 0.13 \begin{bmatrix} 0.23 \\ -0.97 \end{bmatrix} \begin{bmatrix} 0.23 & -0.97 \end{bmatrix} + 0.87 \begin{bmatrix} 0.97 \\ 0.23 \end{bmatrix} \begin{bmatrix} 0.97 & 0.23 \end{bmatrix}$$

It means that, for a different mixture, where the state is $\begin{bmatrix} 0.23 \\ -0.97 \end{bmatrix}$ with prob 0.13, $\begin{bmatrix} 0.97 \\ 0.23 \end{bmatrix}$ with prob 0.87, the density matrix is also ρ.

So, two different mixtures can give rise to the same density matrix.

Example 1 (continued)

If the state is $|0\rangle$ with prob 2/3, $|+\rangle$ with prob 1/3,

is the state same as $\sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|+\rangle$?

Vote: (a) yes, (b) no.

Example 1 (continued)

If the state is $|0\rangle$ with prob 2/3, $|+\rangle$ with prob 1/3,

is the state same as $\sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|+\rangle$?

Vote: (a) yes, (b) no.

Renormalizing $\sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|+\rangle$ gives $0.95|0\rangle + 0.32|1\rangle =: |\psi\rangle$

$$|\psi\rangle\langle\psi| = \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix} \neq \rho = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} \; !$$

Example 1 (continued)

If the state is $|0\rangle$ with prob 2/3, $|+\rangle$ with prob 1/3, is the state same as $\sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|+\rangle$?

Vote: (a) yes, (b) no.

Renormalizing $\sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|+\rangle$ gives $0.95|0\rangle + 0.32|1\rangle =: |\psi\rangle$

$$|\psi\rangle\langle\psi| = \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix} \neq \rho = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} \; !$$

Note: $|\psi\rangle \propto \sqrt{p_0}\,|\psi_0\rangle + \sqrt{p_1}\,|\psi_1\rangle$ and

$\rho = p_0\,|\psi_0\rangle\langle\psi_0| + p_1\,|\psi_1\rangle\langle\psi_1|$ are very different!

A superposition is a sum over vectors that exhibit interference. A mixture is a sum over density matrices, and the summands do not exhibit interference.

Alternative state postulate:

For a d-dim system, states are

- (a) trace 1
- (b) positive semidefinite
- (c) dxd

matrices.

By spectral decomp, states are ensembles of pure states.

Evolution of density matrices by unitaries and meas:

1. For an initial mixture:

   $|\psi_0\rangle$ with prob p_0 , $|\psi_1\rangle$ with prob p_1

   a unitary U evolves it to a new mixture:

   $U|\psi_0\rangle$ with prob p_0 , $U|\psi_1\rangle$ with prob p_1

Evolution of density matrices by unitaries and meas:

1. For an initial mixture:

$|\psi_0\rangle$ with prob p_0 , $|\psi_1\rangle$ with prob p_1

a unitary U evolves it to a new mixture:

$U|\psi_0\rangle$ with prob p_0 , $U|\psi_1\rangle$ with prob p_1

The initial density matrix $\rho = p_0 |\psi_0\rangle\langle\psi_0| + p_1 |\psi_1\rangle\langle\psi_1|$

becomes $p_0 \, U|\psi_0\rangle\langle\psi_0|U^\dagger + p_1 \, U|\psi_1\rangle\langle\psi_1|U^\dagger = U\rho U^\dagger .$

2. For a measurement given by projectors $\{ P_j \}$,

the probability to get outcome j is

$\mathrm{Tr}\, P_j |\psi_0\rangle\langle\psi_0|$ for the initial state $|\psi_0\rangle$

$\mathrm{Tr}\, P_j |\psi_1\rangle\langle\psi_1|$ for the initial state $|\psi_1\rangle$

2. For a measurement given by projectors $\{ P_j \}$,

the probability to get outcome j is

$\mathrm{tr}\, P_j |\psi_0\rangle\langle\psi_0|$ for the initial state $|\psi_0\rangle$

$\mathrm{tr}\, P_j |\psi_1\rangle\langle\psi_1|$ for the initial state $|\psi_1\rangle$

so, for the mixture,

the probability to get outcome j is

$$p_0 \, \mathrm{tr}\, P_j |\psi_0\rangle\langle\psi_0| + p_1 \, \mathrm{tr}\, P_j |\psi_1\rangle\langle\psi_1|$$

2. For a measurement given by projectors $\{ P_j \}$,

the probability to get outcome j is

$\mathrm{Tr}\, P_j |\psi_0\rangle\langle\psi_0|$ for the initial state $|\psi_0\rangle$

$\mathrm{Tr}\, P_j |\psi_1\rangle\langle\psi_1|$ for the initial state $|\psi_1\rangle$

so, for the mixture,

the probability to get outcome j is

$p_0 \,\mathrm{Tr}\, P_j |\psi_0\rangle\langle\psi_0| + p_1 \,\mathrm{Tr}\, P_j |\psi_1\rangle\langle\psi_1|$

which is equal to $\mathrm{Tr}\, \rho P_j$ for $\rho = p_0 |\psi_0\rangle\langle\psi_0| + p_1 |\psi_1\rangle\langle\psi_1|$.

The postmeasurement state given outcome j is

$$\frac{P_j \, |\psi_0\rangle\langle\psi_0| \, P_j}{\mathrm{Tr} \, P_j \, |\psi_0\rangle\langle\psi_0|}$$ for the initial state $|\psi_0\rangle$

$$\frac{P_j \, |\psi_1\rangle\langle\psi_1| \, P_j}{\mathrm{Tr} \, P_j \, |\psi_1\rangle\langle\psi_1|}$$ for the initial state $|\psi_1\rangle$

The postmeasurement state given outcome j is

$$\frac{P_j \, |\psi_0\rangle\langle\psi_0| \, P_j}{\mathrm{Tr}\, P_j \, |\psi_0\rangle\langle\psi_0|}$$ for the initial state $|\psi_0\rangle$

$$\frac{P_j \, |\psi_1\rangle\langle\psi_1| \, P_j}{\mathrm{Tr}\, P_j \, |\psi_1\rangle\langle\psi_1|}$$ for the initial state $|\psi_1\rangle$

The postmeas state given outcome j for the mixture is

$$\frac{p_0 \times \Pr(j \,|\, |\psi_0\rangle) \times \dfrac{P_j \, |\psi_0\rangle\langle\psi_0| \, P_j}{\mathrm{Tr}\, P_j \, |\psi_0\rangle\langle\psi_0|} + p_1 \times \Pr(j \,|\, |\psi_1\rangle) \times \dfrac{P_j \, |\psi_1\rangle\langle\psi_1| \, P_j}{\mathrm{Tr}\, P_j \, |\psi_1\rangle\langle\psi_1|}}{p_0 \times \Pr(j \,|\, |\psi_0\rangle) + p_1 \times \Pr(j \,|\, |\psi_1\rangle)}$$

The postmeasurement state given outcome j is

$$\frac{P_j \, |\psi_0\rangle\langle\psi_0| \, P_j}{\mathrm{Tr}\, P_j \, |\psi_0\rangle\langle\psi_0|} \quad \text{for the initial state } |\psi_0\rangle$$

$$\frac{P_j \, |\psi_1\rangle\langle\psi_1| \, P_j}{\mathrm{Tr}\, P_j \, |\psi_1\rangle\langle\psi_1|} \quad \text{for the initial state } |\psi_1\rangle$$

The postmeas state given outcome j for the mixture is

$$\frac{p_0 \times \Pr(j \mid |\psi_0\rangle) \times \dfrac{P_j \, |\psi_0\rangle\langle\psi_0| \, P_j}{\mathrm{Tr}\, P_j \, |\psi_0\rangle\langle\psi_0|} + p_1 \times \Pr(j \mid |\psi_1\rangle) \times \dfrac{P_j \, |\psi_1\rangle\langle\psi_1| \, P_j}{\mathrm{Tr}\, P_j \, |\psi_1\rangle\langle\psi_1|}}{p_0 \times \Pr(j \mid |\psi_0\rangle) + p_1 \times \Pr(j \mid |\psi_1\rangle)}$$

$$= \frac{p_0 \times P_j \, |\psi_0\rangle\langle\psi_0| \, P_j + p_1 \, P_j \, |\psi_1\rangle\langle\psi_1| \, P_j}{p_0 \times \mathrm{Tr}\, P_j \, |\psi_0\rangle\langle\psi_0| + p_1 \, \mathrm{Tr}\, P_j \, |\psi_1\rangle\langle\psi_1|} = \frac{P_j \, \rho \, P_j}{\mathrm{Tr}\, P_j \, \rho}$$

Remarks:

1. A density matrix corresponds to a pure state if and only if it is rank-1 (in which case also a projector).

2. The overall phase of a vector disappears when we calculate its density matrix, another way to see why it is irrelevant.

Example 1 (continued)

Can the two mixtures:

$|0\rangle$ with prob 2/3, $|+\rangle$ with prob 1/3

$\begin{bmatrix} 0.23 \\ -0.97 \end{bmatrix}$ with prob 0.13, $\begin{bmatrix} 0.97 \\ 0.23 \end{bmatrix}$ with prob 0.87

be distinguished by operating and measuring the given 2-dim system?

Vote: (a) yes, (b) no.

Example 1 (continued)

Can the two mixtures:

$|0\rangle$ with prob 2/3, $|1\rangle$ with prob 1/3

$\begin{bmatrix} 0.23 \\ -0.97 \end{bmatrix}$ with prob 0.13, $\begin{bmatrix} 0.97 \\ 0.23 \end{bmatrix}$ with prob 0.87

be distinguished by operating and measuring the given 2-dim system?

Vote: (a) yes, (b) no.

Answer: no. The subsequent density matrices and measurement outcomes only depend on ρ, so, all decompositions of ρ into a convex sum of states are indistinguishable.

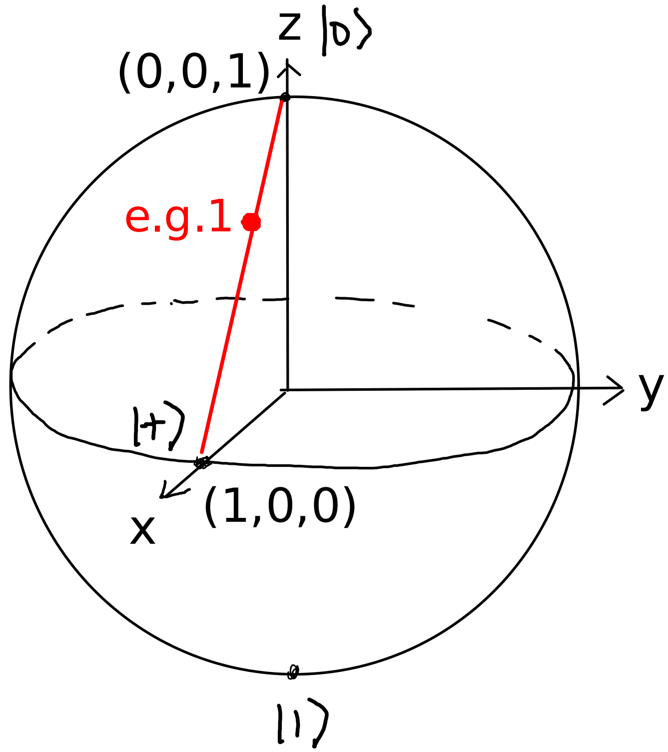
Bloch sphere revisited

For a 2-dimensional quantum system, any density matrix can be written as: $\rho = \frac{1}{2}(I + aX + bY + cZ)$.

Exercise:

Show that $\rho \geq 0$ iff $a^2 + b^2 + c^2 \leq 1$

with rank(ρ) = 1 iff $a^2 + b^2 + c^2 = 1$.



Example: classical marginal distribution

A classical random variable U with range {1,...,d} can be represented as a density matrix $\sum_{u=1}^{d} p_u |u\rangle\langle u|$, $\{|u\rangle\}$ basis.

Classical information is represented by diagonal density matrices !

Let ρ denote the density matrix for a joint distribution on XY and carried by systems AB:

$$\rho = \sum_{x=1}^{d_A} \sum_{y=1}^{d_B} p_{xy} \, |x\rangle\langle x|_A \otimes |y\rangle\langle y|_B$$

Let ρ denote the density matrix for a joint distribution on XY and carried by systems AB:

$$\rho = \sum_{x=1}^{d_A} \sum_{y=1}^{d_B} p_{xy} \, |x\rangle\langle x|_A \otimes |y\rangle\langle y|_B$$

Then,

$$\mathrm{tr}_B \rho = \sum_{i=1}^{d_B} (I \otimes \langle i|) \, \rho \, (I \otimes |i\rangle)$$

$$= \sum_{i=1}^{d_B} (I \otimes \langle i|) \sum_{x=1}^{d_A} \sum_{y=1}^{d_B} p_{xy} \, |x\rangle\langle x|_A \otimes |y\rangle\langle y|_B \, (I \otimes |i\rangle)$$

Let ρ denote the density matrix for a joint distribution on XY and carried by systems AB:

$$\rho = \sum_{x=1}^{d_A} \sum_{y=1}^{d_B} P_{xy} \, |x\rangle\langle x|_A \otimes |y\rangle\langle y|_B$$

Then,

$$\mathrm{tr}_B\,\rho = \sum_{i=1}^{d_B} (I \otimes \langle i|)\,\rho\,(I \otimes |i\rangle)$$

$$= \sum_{i=1}^{d_B} (I \otimes \langle i|) \sum_{x=1}^{d_A} \sum_{y=1}^{d_B} P_{xy} \, |x\rangle\langle x|_A \otimes |y\rangle\langle y|_B \, (I \otimes |i\rangle)$$

$$= \sum_{i=1}^{d_B} \sum_{x=1}^{d_A} \sum_{y=1}^{d_B} P_{xy} \; I\,|x\rangle\langle x|\,I \otimes \underbrace{\langle i|\,|y\rangle\langle y|\,|i\rangle}_{\delta_{iy}}$$

turns B into 1-dim sys

Let ρ denote the density matrix for a joint distribution on XY and carried by systems AB:

$$\rho = \sum_{x=1}^{d_A} \sum_{y=1}^{d_B} P_{xy} \; |x\rangle\langle x|_A \otimes |y\rangle\langle y|_B$$

Then,

$$\mathrm{tr}_B\, \rho = \sum_{i=1}^{d_B} (I \otimes \langle i|)\, \rho\, (I \otimes |i\rangle)$$

$$= \sum_{i=1}^{d_B} (I \otimes \langle i|) \sum_{x=1}^{d_A} \sum_{y=1}^{d_B} P_{xy} \; |x\rangle\langle x|_A \otimes |y\rangle\langle y|_B \; (I \otimes |i\rangle)$$

$$= \sum_{i=1}^{d_B} \sum_{x=1}^{d_A} \sum_{y=1}^{d_B} P_{xy} \; I\, |x\rangle\langle x|\, I \otimes \underbrace{\langle i|\, |y\rangle\langle y|\, |i\rangle}_{\delta_{iy}}$$

turns B into 1-dim sys

$$= \sum_{x=1}^{d_A} \sum_{y=1}^{d_B} P_{xy} \; |x\rangle\langle x|$$

which is indeed the margin distribution X !

Density matrices on a composite system

Consider a bipartite system AB, with dim d_A, d_B.

The most general state on AB is a mixture of pure states on AB, each is $d_A d_B$-dimensional.

Reading exercise for W25?

Density matrices on a composite system

Consider a bipartite system AB, with dim d_A, d_B.

The most general state on AB is a mixture of pure states on AB, each is $d_A d_B$-dimensional.

Special cases:

1. A pure state on AB.

Density matrices on a composite system

Consider a bipartite system AB, with dim d_A, d_B.

The most general state on AB is a mixture of pure states on AB, each is $d_A d_B$-dimensional.

Special cases:

1. A pure state on AB.

2. An uncorrelated state on AB: $\sigma_A \otimes \eta_B$

   where σ_A, η_B are density matrices for sys A, B resp.

Density matrices on a composite system

Consider a bipartite system AB, with dim d_A, d_B.

The most general state on AB is a mixture of pure states on AB, each is $d_A d_B$-dimensional.

Special cases:

1. A pure state on AB.

2. An uncorrelated state on AB: $\sigma_A \otimes \eta_B$

   where σ_A, η_B are density matrices for sys A, B resp.

3. A classical-quantum state on AB:

   $\rho = \sum_x p_x |x\rangle\langle x| \otimes \rho_x$, where $\{|x\rangle\}$ is an orthonormal set of states on A.

Density matrices on a composite system

Consider a bipartite system AB, with dim d_A, d_B.

The most general state on AB is a mixture of pure states on AB, each is $d_A d_B$–dimensional.

Special cases:

1. A pure state on AB.

2. An uncorrelated state on AB: $\sigma_A \otimes \eta_B$
   where σ_A, η_B are density matrices for sys A, B resp.

3. A classical-quantum state on AB:

   $\rho = \sum_x p_x |x\rangle\langle x| \otimes \rho_x$, where $\{|x\rangle\}$ is an orthonormal set of states on A.

   A holds the classical description of the state in B.

Density matrices on a composite system

Consider a bipartite system AB, with dim d_A, d_B.

The most general state on AB is a mixture of pure states on AB, each is $d_A d_B$-dimensional.

Special cases:

1. A pure state on AB.

2. An uncorrelated state on AB: $\sigma_A \otimes \eta_B$

   where σ_A, η_B are density matrices for sys A, B resp.

3. A classical-quantum state on AB:

   $\rho = \sum_x p_x |x\rangle\langle x| \otimes \rho_x$, where $\{|x\rangle\}$ is an orthonormal set of states on A.

   A holds the classical description of the state in B.

   These states arises if Alice draws x with prob p_x records x in system A (classical information), prepares ρ_x on system B and gives it to Bob.

Exercise:

Show that if $\rho = \sum_x p_x |x\rangle\langle x| \otimes \rho_x$

then $\mathrm{tr}_A \rho = \underbrace{\sum_x p_x \rho_x}_{\text{density matrix for the mixture } \rho_x \text{ wp } p_x.}$