8. Modelling noise: mixed state QM & Q operations

States:

(b) States on composite systems (NC 8.3.1, 2.5, KLM 3.5.2)

(a) Noisy quantum data (NC 2.4, KLM 3.5.1)

Evolution:

(c) Most general (noisy) quantum dynamics

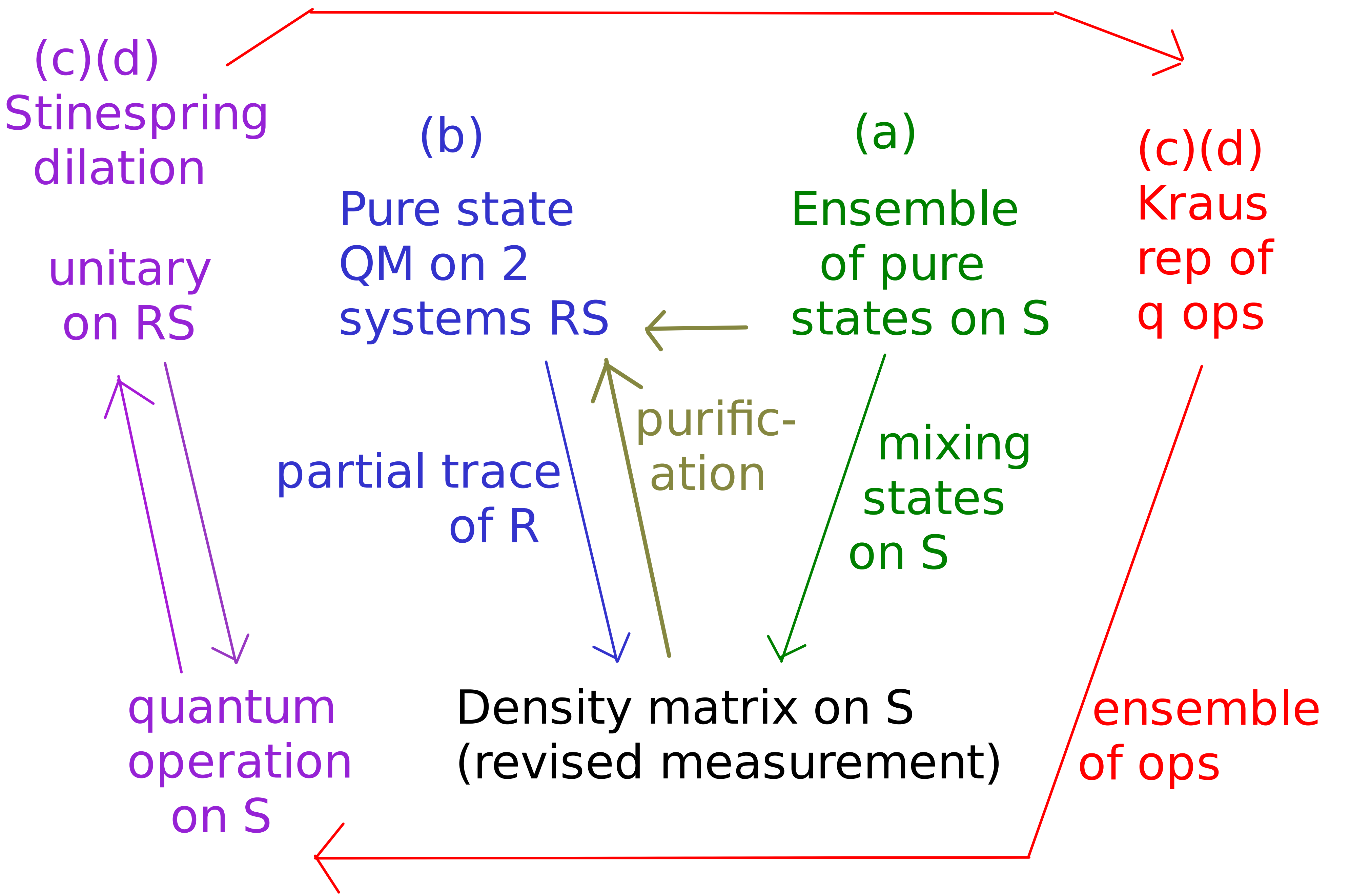
(d) Characterizations (NC 8.2, KLM 3.5.3)

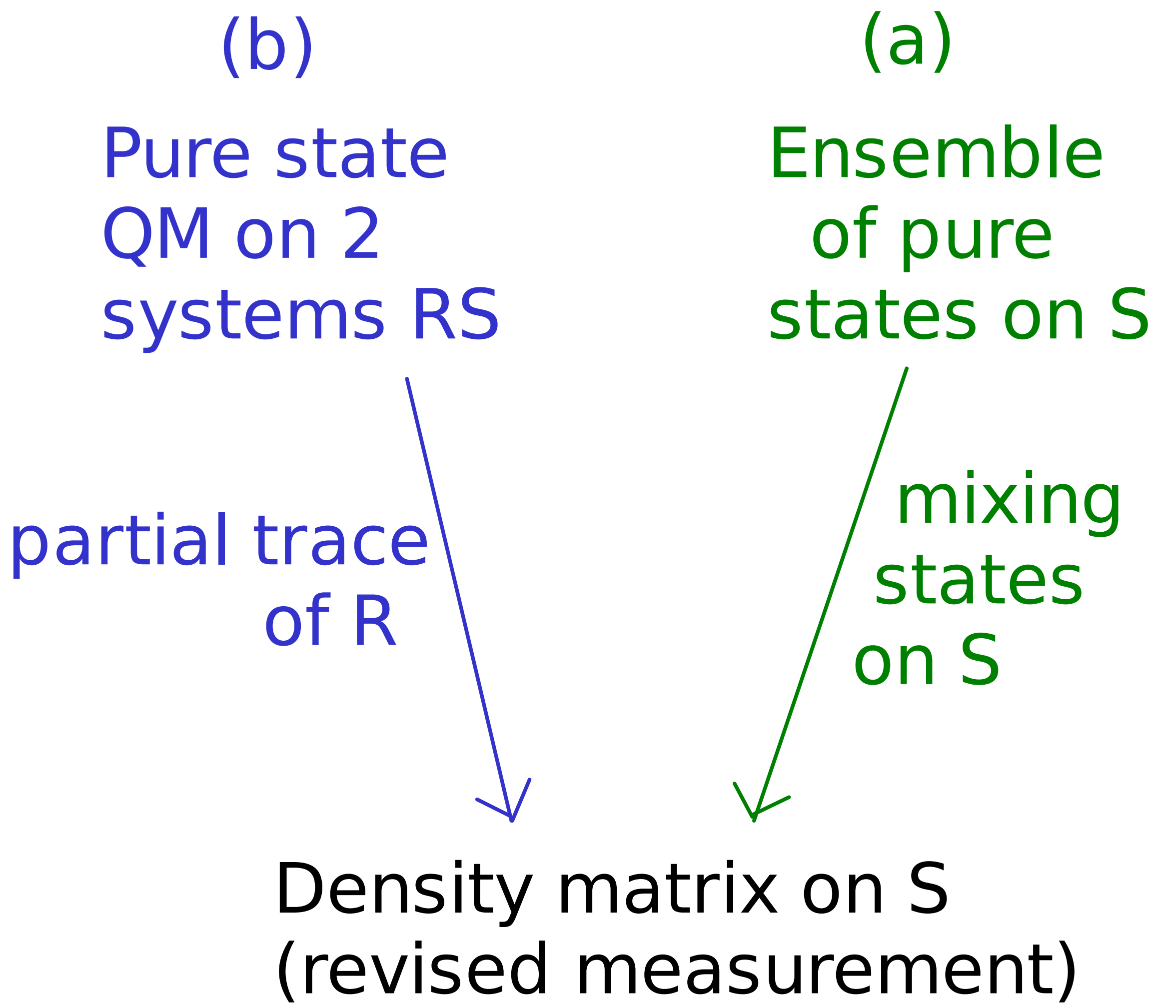
(e) Important examples (NC 8.3)

Measurements:

(f) POVM measurements (NC 2.2.6, KLM A8)

(g) Trace distance, indistinguishability,
Helstrom-Holevo theorem (NC 9.2, KLM A8)


(c)(d)
Stinespring
dilation
unitary
on RS
(b)
Pure state
QM on 2
systems RS
(a)
Ensemble
of pure
states on S
(c)(d)
Kraus
rep of
q ops
partial trace
of R
purific-
ation
mixing
states
on S
quantum
operation
on S
Density matrix on S
(revised measurement)
ensemble
of ops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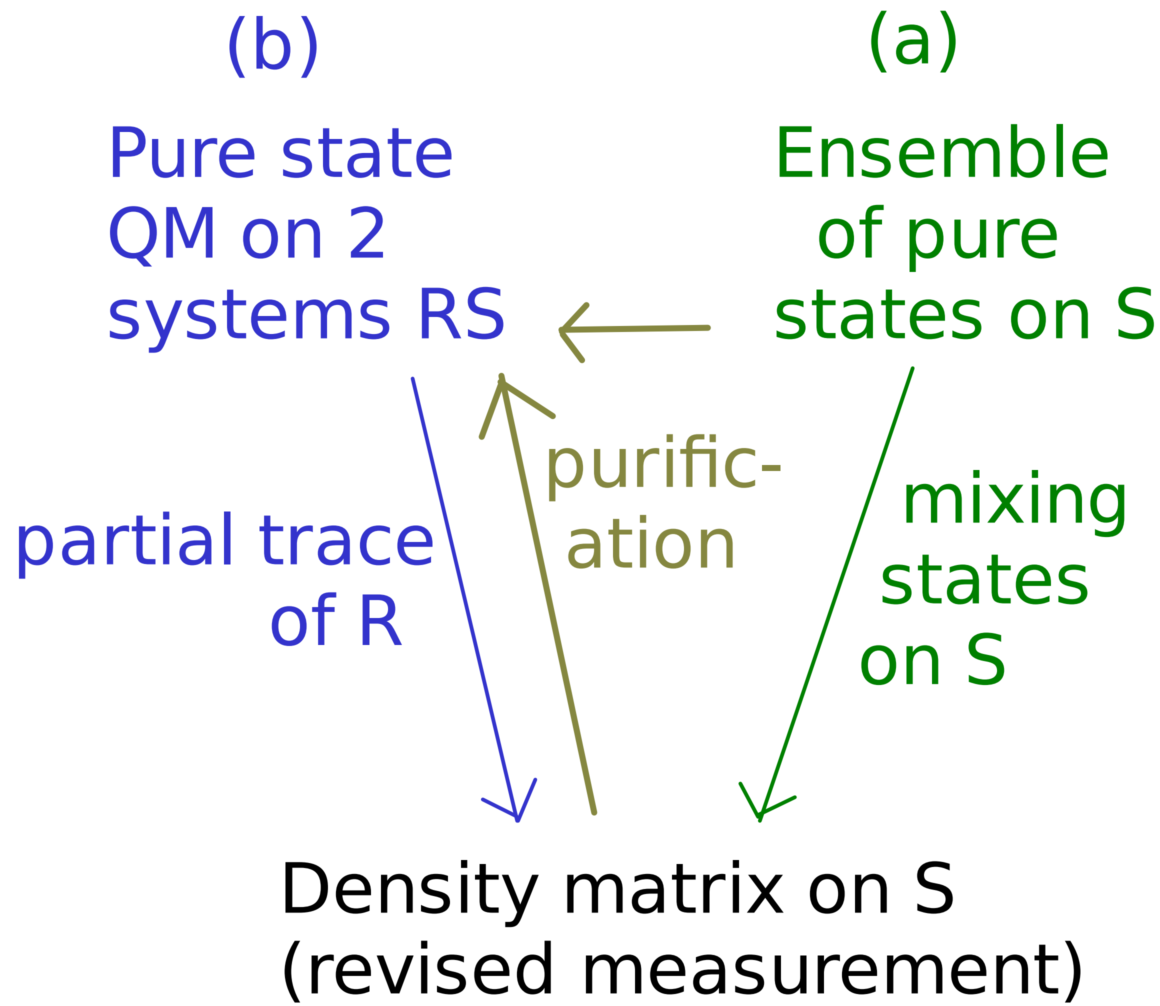

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Concept: accessing 1 out of 2 systems

Physically irreversible unless discarding an uncorrelated system (QECC)

Pure state on RS $\xrightarrow{\text{partial trace}}$ Mixed state on S

Concept: accessing 1 out of 2 systems

Physically irreversible unless discarding an uncorrelated system (QECC)

$$\text{Pure state on RS} \; \underset{\text{purification}}{\overset{\text{partial trace}}{\rightleftarrows}} \; \text{Mixed state on S}$$

Concept: given a state on system S, what pure states in systems RS, if exist, could have given rise to it?

Concept: accessing 1 out of 2 systems

Physically irreversible unless discarding an uncorrelated system (QECC)

$$
\begin{tikzcd}
\text{Pure state on RS} \arrow[r, bend left, "\text{partial trace}"] & \text{Mixed state on S} \arrow[l, bend left, "\text{purification}"]
\end{tikzcd}
$$

Concept: given a state on system S, what pure states in systems RS, if exist, could have given rise to it?

Conceptual inverse of partial trace, not physically possible, crucial in cryptograhy and extremely useful in general.

Definition: given a density matrix ρ on a system S, a purification of ρ is a pure state $|\psi\rangle$ on S and an auxiliary system (say, R), such that $\mathrm{tr}_R |\psi\rangle\langle\psi| = \rho$.

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Given a density matrix,

(1) is there always a purification?

(2) how big does the "purifying system" (R) have to be?

(3) if purifications exist, how are they related?

Example:

Recall that for any basis $\{|e_i\rangle\}$ on R

any unit vectors $|\eta_i\rangle$ on S

$$|\psi\rangle = \sum_i a_i |e_i\rangle|\eta_i\rangle \text{ on RS}$$

$$\mathrm{tr}_R |\psi\rangle\langle\psi| = \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

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e.g. $\rho = \begin{bmatrix} \frac{3}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix} = \lambda_1 |e_1\rangle\langle e_1| + \lambda_2 |e_2\rangle\langle e_2|$

$$\lambda_1 \doteq 0.80 \quad , \quad \lambda_2 \doteq 0.20$$

$$|e_1\rangle \doteq \begin{bmatrix} 0.96 \\ 0.29 \end{bmatrix} \quad , \quad |e_2\rangle \doteq \begin{bmatrix} 0.29 \\ -0.96 \end{bmatrix}$$

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Possible purification: $|\psi\rangle = \sqrt{\lambda_1}\,|1\rangle|e_1\rangle + \sqrt{\lambda_2}\,|2\rangle|e_2\rangle$

Theorem: Let ρ be a density matrix in system S.

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Extension of (ii): Suppose

$|\psi_1\rangle$:purification of ρ with purifying system R1

$|\psi_2\rangle$:purification of ρ with purifying system R2

WLOG, dim(R2) is no less than dim(R1).

Then, $|\psi_2\rangle = U \otimes I_S \, |\psi_1\rangle$ for some isometry U from R1 to R2.

The proof for R1=R2 holds for this extension (modifications needed will be made in green).

<u>Theorem:</u> Let ρ be a density matrix in system S.

(i) ρ has a purification on sys R iff dim(R) $\geqslant$ rank(ρ).

Proof (if): By the spectral theorem $\rho = \sum_{k=1}^{rk(\rho)} \lambda_k |e_k\rangle\langle e_k|$

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Partial-trace of R gives a sum of at most r rank-1 projectors, with rank at most r. So, rank(ρ) $\leq r$.

We saw earlier that if $|\Psi\rangle$ is a purification of ρ, so is $U \otimes I \, |\Psi\rangle_{RS}$ for any unitary U on R.

(ii) says there are no other purifications.

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To see (ii), consider partial trace with the 1st rep of bipartite states on RS.

Let $|\psi\rangle = \sum_{ij} \alpha_{ij} |i\rangle |j\rangle$.

$$\sum_{ij} = \sum_{i=1}^{r} \sum_{j=1}^{s}$$

dim(R) dim(S)

Let $M = \sum_{ij} \alpha_{ij} |j\rangle\langle i|$.

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with r = dim(R) and s = dim(S).

Let $M = \sum_{ij} \alpha_{ij} |j\rangle\langle i|$.

The transformation $|\Psi\rangle \longrightarrow M$ defines a bijection from rs-dim vectors to r-by-s matrices.

Ex: check that

if $|\psi\rangle = s\updownarrow \left.\begin{bmatrix} c_1 \\ \hline c_2 \\ \hline \vdots \\ \\ \hline c_r \end{bmatrix}\right\}$ r blocks

then M = $s\updownarrow \overbrace{\left[\begin{array}{c|c|c|c} c_1 & c_2 & \cdots & \end{array}\right]}^{r}$.

call this Mat($|\psi\rangle$)

$$|\psi\rangle_{RS} = \sum_{ij} \alpha_{ij} |i\rangle|j\rangle. \quad M = \sum_{ij} \alpha_{ij} |j\rangle\langle i|.$$

Lemma: $\mathrm{tr}_R |\psi\rangle\langle\psi| = M M^\dagger$

The proof is a useful exercise for W25.
Answer in the next few pages.

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Proof: LHS = $\sum_{k=1}^{r} \langle k| \otimes I \, |\psi\rangle\langle\psi| \, |k\rangle \otimes I$

$\langle k| \otimes I |\psi\rangle = \sum_j \alpha_{kj} |j\rangle$

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$\delta_{ii'}$

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$$= \sum_i \sum_j \alpha_{ij} |j\rangle \sum_{j'} \alpha^*_{ij'} \langle j'| = \text{LHS}$$

Theorem: (ii) if $|\psi_1\rangle$, $|\psi_2\rangle$ are two purifications of ρ with purifying system R, then, $|\psi_2\rangle = U_R \otimes I_S \; |\psi_1\rangle$ for some unitary U acting on R.

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lemma

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Singular value decomp (SVD) : $M = \sqrt{M M^\dagger}\, V$ (V unitary)

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$$M_1 = \sum_{k=1}^{rk(\rho)} \sqrt{\lambda_k}\, |e_k\rangle\langle f_k| = \sqrt{\rho}\, U_1 \, , \quad \text{where } U_1 = \sum_k |e_k\rangle\langle f_k|$$

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Let $|\psi_1\rangle = \sum_{ij} \alpha_{ij} |i\rangle|j\rangle$, $M_1 = \sum_{ij} \alpha_{ij} |j\rangle\langle i|$.

$$M_2 = M_1 U_1^\dagger U_2 = \sum_{ij} \alpha_{ij} |j\rangle \left(\langle i| U_1^\dagger U_2 \right) \leftarrow \text{linear comb of } \langle \ell|\text{'s}$$

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---

Let $V = (U_1^\dagger U_2)^\dagger = U_2^\dagger U_1$

$$V|i\rangle = \sum_\ell v_{i\ell} |\ell\rangle$$

$$\langle i| V^\dagger = \sum_\ell v_{i\ell}^* \langle \ell|$$

$$\sum_{ij} \alpha_{ij} |j\rangle \left(\langle i| U_1^\dagger U_2\right) = \sum_{ij} \alpha_{ij} |j\rangle \sum_\ell v_{i\ell}^* \langle \ell|$$

Inverting Mat, $|\psi_2\rangle = \sum_{ij} \alpha_{ij} \sum_\ell v_{i\ell}^* |\ell\rangle |j\rangle$

$$= \sum_{ij} \alpha_{ij} V^* |i\rangle |j\rangle$$

$$= \sum_{ij} \alpha_{ij} \left(U_2^\dagger U_1^* |i\rangle\right) |j\rangle$$

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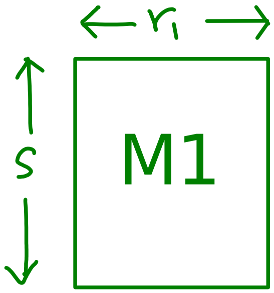
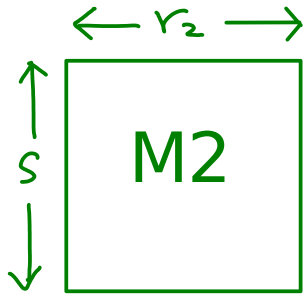
$$= \left(U_2^T U_1^* \otimes I\right) \sum_{ij} \alpha_{ij} |i\rangle|j\rangle$$

$$= \left(U_2^T U_1^* \otimes I\right) |\psi_1\rangle \qquad \square$$

For the extension dim(R1)=r1, dim(R2)=r2, r1 $\leqslant$ r2:





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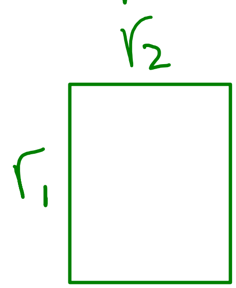
$$U_1 U_1^\dagger = I_s$$

$$M_2 = \sum_{k=1}^{rk(\rho)} \sqrt{\lambda_k}\, |e_k\rangle\langle g_k| = \sqrt{\rho}\, U_2 , \quad \text{where } U_2 = \sum_k |e_k\rangle\langle g_k|$$

$$= \sqrt{\rho}\, U_1 U_1^\dagger U_2 = M_1 \underbrace{U_1^\dagger U_2}$$



Rest of proof
as before.

A purification is a pure state on 2 systems.

Returning to topic03-02 briefly ...

On the 3rd representation of bipartite states,
the Schmidt decomposition.

Assigned as reading exercise for W25?

3 ways to write down a pure state on 2 systems A & B:

1. Take any basis $\{|e_i\rangle\}$ for A, and basis $\{|f_j\rangle\}$ for B.

$$|\psi\rangle = \sum_{ij} \alpha_{ij} |e_i\rangle |f_j\rangle$$ for unique α_{ij}, $\sum_{ij} |\alpha_{ij}|^2 = 1$.

2. Take any basis $\{|e_i\rangle\}$ for A, and basis $\{|f_j\rangle\}$ for B.

$$|\psi\rangle = \sum_{ij} \alpha_{ij} |e_i\rangle |f_j\rangle = \sum_i |e_i\rangle \sum_j \alpha_{ij} |f_j\rangle$$

$$= \sum_i c_i |e_i\rangle |u_i\rangle$$

where $c_i = \sqrt{\sum_j |\alpha_{ij}|^2}$, $|u_i\rangle = \frac{1}{c_i} \sum_j \alpha_{ij} |f_j\rangle$ unit vectors, not necessarily ortho

Similarly with AB interchanged.

3. The Schmidt decomposition (singular value decomp)

KLM A7
NC 2.5

Starting from $|\psi\rangle = \sum_{ij} \alpha_{ij} |e_i\rangle |f_j\rangle$

Define matrix M with (i,j)-entry being α_{ij}.

From the singular value decomposition M = UDV
where D is diagonal with non-negative entries, and

U,V are unitary. So, $\alpha_{ij} = \sum_{k,l} U_{ik} D_{kl} V_{lj}$. Thus

$$|\psi\rangle = \sum_{ij} \alpha_{ij} |e_i\rangle |f_j\rangle = \sum_{ij} \sum_{k,l} U_{ik} D_{kl} V_{lj} |e_i\rangle |f_j\rangle$$

$$= \sum_{k} D_{kk} \underbrace{\sum_{i} U_{ik} |e_i\rangle}_{|a_k\rangle} \underbrace{\sum_{j} V_{kj} |f_j\rangle}_{|b_k\rangle} = \sum_{k} D_{kk} |a_k\rangle |b_k\rangle$$

D diagonal,
so, l = k.

no cross terms!

Ex: check that $|a_k\rangle$'s $(|b_k\rangle$'s$)$ orthonormal, by unitarity of U (V).

NB The Schmidt decomposition is like the 2nd representation but the expression is in terms of a basis for A and also a basis for B! We pay a price -- in the 2nd representation, we can choose any basis for A, here we do not get to choose either basis.

NB The singular values of M, D_{kk} , are called the Schmidt coefficients of $|\psi\rangle$. The rank of M, which is the number of terms in the Schmidt decomposition, is called the Schmidt rank. The bases $\{|a_k\rangle\}$, $\{|b_k\rangle\}$ are called the Schmidt bases of $|\psi\rangle$.

Exercise: show that the Schmidt coefficients are invariant under local unitaries acting on A and B.

They characterize the entanglement of $|\psi\rangle$.

Example:

$$|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle$$

is already in a Schmidt decomposition.

Example: $d_A = 2, \quad d_B = 3$

$$|\psi\rangle = \frac{1}{\sqrt{91}}\left(|00\rangle + 2|01\rangle + 3|02\rangle + 4|10\rangle + 5|11\rangle + 6|12\rangle\right)$$

$$M = \frac{1}{\sqrt{91}}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\text{Singular decomposition of } M = \underbrace{\overbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}^{\longleftarrow 2 \longrightarrow}}_{U}\ \underbrace{\overbrace{\left[\begin{array}{cc|c} \quad & \quad & 0 \\ \quad & \quad & 0 \end{array}\right]}^{\longleftarrow 3 \longrightarrow}}_{D}\ \underbrace{\overbrace{\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}}^{\longleftarrow 3 \longrightarrow}}_{V}$$

$$M^{\dagger} M = V^{\dagger} D^{\dagger} D V$$

Performing a spectral decomposition:

$$M^{\dagger} M = \frac{1}{91} \begin{bmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{bmatrix}$$

$$= \begin{bmatrix} 0.43 & 0.81 & -0.41 \\ 0.57 & 0.11 & 0.82 \\ 0.70 & -0.58 & -0.41 \end{bmatrix} \underbrace{\frac{1}{91} \begin{bmatrix} 90.40 & 0 & 0 \\ 0 & 0.60 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D^{\dagger} D} \underbrace{\begin{bmatrix} 0.43 & 0.57 & 0.70 \\ 0.81 & 0.11 & -0.58 \\ -0.41 & 0.82 & -0.41 \end{bmatrix}}_{V}$$

$$MM^{+} = U D D^{+} U^{+}$$

Performing a spectral decomposition:

$$MM^{+} = \frac{1}{91}\begin{bmatrix} 14 & 32 \\ 32 & 77 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.37 & -0.92 \\ 0.92 & 0.37 \end{bmatrix}}_{U} \underbrace{\frac{1}{91}\begin{bmatrix} 90.40 & 0 \\ 0 & 0.60 \end{bmatrix}}_{DD^{+}} \begin{bmatrix} 0.37 & 0.92 \\ -0.92 & 0.37 \end{bmatrix}$$

$$\because D^{+}D = \frac{1}{91}\begin{bmatrix} 90.40 & 0 & 0 \\ 0 & 0.60 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D = \frac{1}{\sqrt{91}}\begin{bmatrix} \sqrt{90.40} & 0 & 0 \\ 0 & \sqrt{0.60} & 0 \end{bmatrix}$$

$$\therefore M = UDV = \underbrace{\begin{bmatrix} 0.37 & -0.92 \\ 0.92 & 0.37 \end{bmatrix}}_{U} \underbrace{\frac{1}{\sqrt{91}}\begin{bmatrix} \sqrt{90.40} & 0 & 0 \\ 0 & \sqrt{0.60} & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 0.43 & 0.57 & 0.70 \\ 0.81 & 0.11 & -0.58 \\ -0.41 & 0.82 & -0.41 \end{bmatrix}}_{V}$$

$$|\psi\rangle = \sum_k D_{kk} \underbrace{\sum_i U_{ik} |e_i\rangle}_{|a_k\rangle} \underbrace{\sum_j V_{kj} |f_j\rangle}_{|b_k\rangle} = \sum_k D_{kk} |a_k\rangle |b_k\rangle$$

$$D_{11} = \frac{\sqrt{90.40}}{\sqrt{91}} = 0.99671 , \quad D_{22} = \frac{\sqrt{0.60}}{\sqrt{91}} = 0.08102$$

$(K=1)$ $$|a_1\rangle = U_{11}|0\rangle + U_{21}|1\rangle = 0.37|0\rangle + 0.92|1\rangle$$

$(K=2)$ $$|a_2\rangle = U_{12}|0\rangle + U_{22}|1\rangle = -0.92|0\rangle + 0.37|1\rangle$$

$(K=1)$ $$|b_1\rangle = V_{11}|0\rangle + V_{12}|1\rangle + V_{13}|2\rangle$$
$$= 0.43|0\rangle + 0.57|1\rangle + 0.70|2\rangle$$

$(K=2)$ $$|b_2\rangle = V_{21}|0\rangle + V_{22}|1\rangle + V_{23}|2\rangle$$
$$= 0.81|0\rangle + 0.11|1\rangle - 0.58|2\rangle$$

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

$$V = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}$$

$$|\psi\rangle = 0.99671\left(0.37|0\rangle + 0.92|1\rangle\right)\left(0.43|0\rangle + 0.57|1\rangle + 0.70|2\rangle\right)$$

$$+ 0.08102\left(-0.92|0\rangle + 0.37|1\rangle\right)\left(0.81|0\rangle + 0.11|1\rangle - 0.58|2\rangle\right)$$

End of reading exercise for Schmidt decomposition.

Partial tracing a pure bipartite state: say, of system A

1. $|\psi\rangle = \sum_{ij} \alpha_{ij} |e_i\rangle |f_j\rangle$

2. $|\psi\rangle = \sum_{i} c_i |e_i\rangle |u_i\rangle$

3. $|\psi\rangle = \sum_{k} D_{kk} |a_k\rangle |b_k\rangle$

Recall we can take the partial in any basis.

From 1: $\mathrm{tr}_A |\psi\rangle\langle\psi| = \sum_k \langle e_k| \otimes I \ |\psi\rangle\langle\psi| \ |e_k\rangle \otimes I$

$$= \sum_k \sum_j \alpha_{kj} |f_j\rangle \langle\psi| \ |e_k\rangle \otimes I$$

$$= \sum_k \sum_j \alpha_{kj} |f_j\rangle \sum_{j'} \alpha^*_{kj'} \langle f_{j'}|$$

$$= \sum_j \sum_{j'} \left(\sum_k \alpha_{kj} \alpha^*_{kj'} \right) |f_j\rangle\langle f_{j'}|$$

matrix in $|f_j\rangle$ basis

From 2: $\mathrm{tr}_A |\psi\rangle\langle\psi| = \sum_k \langle e_k| \otimes I \; |\psi\rangle\langle\psi| \; |e_k\rangle \otimes I$

$$= \sum_k c_k |u_k\rangle \quad \langle\psi| \; |e_k\rangle \otimes I$$

$$= \sum_k c_k c_k^* |u_k\rangle\langle u_k|$$

convex combination of pure states $|u_k\rangle\langle u_k|$'s

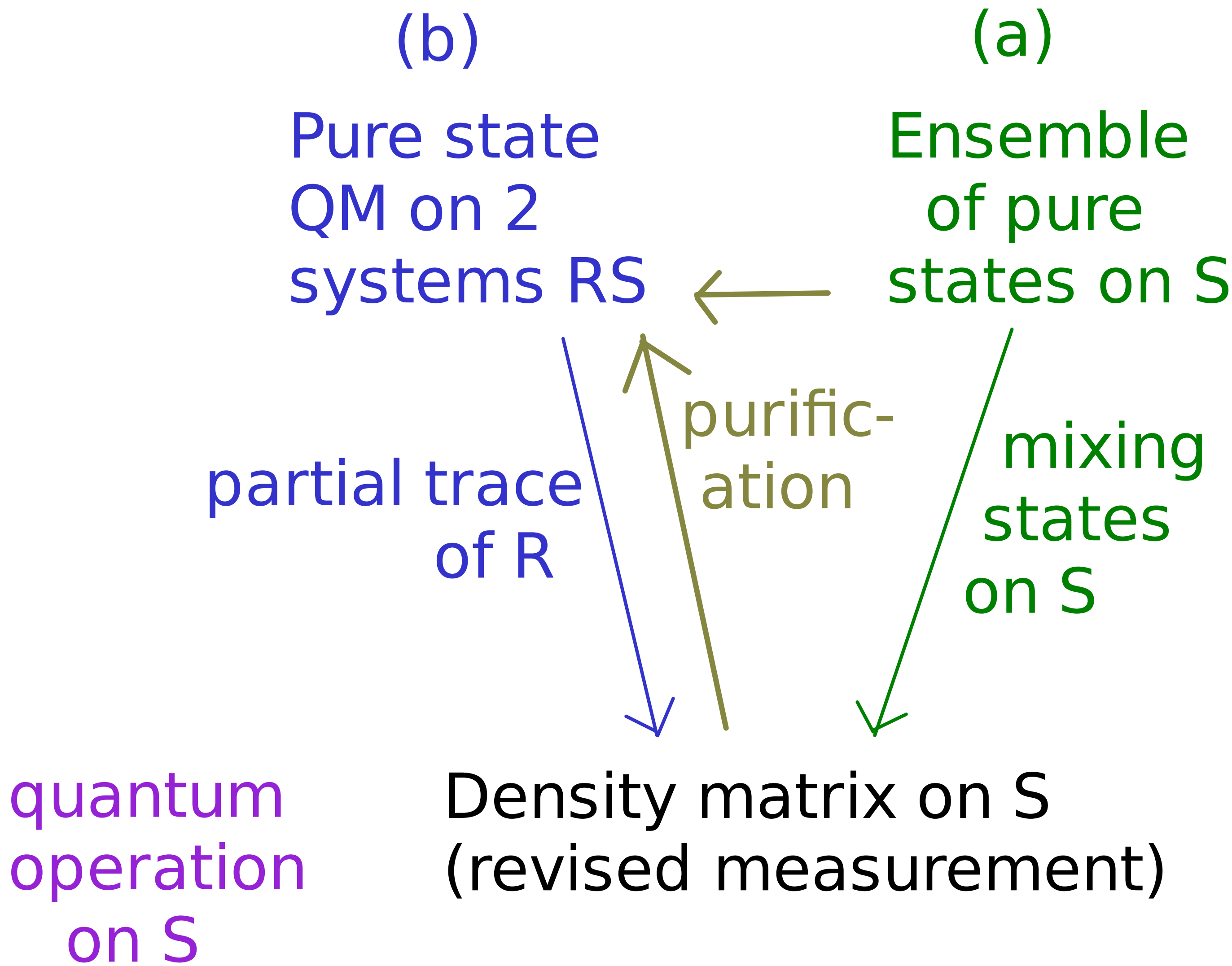
From 3: $\mathrm{tr}_A |\psi\rangle\langle\psi| = \sum_i \langle a_i| \otimes I \; |\psi\rangle\langle\psi| \; |a_i\rangle \otimes I$

$$= \sum_i D_{ii} |b_i\rangle \langle\psi| \; |a_i\rangle \otimes I$$

$$= \sum_i D_{ii}^2 |b_i\rangle\langle b_i|$$

spectral decomp of $\mathrm{tr}_A |\psi\rangle\langle\psi|$ $\because$ $|b_i\rangle$'s orthonormal.

Note that all 3 answers are the same -- we obtain different expressions for the same density matrix !!


(b)
(a)
Pure state QM on 2 systems RS
Ensemble of pure states on S
purific-ation
partial trace of R
mixing states on S
quantum operation on S
Density matrix on S (revised measurement)

Quantum Operations

What is the most general transformation allowed by QM?

Any reasonable transformation N should take quantum states to quantum states !

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Any reasonable transformation N should take quantum states to quantum states !

Viewing N as a mapping from matrices to matrices:

(1) N is linear (QM is)

(2) N is trace preserving: tr(N(M)) = tr(M)
(conservation of probability when M = ρ)

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N applied to 1 out of 2 systems takes a valid initial joint state $\rho \geq 0$ to a valid new joint state $I \otimes N(\rho) \geq 0$.

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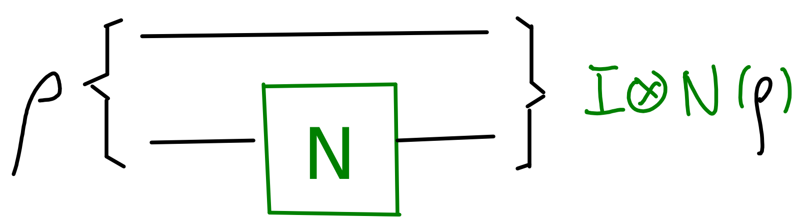

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N applied to 1 out of 2 systems takes a valid initial joint state $\rho \geq 0$ to a valid new joint state $I \otimes N(\rho) \geq 0$.

e.g., conjugation by a unitary is CP
e.g., partial trace is CP

$$\boxed{\text{Definition: a quantum operation is a mapping from matrices to matrices that is (1) linear, (2) trace-preserving, and (3) completely positive.}}$$

Synonyms: quantum channel, TCP map ...

Question:

Define the transpose map as $T(M) = M^T$.

Is the transpose map a quantum channel?

(a) yes, (b) no

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Define the transpose map as $T(M) = M^T$.

Is the transpose map a quantum channel?

(a) yes, (b) no

The transpose is "positive" : M ≥ 0 implies T(M) ≥ 0, but not completely positive. Let $|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

$$|\Phi\rangle\langle\Phi| = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 11| + |11\rangle\langle 00|\right)$$

$$I\otimes T\left(|\Phi\rangle\langle\Phi|\right) = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01|\right)$$

$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$: eigenvector with negative eigenvalue.

The identity map:

Consider the map $I(M) = M$. It is linear, trace preserving and completely positive. It represents the evolution in which nothing happens.

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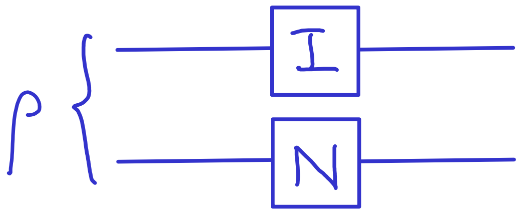

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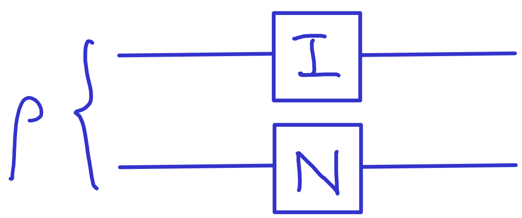

On a tensor product input, $\mathcal{I}\otimes\mathcal{N}(\sigma\otimes\xi) = \sigma\otimes\mathcal{N}(\xi)$.

Then, linearity allows the most general $\mathcal{I}\otimes\mathcal{N}(\rho)$ to be computed.

Definition: a quantum operation is a mapping from matrices to matrices that is (1) linear, (2) trace-preserving, and (3) completely positive.

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Implied from the definition:

1. Composition of two quantum ops is a quantum op.
   (All 3 properties are preserved by composition.)

2. Tensor product of two quantum ops (applied to two disjoint systems) is a quantum op.

Example 1: Conjugation by unitary $N(\rho) = U\rho U^\dagger$

Example 2: Partial trace $N(\rho) = \mathrm{tr}_R\, \rho_{RS}$.

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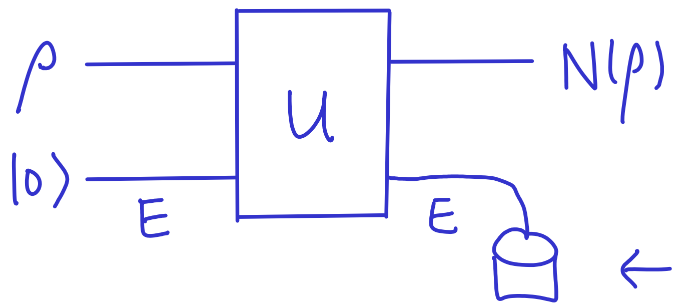
Example 3: $N(\rho) = \mathrm{tr}_E\left(U\, \rho \otimes |0\rangle\langle 0|_E\, U^\dagger\right)$ is a quantum operation for any system E and any U.



← partial trace

Proof: by examples 1-2 and composition.

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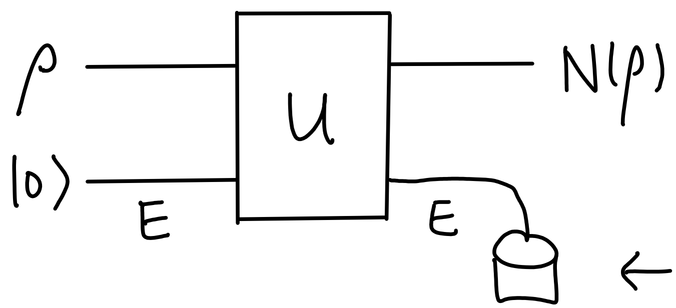


$\leftarrow$ partial trace

Proof: by examples 1-2 and composition.

Extensions: E can start in any other density matrix uncorrelated with ρ , and partial trace can be taken over a system E' of any size.

Surprise: this makes up all quantum operations!

Theorem: any quantum operation N from system A to system B can be represented as $N(\rho) = tr_E(U \rho U^\dagger)$ for some system E and some Stinespring dilation U.

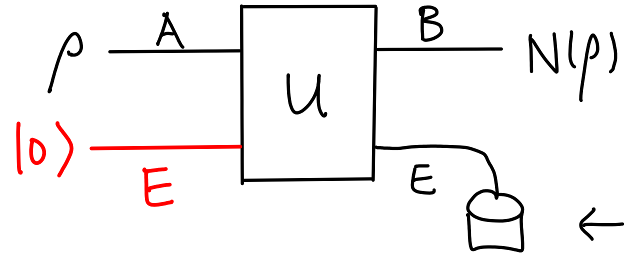
Proof: out of syllabus. For the interested, here's a write-up you already have the background to read:

arxiv.org/abs/quant-ph/0201119

Representations of quantum operations:

✓ 1. Unitary representation

$$N(\rho) = \mathrm{tr}_E(U \rho U^{+})$$



← partial trace

can skip

U is then an isometry.
U is called the Stinespring dilation for N.

Example: amplitude damping channel

We can define U by its action on a pure qubit state:

$$U\left(a|0\rangle + b|1\rangle\right)_A = a|00\rangle_{EB} + b\left(\sqrt{1-\gamma}\,|01\rangle + \sqrt{\gamma}\,|10\rangle\right)_{EB}$$

the excitation is transfered from A to E

NB A, B, E all 2-dim.

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$$U = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \\ 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

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$$U = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \\ 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

the excitation is transfered from A to E

NB A, B, E all 2-dim.

On a general density matrix $\rho = \begin{bmatrix} c & d \\ e & f \end{bmatrix}$,

$$U\rho U^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \\ 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & \sqrt{\gamma} & 0 \end{bmatrix} = \begin{pmatrix} c & \sqrt{1-\gamma}\,d & \sqrt{\gamma}\,d & 0 \\ \sqrt{1-\gamma}\,e & (1-\gamma)f & \sqrt{\gamma}\sqrt{1-\gamma}\,f & 0 \\ \sqrt{\gamma}\,e & \sqrt{\gamma}\sqrt{1-\gamma}\,f & \gamma f & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{tr}_E\, U\rho U^\dagger = \mathrm{tr}_E \begin{pmatrix} c & \sqrt{1-\gamma}\, d & \sqrt{\gamma}\, d & 0 \\ \sqrt{1-\gamma}\, e & (1-\gamma) f & \sqrt{\gamma}\sqrt{1-\gamma}\, f & 0 \\ \sqrt{\gamma}\, e & \sqrt{\gamma}\sqrt{1-\gamma}\, f & \gamma f & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} c & \sqrt{1-\gamma}\, d \\ \sqrt{1-\gamma}\, e & (1-\gamma) f \end{bmatrix} + \begin{bmatrix} \gamma f & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c + \gamma f & \sqrt{1-\gamma}\, d \\ \sqrt{1-\gamma}\, e & (1-\gamma) f \end{bmatrix}$$

$$\mathrm{tr}_E\, U\rho U^\dagger = \mathrm{tr}_E \begin{pmatrix} c & \sqrt{1-\gamma}\, d & \sqrt{\gamma}\, d & 0 \\ \sqrt{1-\gamma}\, e & (1-\gamma) f & \sqrt{\gamma}\sqrt{1-\gamma}\, f & 0 \\ \sqrt{\gamma}\, e & \sqrt{\gamma}\sqrt{1-\gamma}\, f & \gamma f & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} c & \sqrt{1-\gamma}\, d \\ \sqrt{1-\gamma}\, e & (1-\gamma) f \end{bmatrix} + \begin{bmatrix} \gamma f & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c+\gamma f & \sqrt{1-\gamma}\, d \\ \sqrt{1-\gamma}\, e & (1-\gamma) f \end{bmatrix}$$

So, the channel takes $\rho = \begin{bmatrix} c & d \\ e & f \end{bmatrix}$ to $\begin{bmatrix} c+\gamma f & \sqrt{1-\gamma}\, d \\ \sqrt{1-\gamma}\, e & (1-\gamma) f \end{bmatrix}$

A fraction γ of the (1,1) entry is moved to the (0,0) entry, and the off diagonal terms are diminished.

Representations of quantum operations:

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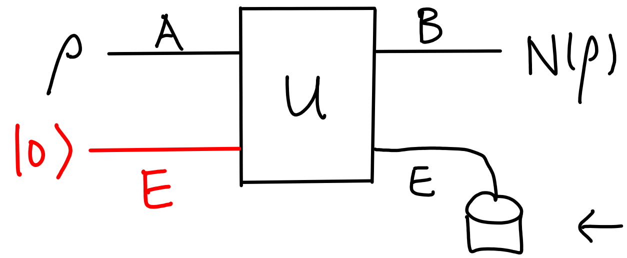
$$N(\rho) = \mathrm{tr}_E (U \rho U^{+})$$



← partial trace

can skip

U is then an isometry.
U is called the Stinespring dilation for N.

What is $\mathcal{N}(\rho)$ in terms of U?

Let $$U = \sum_{j=0}^{d_E-1}\sum_{k=0}^{d_E-1} |j\rangle\langle k|_E \otimes U_{jk} = \left[\begin{array}{c|c|c} U_{00} & U_{01} & U_{02} \;\cdots \\ \hline U_{10} & U_{11} & U_{12} \;\cdots \\ \hline U_{20} & U_{21} & U_{22} \\ \vdots & \vdots & \quad\ddots \end{array}\right]$$

E: 1st register.

can skip

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E: 1st register.

$$N(\rho) = tr_E\left(U \rho \otimes |0\rangle\langle 0|_E U^\dagger \right)$$

$$= tr_E\left(\sum_{j=0}^{d_E-1}\sum_{k=0}^{d_E-1} |j\rangle\langle k|_E \otimes U_{jk}\right)\left(|0\rangle\langle 0|_E \otimes \rho\right)\left(\sum_{j'=0}^{d_E-1}\sum_{k'=0}^{d_E-1} |k'\rangle\langle j|_E \otimes U_{j'k'}^\dagger\right)$$

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E: 1st register.

$$N(\rho) = \mathrm{tr}_E\left(U \rho \otimes |0\rangle\langle 0|_E U^\dagger\right)$$

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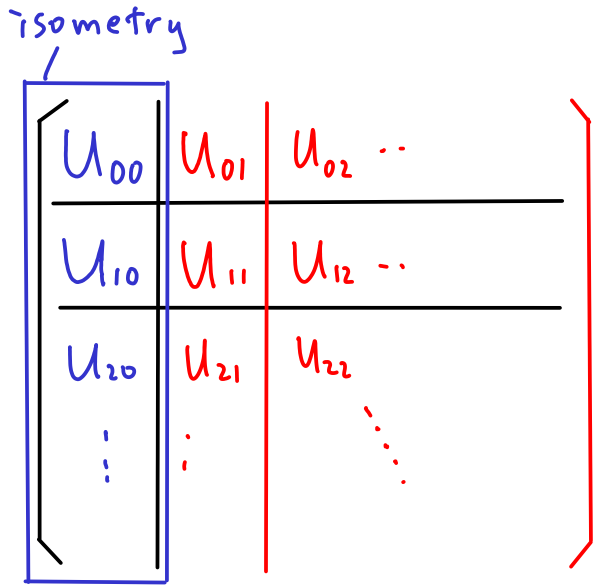
$$= \mathrm{tr}_E\underbrace{\left(\sum_{j=0}^{d_E-1} |j\rangle_E \otimes U_{j0}\right)}_{\text{isometry}}\left(\underbrace{1}_{\text{1-dim}} \otimes \rho\right)\left(\sum_{j'=0}^{d_E-1} \langle j'|_E \otimes U_{j'0}^\dagger\right)$$

can be omitted

What is $N(\rho)$ in terms of U?

Let $U = \sum_{j=0}^{d_E-1}\sum_{k=0}^{d_E-1} |j\rangle\langle k|_E \otimes U_{jk} = $

isometry



$$\begin{bmatrix} U_{00} & U_{01} & U_{02} & \cdots \\ U_{10} & U_{11} & U_{12} & \cdots \\ U_{20} & U_{21} & U_{22} & \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

E: 1st register.

$$N(\rho) = \mathrm{tr}_E\left(U\, \rho \otimes |0\rangle\langle 0|_E\, U^\dagger\right)$$

$$= \mathrm{tr}_E\left(\sum_{j=0}^{d_E-1}\sum_{k=0}^{d_E-1} |j\rangle\langle k|_E \otimes U_{jk}\right)\left(|0\rangle\langle 0|_E \otimes \rho\right)\left(\sum_{j'=0}^{d_E-1}\sum_{k'=0}^{d_E-1} |k'\rangle\langle j|_E \otimes U^\dagger_{j'k'}\right)$$

$$= \mathrm{tr}_E\Big(\underbrace{\sum_{j=0}^{d_E-1} |j\rangle_E \otimes U_{j0}}_{\text{isometry}}\Big)\Big(\underset{\text{1-dim}}{1} \otimes \rho\Big)\Big(\sum_{j'=0}^{d_E-1} \langle j'|_E \otimes U^\dagger_{j'0}\Big)$$

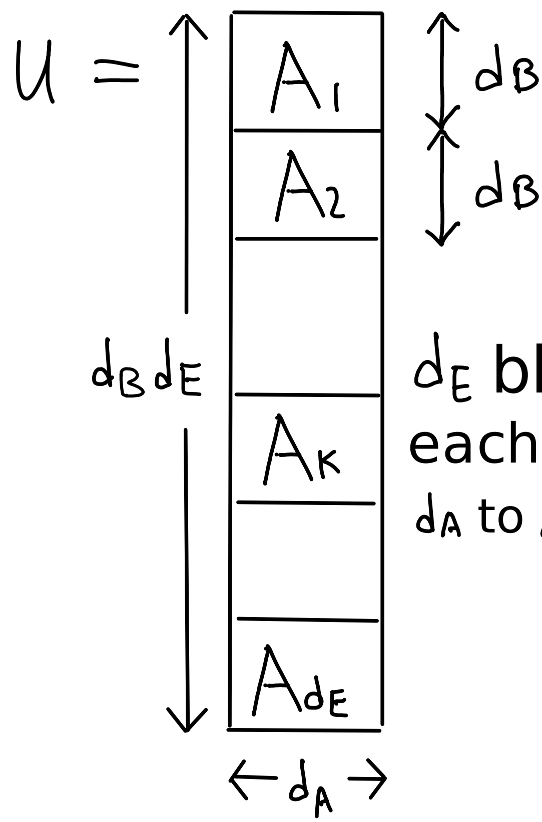
can be omitted

$$= \sum_{j=0}^{d_E-1} U_{j0}\, \rho\, U_{j0}^\dagger$$

mixture of states $\dfrac{U_{j0}\,\rho\, U_{j0}^\dagger}{\mathrm{tr}\, U_{j0}^\dagger U_{j0}\rho}$ wp $\mathrm{tr}\, U_{j0}^\dagger U_{j0} \rho$

not necessarily unitary

More generally, let U be an isometry taking system A to system BE (dims of A, B, and E are arbitrary).



d_E blocks each taking d_A to d_B dims

$$U = \sum_{k=1}^{d_E} |k\rangle_E \otimes A_k$$

Stinespring dilation, isometric extension

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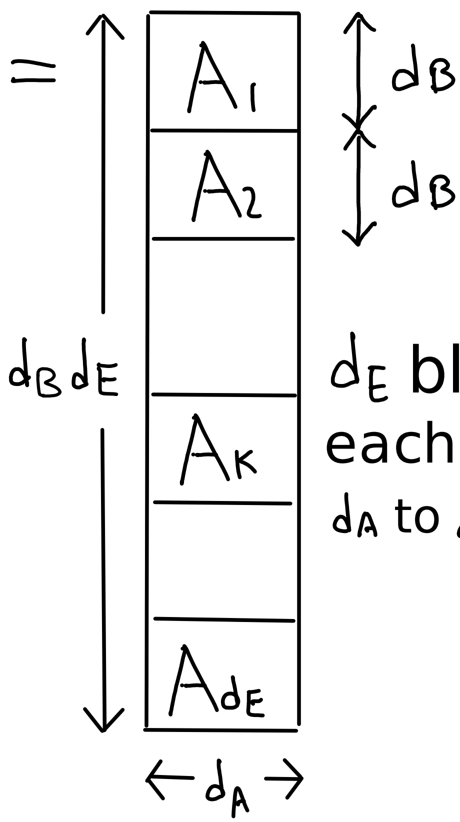
$U =$



d_E blocks each taking d_A to d_B dims

$$N(\rho) = tr_E(U \rho U^\dagger) = \sum_{k=1}^{d_E} A_k \rho A_k^\dagger$$

Kraus representation of N

A_k's : Kraus operators

not $A_k^\dagger$'s

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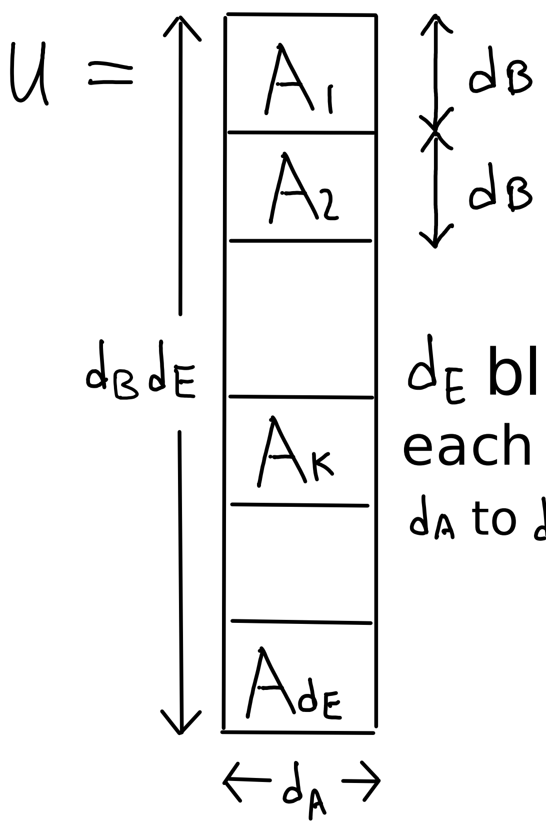

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* A map w/ Kraus representation is linear and completely positive

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$$U = \begin{array}{c} \\ d_B d_E \\ \\ \end{array} \underbrace{\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \\ \vdots \\ A_{d_E} \end{bmatrix}}_{d_A} \begin{array}{l} \} \, d_B \\ \} \, d_B \\ \\ \\ \\ \\ \end{array}$$

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Stinespring dilation, isometric extension

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Kraus representation of N

A_k's : Kraus operators

* A map w/ Kraus representation is linear and completely positive

* U isometry $\Leftrightarrow U^\dagger U = I_A$

$\Leftrightarrow \sum_{k=1}^{d_E} A_k^\dagger A_k = I_A$

$\Leftrightarrow N$ trace preserving

Example: amplitude damping channel

$$U = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \\ \hline 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \qquad A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$$

$$N(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger$$

Ex: check $A_0^\dagger A_0 + A_1^\dagger A_1 = I$

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If the initial state is $|\psi\rangle = a|0\rangle + b|1\rangle$ $\quad (\rho = |\psi\rangle\langle\psi|)$

output is the mixture of two unnormalized states:

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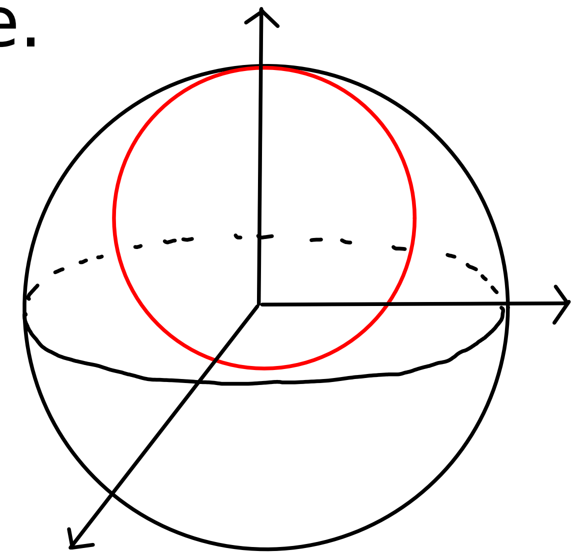
Interpretation: $|0\rangle$: ground state

$|1\rangle$: excited state

A_1 : de-excitation (with prob γ)

A_0 : no de-excitation, but diminished amplitude for $|1\rangle$

Execise: evaluate $N(\frac{1}{2}I + aX + bY + cZ)$ and find how N transform the Bloch sphere.

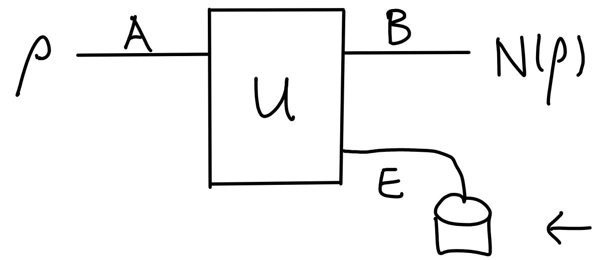


The ground state $|0\rangle\langle 0|$ is a fixed point of N.
N is not unital (taking the identity matrix to itself).

<u>Representations of quantum operations:</u>

✓ 1. Unitary representation

$$N(\rho) = tr_E(U \rho U^+)$$



← partial trace

✓ 2a. Kraus rep: $N(\rho) = \sum_{k=1}^{d_E} A_k \rho A_k^+ \, , \quad \sum_{k=1}^{d_E} A_k^+ A_k = I_A$

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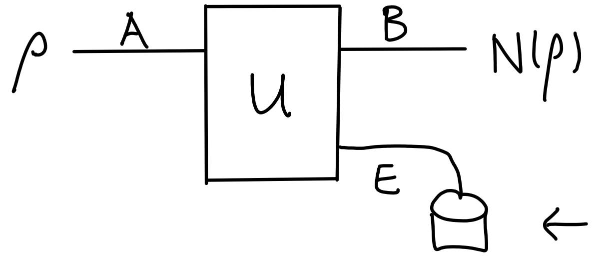

← partial trace

2a. Kraus rep: $N(\rho) = \sum_{k=1}^{d_E} A_k \rho A_k^\dagger$, $\sum_{k=1}^{d_E} A_k^\dagger A_k = I_A$

2b. Conversely, given d_E operators A_k mapping from system A to B satisfying $\sum_{k=1}^{d_E} A_k^\dagger A_k = I_A$, $U = \sum_{k=1}^{d_E} |k\rangle_E \otimes A_k$ is an isometry, and $\mathrm{tr}_E\left(U \rho U^\dagger \right) = \sum_{k=1}^{d_E} A_k \rho A_k^\dagger$.

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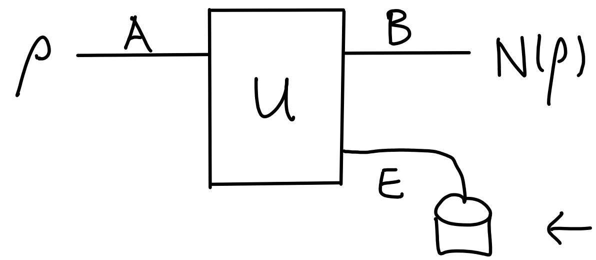

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3. $N(\rho)$ as an explicit function of ρ e.g. $\begin{bmatrix} c & d \\ e & f \end{bmatrix} \rightarrow \begin{bmatrix} c+\gamma f & \sqrt{1-\gamma}\, d \\ \sqrt{1-\gamma}\, e & (1-\gamma) f \end{bmatrix}$

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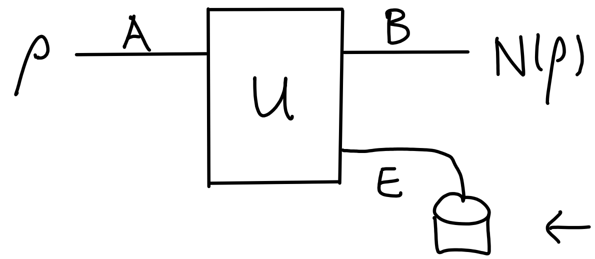

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4. Choi matrix (see arxiv.org/abs/quant-ph/0201119)

Example: qubit depolarizing channel w/ noise rate p

$A = B = \mathbb{C}^2$. $N_p(\rho) = (1-p)\rho + p\frac{I}{2}\operatorname{tr}(\rho)$.

Specified as a linear map, we do not know upfront if the map is a quantum operation. We will derive a Kraus representation which verifies that N_p is a q op.

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$A = B = \mathbb{C}^2$. $N_p(\rho) = (1-p)\rho + p\frac{I}{2}\operatorname{tr}(\rho)$.

Let $R(\rho) = \operatorname{tr}(\rho)\frac{I}{2}$, the randomization map (p=1).

Claim: $R(\rho) = \frac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z)$. (Kraus rep)

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Proof: Let $\rho = \frac{1}{2}(\mathrm{tr}(\rho) I + aX + bY + cZ)$

$X\rho X = \frac{1}{2}(\mathrm{tr}(\rho) I + aX - bY - cZ)$

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\end{aligned}\right\} \frac{\text{sum}}{4} = \mathrm{tr}(\rho)\frac{I}{2} = R(\rho)$$

NB $\mathcal{R}$ can be interpreted as an evolution in which one of the unitaries I, X, Y, Z are picked at random and applied to the input.

Example: qubit depolarizing channel w/ noise rate p

$A = B = \mathbb{C}^2$. $N_p(\rho) = (1-p)\rho + p\frac{I}{2}\operatorname{tr}(\rho)$.

Let $R(\rho) = \operatorname{tr}(\rho)\frac{I}{2}$, the randomization map.

Lemma: $R(\rho) = \frac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z)$. (Kraus rep)

$$\therefore N(\rho) = (1-p)\rho + p\frac{I}{2}\operatorname{tr}(\rho)$$

$$= (1-p)\rho + p\frac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z)$$

$$= \left(1-\frac{3p}{4}\right)\rho + \frac{p}{4}X\rho X + \frac{p}{4}Y\rho Y + \frac{p}{4}Z\rho Z$$ (Kraus rep)

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NB The qubit depolarizing channel w/ noise rate p can be interpreted as a noise process in which X, Y, and Z each happens with prob p/4, and I happens otherwise.

For a qubit density matrix $\rho = \frac{1}{2}(I + aX + bY + cZ)$

$$N(\rho) = (1-p)\rho + p\frac{I}{2}\mathrm{tr}(\rho) = (1-p)\frac{1}{2}(I + aX + bY + cZ) + p\frac{I}{2}$$

$$= \frac{1}{2}(I + (1-p)(aX + bY + cZ))$$

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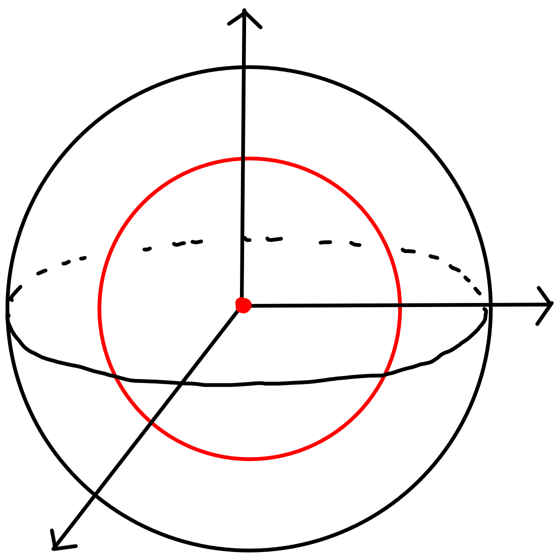

The depolarizing channel shrinks the Bloch vector by a factor of (1-p).

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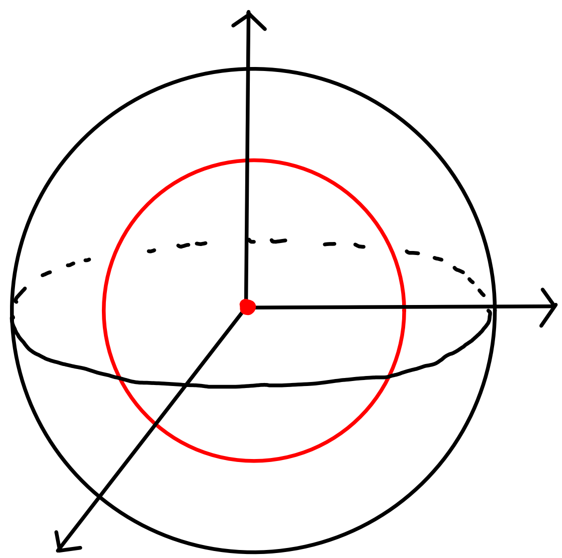

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The randomization map sends any input to the center.

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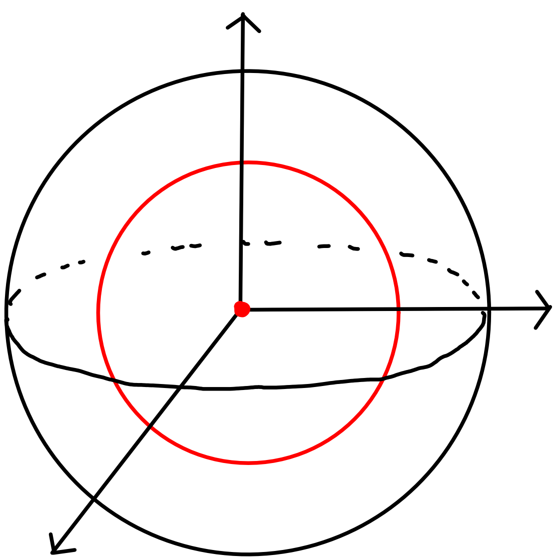
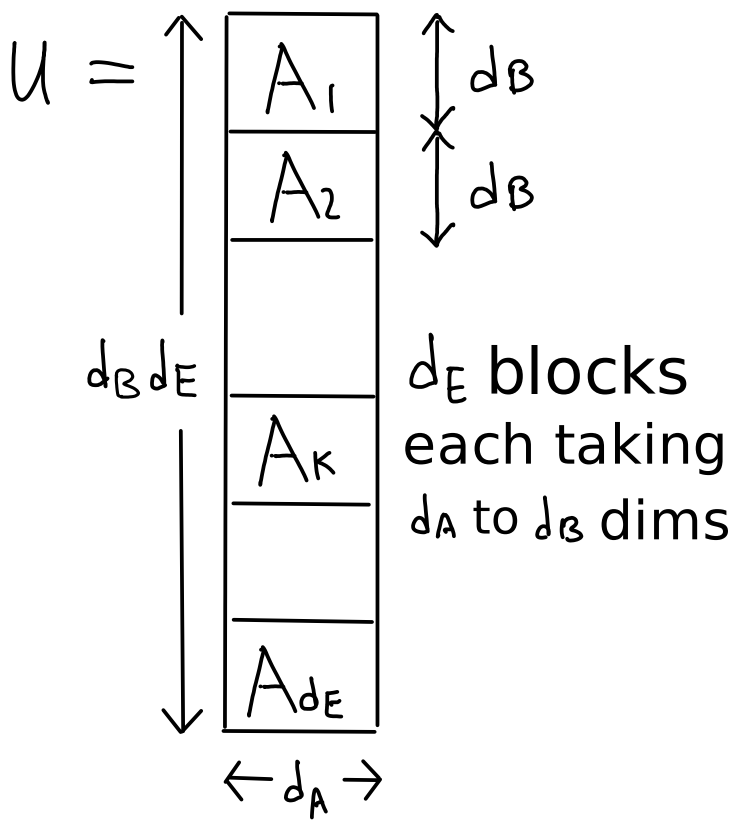


The depolarizing channel shrinks the Bloch vector by a factor of (1-p).

The randomization map sends any input to the center.

Unlike the amplitude damping channel, the depolarizing channel is unital and maps I to I.

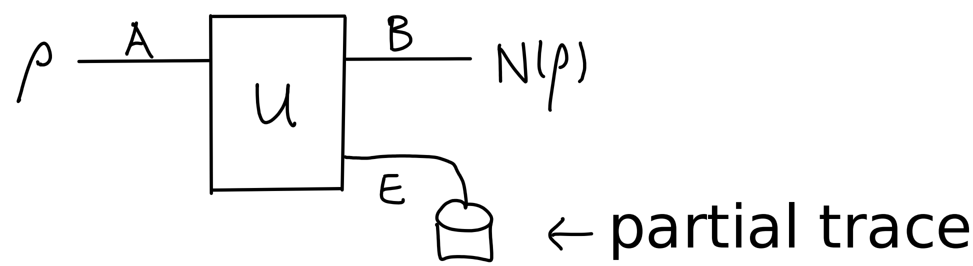
Degree of freedom in the Kraus operators:



$$U = \sum_{k=1}^{d_E} |k\rangle_E \otimes A_k$$

$$A_k = (\langle k|_E \otimes I)\, U$$

$$N(\rho) = \mathrm{tr}_E(U \rho U^\dagger) = \sum_{k=1}^{d_E} A_k \rho A_k^\dagger$$



Question: if we apply partial trace of E in a basis different from $\{|k\rangle\}$ do we:

(a) get the same map N, same Kraus operators A_k's.

(b) get the same map N but different Kraus operators

(c) get a different map ?

A4 Q2:

Detailed study of decoherence, a quantum operation that fixes the diagonal and shrinks the off-diagonal entries.

$$D\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & (1-\lambda)b \\ (1-\lambda)c & d \end{bmatrix}$$

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You will show that two different Kraus representations correspond to the same map.

One Kraus rep applies $e^{i\theta Z}$, $e^{-i\theta Z}$ at random.

One Kraus rep applies I with prob 1-p, Z with prob p.

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One Kraus rep applies I with prob 1-p, Z with prob p.

You will see a Stinespring dilation, and you have to find a change in the basis of the partial trace that transforms the second set of Kraus operators to the first.

A4 Q2

In the extreme case:

the map $\mathcal{D}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

corresponds to someone measuring the qubit.

e.g., with Stinespring dilation

$$\sqrt{a}|0\rangle + \sqrt{d}|1\rangle \longrightarrow \left(\sqrt{a}|00\rangle + \sqrt{d}|11\rangle\right)_{BE}$$

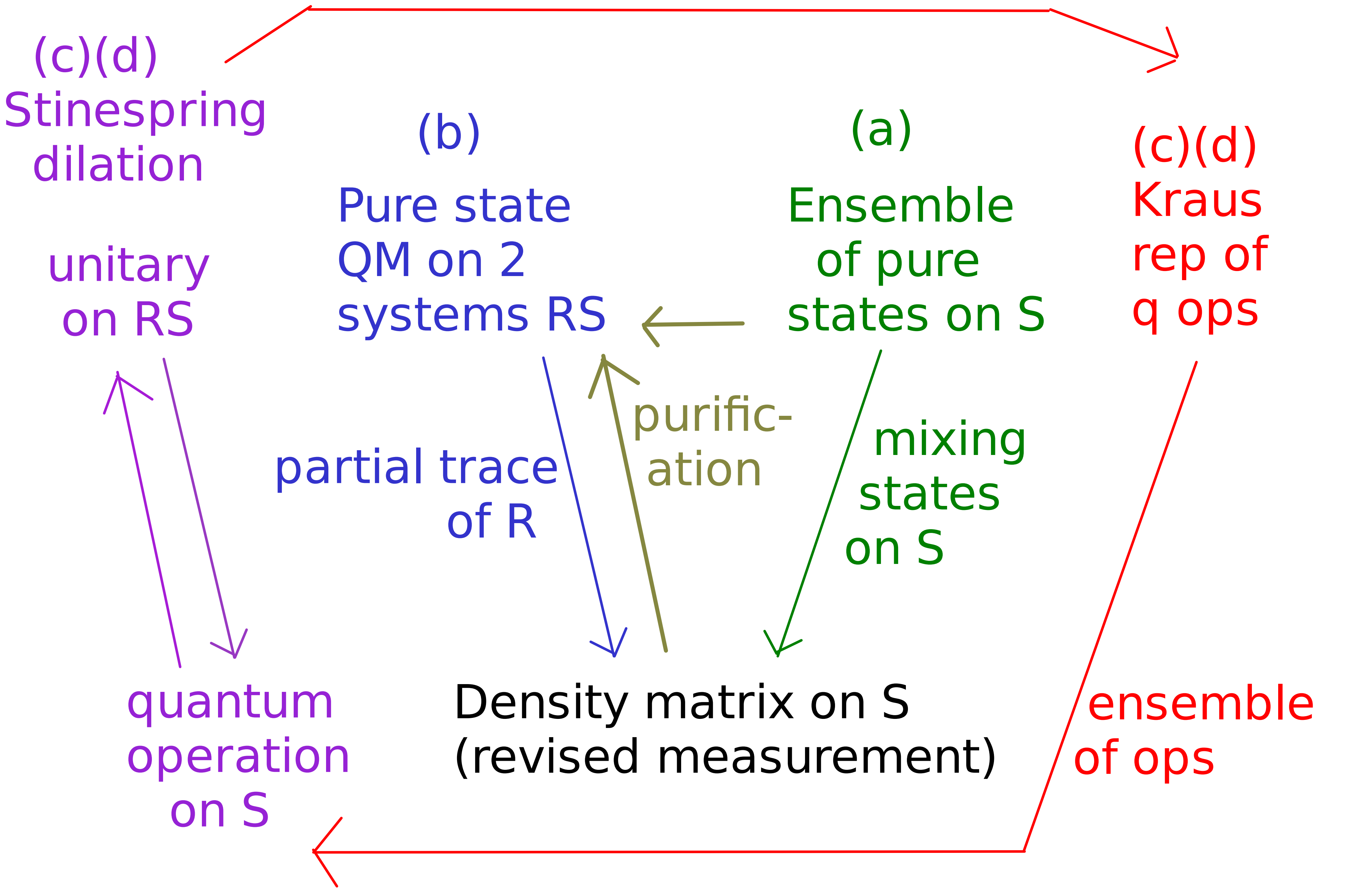
followed by partial trace of E.

Ex: check that decoherence can arise from a small probability of having the qubit measured and losing the outcome (a 3rd interpretation).

A4 Q3:

While quantum operations are not reversible in general, we characterize conditions for reversal in this question.

The question challenges your understanding of partial trace, purification, and quantum operations.


(c)(d)
Stinespring
dilation
unitary
on RS
(b)
Pure state
QM on 2
systems RS
(a)
Ensemble
of pure
states on S
(c)(d)
Kraus
rep of
q ops
partial trace
of R
purific-
ation
mixing
states
on S
quantum
operation
on S
Density matrix on S
(revised measurement)
ensemble
of ops

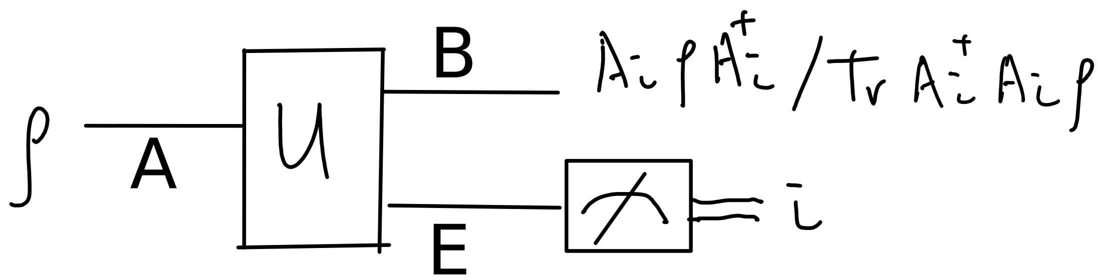
POVM measurements

Consider the following measurement on system A:

1. Apply an isometry U to system A, mapping it to systems BE.

$$U = \sum_{k=1}^{d_E} |k\rangle \otimes A_k .$$

2. Apply a complete projective measurement along the computational basis on E.



Projective measurement

$\{P_k\}$

$\sum_k P_k = I$

$\text{Prob}(k) = \text{tr}\, P_k \rho$

postmeas state

$$= \frac{P_k \rho P_k}{\text{tr}\, P_k \rho} = \frac{\sqrt{P_k}\, \rho \sqrt{P_k}}{\text{tr}\, P_k \rho}$$

POVM measurement

$E_k = A_k^\dagger A_k \geq 0$

$\sum_k E_k = I$

$\text{Prob}(k) = \text{tr}\, E_k \rho$

postmeas state

$$= \frac{A_k \rho A_k^\dagger}{\text{tr}\, E_k \rho} = \frac{\sqrt{E_k}\, \rho \sqrt{E_k}}{\text{tr}\, E_k \rho}$$

$P_k \to E_k$, orthogonality condition on $\{P_j\}$ lifted, and it is possible to have more outcomes than the dim.

NB POVM measurement on A is projective on BE.

Quick recap of mixed state quantum mechanics:

States: Density matrices: rank 1, positive semidefinite

Interpretation / characterization :
Convex combination of rank 1 projectors
Probabilistic mixture of pure states (outer products)

Transformations:

Mappings $\mathcal{N}$ from square matrices to square matrices that're linear, trace-preserving, & completely positive

Interpretations / characterizations :

1. Stinespring dilation: $\mathcal{N}(\rho) = \mathrm{tr}_2\, U \rho U^\dagger$

   Conjugate input matrix by an isometry (reversible) into matrix in 2 systems. Then, apply partial trace (irreversible) to one system.

2. Kraus representation: $\mathcal{N}(\rho) = \sum_k A_k \rho A_k^\dagger$

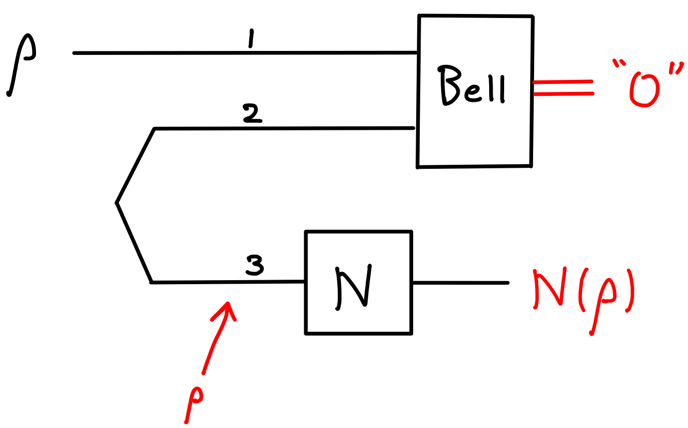
Conjugate input matrix by "Kraus operators" and sum the resulting matrices. The Kraus operators A_k need not be unitary. They satisfy: $\sum_k A_k^\dagger A_k = I$.

3. Choi representation:

Define the Choi matrix of $\mathcal{N}$ as

$$J(\mathcal{N}) = I \otimes \mathcal{N}(|v\rangle\langle v|) = \sum_{i,j} |i\rangle\langle j| \otimes \mathcal{N}(|i\rangle\langle j|)$$

where $|v\rangle = \sum_i |i\rangle_R |i\rangle_A$.



$$\mathcal{N}(\rho) = \mathrm{tr}_{12}\left(|v\rangle\langle v|_{12} \otimes I_3\right)\left(\rho_1 \otimes J(\mathcal{N})_{23}\right)$$

Ex: verify using Dirac notation

$$\mathcal{N}(\rho) = \mathrm{tr}_2\left(\rho_2^T \otimes I_3\right) J(\mathcal{N})_{23}.$$

Measurements:

Measurements described by a POVM: $\{E_k\}$

$$E_k \geq 0, \sum_k E_k = I.$$

Interpretations / characterizations :

1. Stinespring dilation:

   Conjugate input matrix by an isometry (reversible) followed by a projective measurement.

2. Kraus representation: $\mathcal{M}(\rho) = \sum_k A_k \rho A_k^\dagger \otimes |k\rangle\langle k|$

$$E_k = A_k^\dagger A_k$$

Crucial concept: partial trace

Pure state $\xrightarrow{\text{partial trace}}$ Mixed state

Fixed pure state + Unitary $\xrightarrow{\text{partial trace}}$ Quantum operation

Fixed pure state + Unitary + Projective measurement $\xrightarrow{\text{no partial trace}}$ POVM measurement