9. Combating noise: quantum error correcting codes

(NC 10.1-10.3, 10.5, M 5, KLM 10)

(a) Classical noise model

(b) 3-bit repetition code

(c) Quantum noise model

(d) Quantum 3-bit repetition code for X errors

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(g) Discretization and sufficient conditions for QECC

(h) Stabilizer formalism -- quantum parity checks !

(i) Shor 9-bit code reloaded

(j) Sufficient conditions for QECC for stabilizer codes

(l) 7-bit Steane code

(m) Erasure errors, q secret sharing, AdS/CFT corr

(a) Classical noise model

e.g.1. bit flip error: $0 \longleftrightarrow 1$

Adversarial model: at most t bit flips out of n

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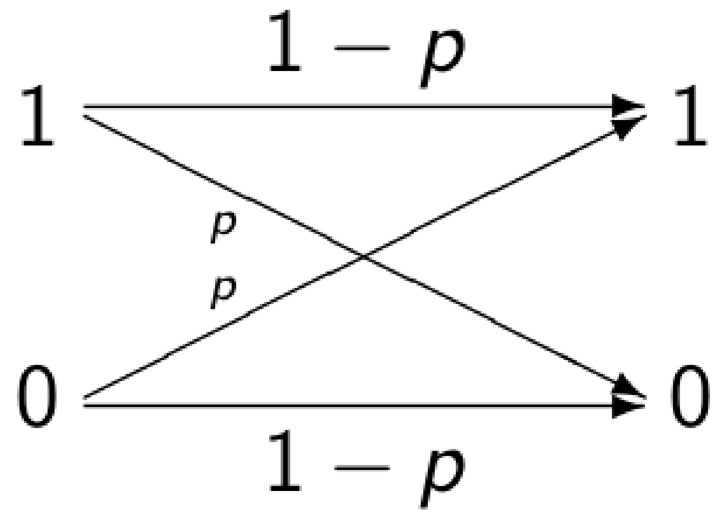
Probabilistic model: each bit flips wp p, independently

binary symmetric channel (BSC)



For large n, roughly np bit flips.

(a) Classical noise model

e.g.2. erasure error: 0,1 $\longrightarrow$ E

Adversarial model: at most t erasures out of n

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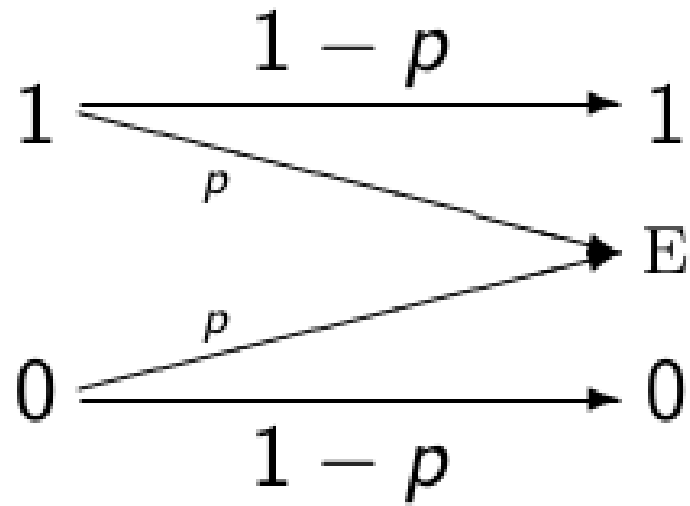
erasure channel



For large n, roughly np erasures.

(b) Classical 3-bit repetition code

logical data	code word
0	000
1	111

C0

(b) Classical 3-bit repetition code

logical data	code word	after X1
0	000	100
1	111	011
	C0	C1

(b) Classical 3-bit repetition code

logical data	code word	after X1	after X2	after X3
0	000	100	010	001
1	111	011	101	110
	C0	C1	C2	C3

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logical data	code word	after X1	after X2	after X3
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← events up to 1 error

← 4 disjoint sets

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← events up to 1 error

← 4 disjoint sets

Suppose up to 1 error occurs, resulting in $y_1 y_2 y_3$.

Determining which C_i contains $y_1 y_2 y_3$ reveals the event, and the error reverted, without learning $y_1 y_2 y_3$.

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logical data	code word	after X1	after X2	after X3
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← events up to 1 error (after X1, X2, X3)

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$$y_1 y_2 y_3 \in C_0, C_1, C_2, C_3$$

$$\Leftrightarrow \quad s_1 s_2 = 00, 10, 11, 01$$

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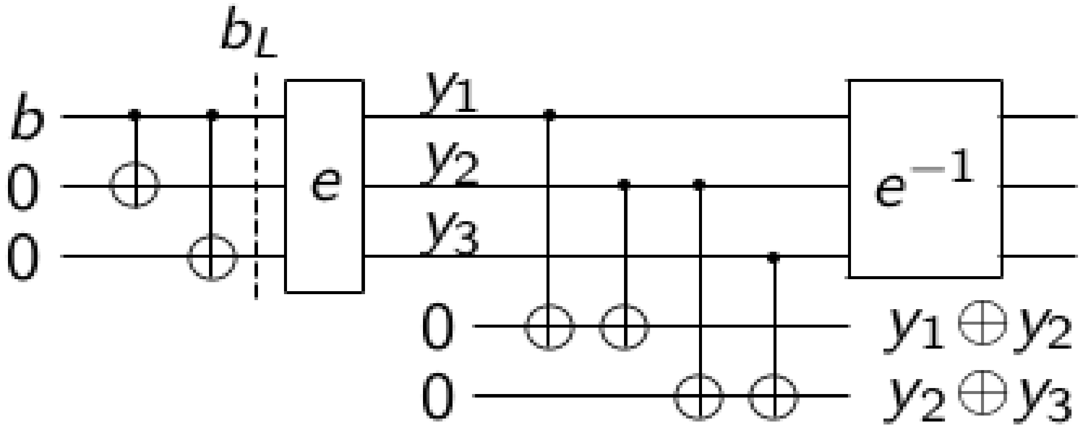
$s_1 s_2$:syndrome (singular) that identifies the error

(b) Classical 3-bit repetition code $0 \longrightarrow 000,\ 1 \longrightarrow 111$

Encoding and decoding circuits



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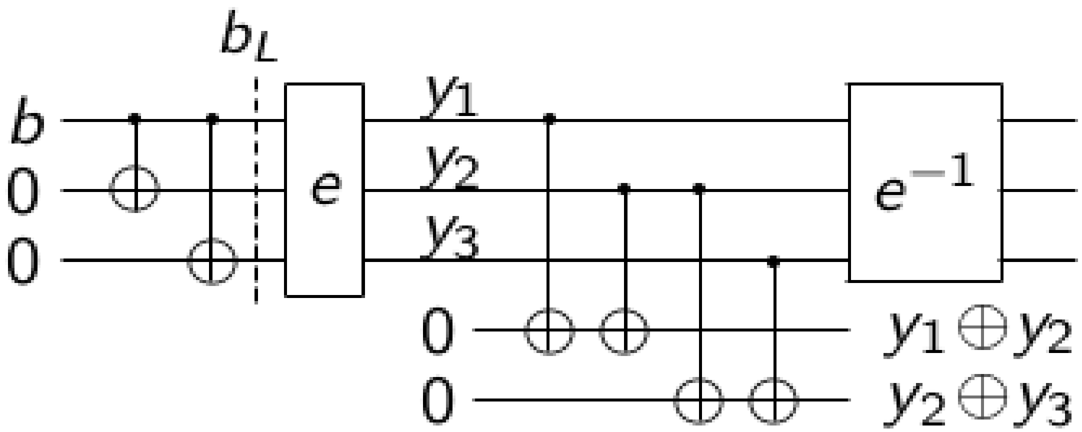

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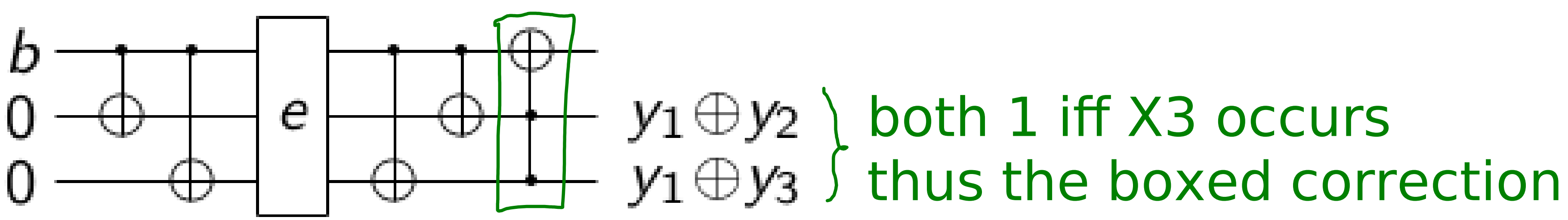

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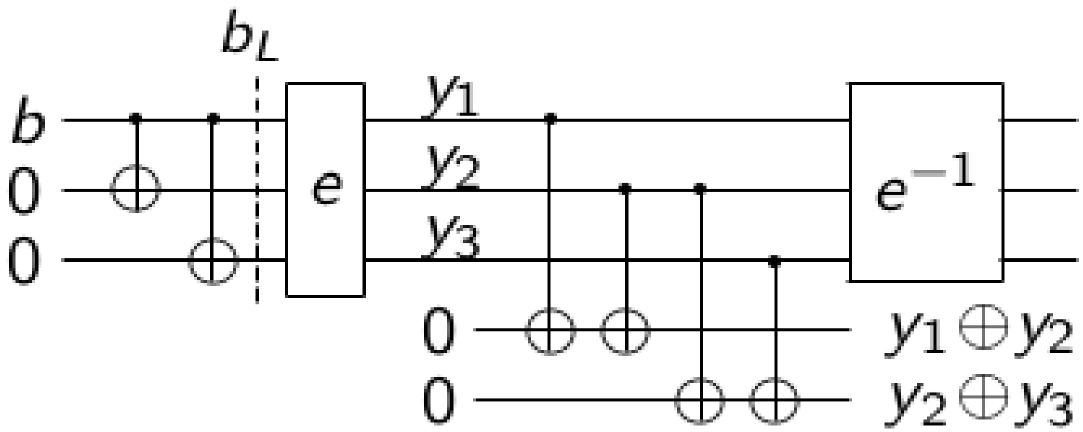

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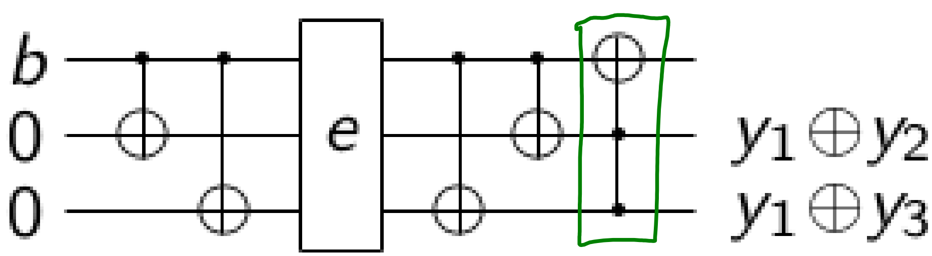

both 1 iff X3 occurs
thus the boxed correction

Corrects 1 error, or in the prob error model, reduces the error rate from p to 1-O(p^2). The "rate" is 1/3.

(c) Quantum noise model

e.g.1. X error:

$$X(a|0\rangle + b|1\rangle) = a|1\rangle + b|0\rangle$$

$$X \otimes I\,(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle) = a|10\rangle + b|11\rangle + c|00\rangle + d|01\rangle$$

$$I \otimes X\,(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle) = a|01\rangle + b|00\rangle + c|11\rangle + d|10\rangle$$

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Adversarial model: at most t X errors out of n

Kraus rep (for n=3, t=1):

$$A_x(\rho) = p_0\,\rho + p_1\, XII\,\rho\, XII + p_2\, IXI\,\rho\, IXI + p_3\, IIX\,\rho\, IIX$$

(XII = $X\otimes I\otimes I$)

where $p_{0,1,2,3} \geq 0$, $p_0 + p_1 + p_2 + p_3 = 1$, unknown otherwise.

(c) Quantum noise model

e.g.1. X error:

Reminder: if $\mathcal{E}_1(\rho) = \sum_k A_k \rho A_k^\dagger$, $\mathcal{E}_2(\sigma) = \sum_j B_j \sigma B_j^\dagger$

then $\mathcal{E}_1 \otimes \mathcal{E}_2(\eta) = \sum_k \sum_j A_k \otimes B_j \, \eta \, A_k^\dagger \otimes B_j^\dagger$

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Probabilistic model: the n qubits are evolved by $\mathcal{N}_X^{\otimes n}$

where $\mathcal{N}_X(\rho) = (1-p)\rho + p X \rho X$.

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Probabilistic model: the n qubits are evolved by $\mathcal{N}_X^{\otimes n}$

where $\mathcal{N}_X(\rho) = (1-p)\rho + p X \rho X$.

e.g., $\mathcal{N}_X^{\otimes 3}(\eta) = (1-p)^3 \rho$

$$+ (1-p)^2 p \left(X11\rho X11 + 1X1 \rho 1X1 + 11X \rho 11X \right)$$

$$+ (1-p) p^2 \left(XX1 \rho XX1 + X1X \rho X1X + 1XX \rho 1XX \right)$$

$$+ p^3 \, XXX \rho XXX$$

(b) Quantum 3-bit repetition code

logical data	code word
$a\|0\rangle$	$a\|000\rangle$
$+b\|1\rangle$	$+b\|111\rangle$
	C0

(b) Quantum 3-bit repetition code

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$a\|0\rangle$	$a\|000\rangle$	$a\|100\rangle$
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	C0	C1

(b) Quantum 3-bit repetition code

logical data	code word	after X1	after X2	after X3
$a\|0\rangle$	$a\|000\rangle$	$a\|100\rangle$	$a\|010\rangle$	$a\|001\rangle$
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	C0	C1	C2	C3

(b) Quantum 3-bit repetition code

logical data	code word	after X1	after X2	after X3	← Kraus ops
$a\|0\rangle + b\|1\rangle$	$a\|000\rangle + b\|111\rangle$	$a\|100\rangle + b\|011\rangle$	$a\|010\rangle + b\|101\rangle$	$a\|001\rangle + b\|110\rangle$	
	C0	C1	C2	C3	← 4 ortho sub spaces

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Suppose one of the above occurs. The 4 ortho sub spaces can be distinguished by a measurement and and X applied to revert the error.

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logical data	code word	after X1	after X2	after X3	← Kraus ops
$a\|0\rangle$	$a\|000\rangle$	$a\|100\rangle$	$a\|010\rangle$	$a\|001\rangle$	
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Algorithm: compute $s_1 = y_1 \oplus y_2$, $s_2 = y_2 \oplus y_3$ from $|y_1 y_2 y_3\rangle$ WITHOUT looking at any of y_1, y_2, y_3.

State in C0, C1, C2, C3 $\Leftrightarrow$ $s_1 s_2$ = 00, 10, 11, 01.

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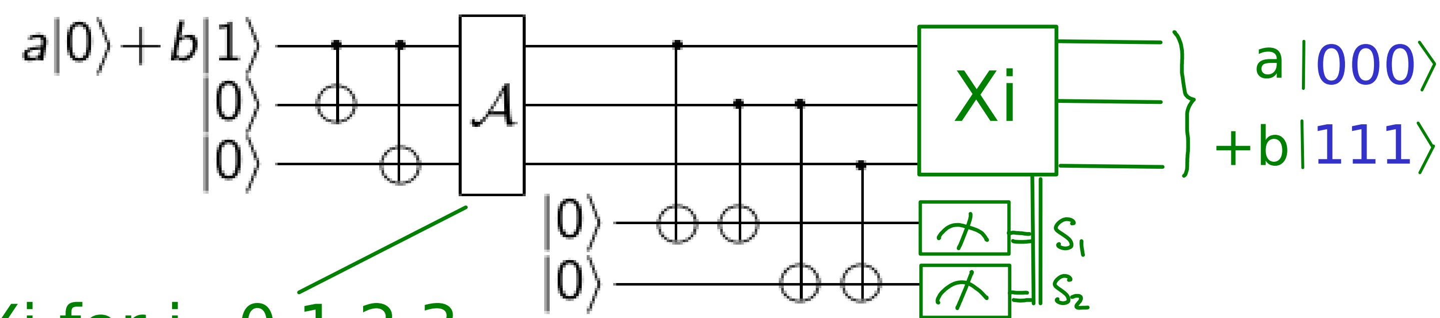

Xi for i=0,1,2,3

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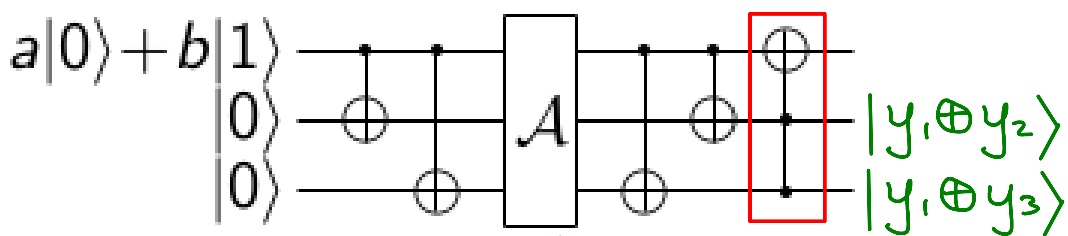
Alternative:



Xi for i=0,1,2,3

X1 iff both $y_1 \oplus y_2, y_1 \oplus y_3 = 1$

appropriate correction

Useful general observations

1. Parity measurement in terms of Pauli's

$$\text{Recall } ZZ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The even parity space, span$\{|00\rangle, |11\rangle\}$, is the $+1$ eigenspace of ZZ.

The odd parity space, span$\{|00\rangle, |11\rangle\}$, is the -1 eigenspace.

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Projector onto

even subspace: $\Pi_{+} = |00\rangle\langle 00| + |11\rangle\langle 11| = \frac{1}{2}(\mathbb{1} + ZZ)$

odd subspace: $\Pi_{-} = |01\rangle\langle 01| + |10\rangle\langle 10| = \frac{1}{2}(\mathbb{1} - ZZ)$

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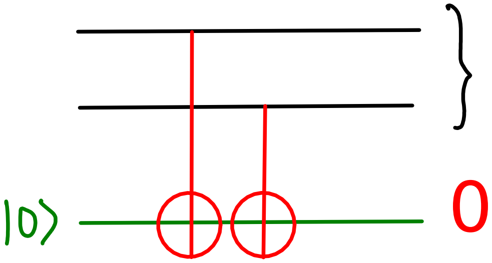
NB: if M has eigenvalues +/-1, then the projectors onto the +/-1 eigenspaces are $\frac{1}{2}(I \pm M)$.

Useful general observations

2. How to measure the eigenvalue of ZZ?



two qubits where ZZ is to be measured

ancilla

0 (if +1 eigenvalue)

1 (if -1 eigenvalue)

Useful general observations

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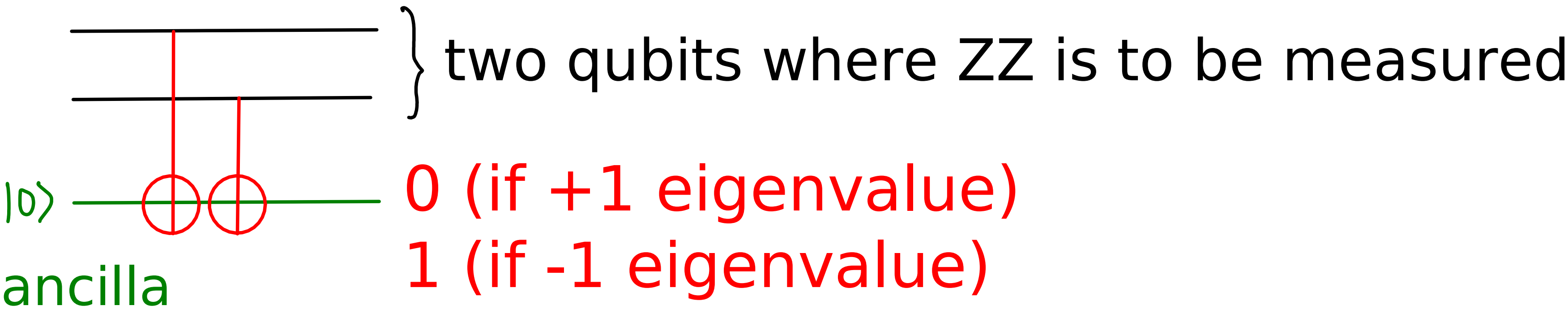
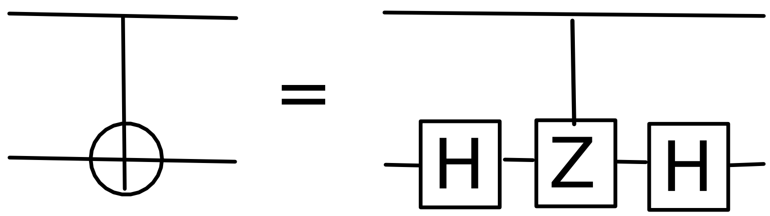


From A2:



X = HZH

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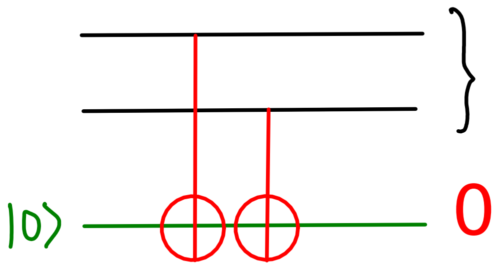

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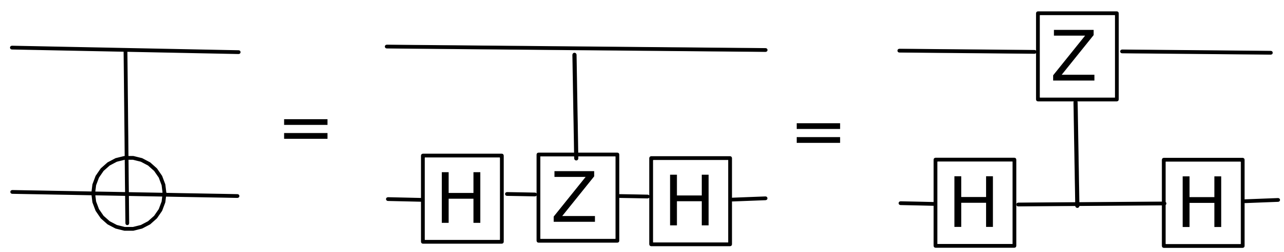

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controlled-Z symmetric

Useful general observations

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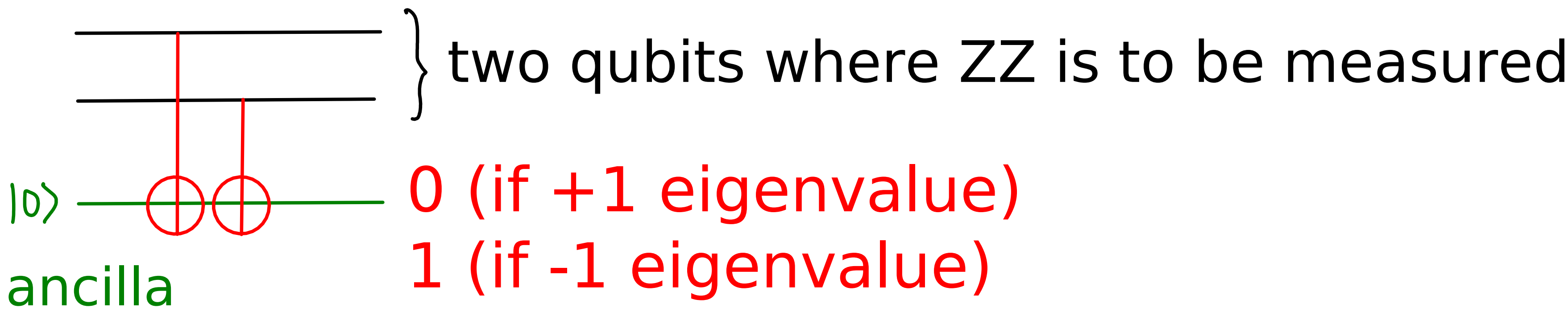

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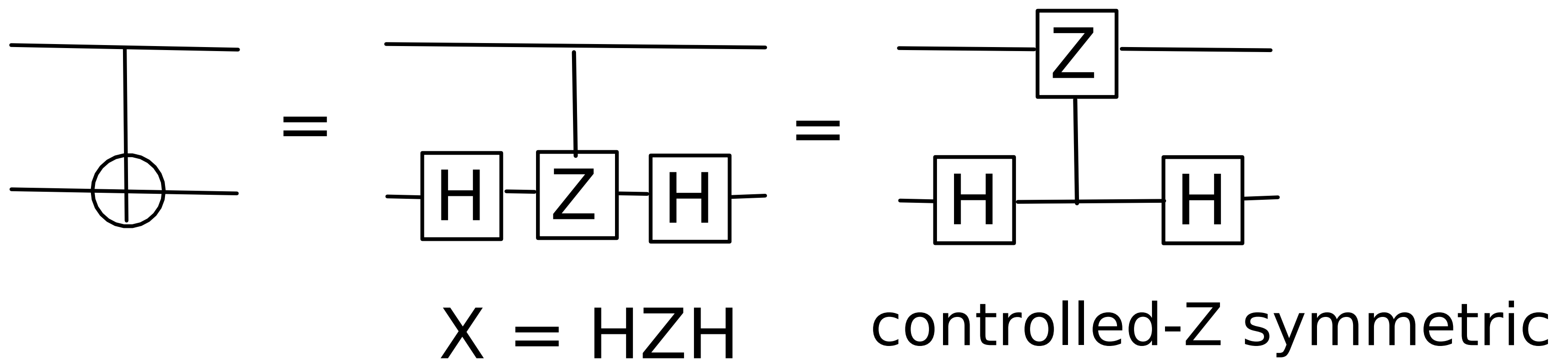

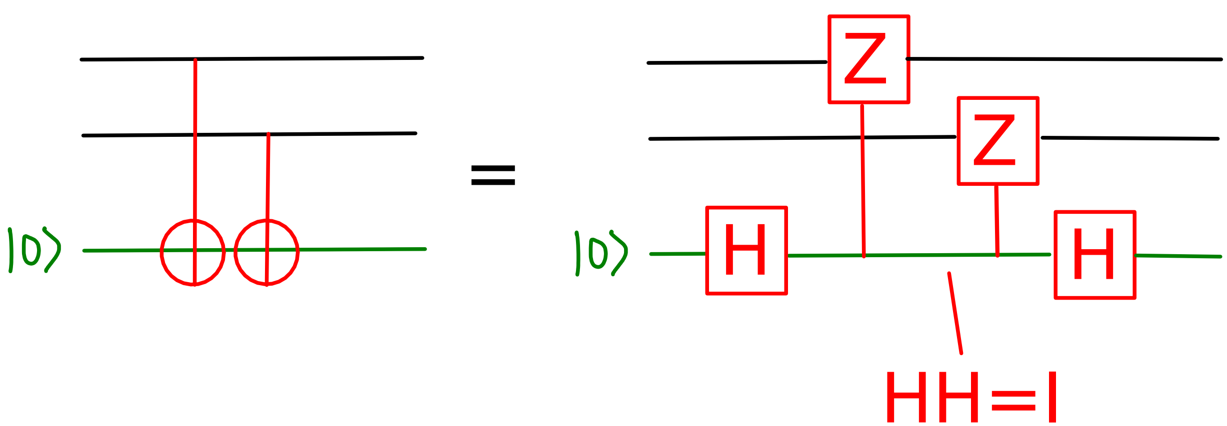

<u>Useful general observations</u>

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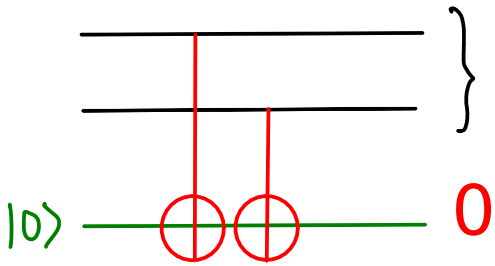

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ancilla

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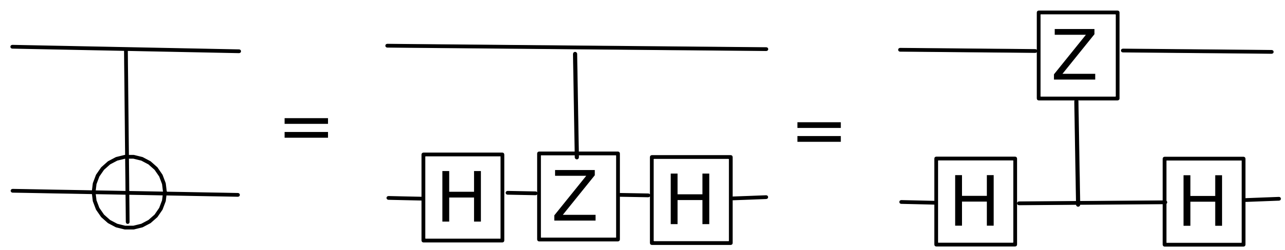

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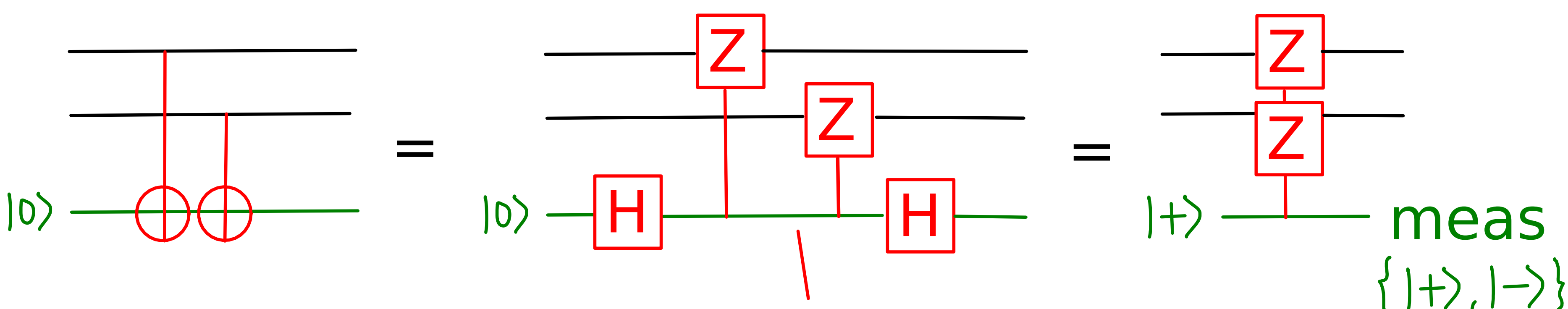

HH=I

Useful general observations

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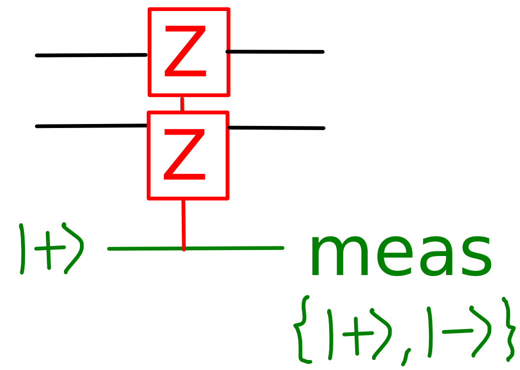
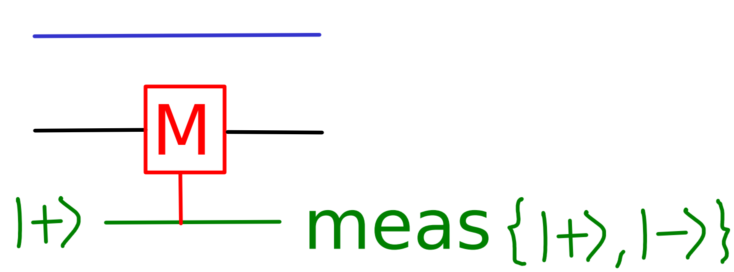


If M has eigenvalues +/-1, how to measure it?

R: system not acted on by M

S: system acted on by M



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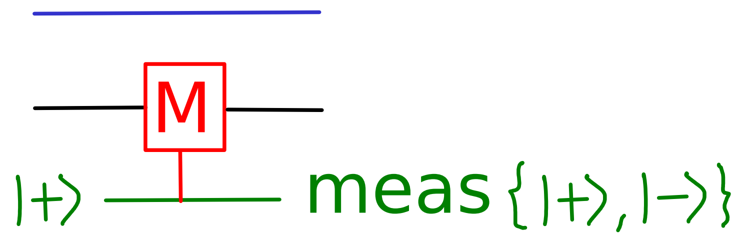

Proof: let $|\Psi\rangle_{RS}$ be the pre-measurement state.

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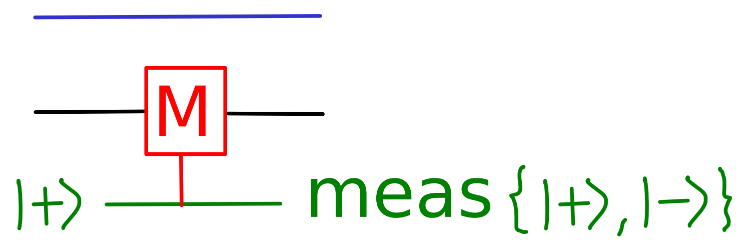

Proof: let $|\Psi\rangle_{RS}$ be the pre-measurement state.

$$|\Psi\rangle_{RS} = \sum_i \alpha_i |\eta_i\rangle_R |e_i\rangle_S + \sum_j \beta_j |\xi_j\rangle_R |f_j\rangle_S$$

$|e_i\rangle_S$: +1 eigenvectors of M

$|f_j\rangle_S$: -1 eigenvectors of M

the union form a basis of S

Useful general observations

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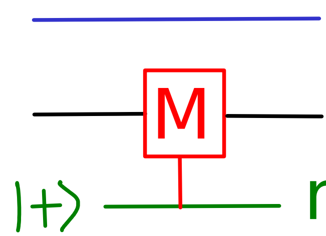


meas $\{|+\rangle, |-\rangle\}$

Proof: let $|\Psi\rangle_{RS}$ be the pre-measurement state.

$$|\Psi\rangle_{RS} = \sum_i \alpha_i |\eta_i\rangle_R |e_i\rangle_S + \sum_j \beta_j |\xi_j\rangle_R |f_j\rangle_S$$

+1 eigenvectors of M (the $|e_i\rangle_S$)

-1 eigenvectors of M (the $|f_j\rangle_S$)

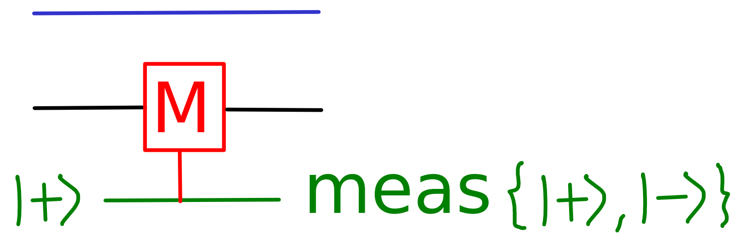
$$|\Psi\rangle_{RS} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \longrightarrow \frac{1}{\sqrt{2}}\left(|\Psi\rangle_{RS}|0\rangle + M|\Psi\rangle_{RS}|1\rangle\right)$$

Useful general observations

If M has eigenvalues +/-1, how to measure it?

R: system not acted on by M

S: system acted on by M



Proof: let $|\Psi\rangle_{RS}$ be the pre-measurement state.

$$|\Psi\rangle_{RS} = \sum_i \alpha_i |\eta_i\rangle_R |e_i\rangle_S + \sum_j \beta_j |\xi_j\rangle_R |f_j\rangle_S$$

+1 eigenvectors of M (the $|e_i\rangle_S$) — -1 eigenvectors of M (the $|f_j\rangle_S$)

$$|\Psi\rangle_{RS} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \longrightarrow \frac{1}{\sqrt{2}}\left(|\Psi\rangle_{RS}|0\rangle + M|\Psi\rangle_{RS}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\sum_i \alpha_i |\eta_i\rangle_R |e_i\rangle_S + \sum_j \beta_j |\xi_j\rangle_R |f_j\rangle_S\right)|0\rangle$$

$$+ \frac{1}{\sqrt{2}}\left(\sum_i \alpha_i |\eta_i\rangle_R |e_i\rangle_S - \sum_j \beta_j |\xi_j\rangle_R |f_j\rangle_S\right)|1\rangle$$

Useful general observations

If M has eigenvalues +/-1, how to measure it?

R: system not acted on by M

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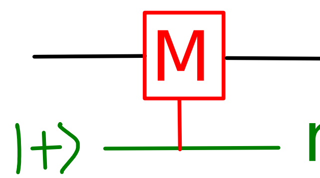

meas $\{|+\rangle, |-\rangle\}$

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$$= \sum_i \alpha_i |\eta_i\rangle_R |e_i\rangle_S |+\rangle + \sum_j \beta_j |\xi_j\rangle_R |f_j\rangle_S |-\rangle$$

So, measuring the ancilla in the $\{|+\rangle, |-\rangle\}$ basis projects the joint state the same way as a projector onto the +/- 1 eigenspaces of M. $\square$

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3. If a set of unitary errors E_i's take the codespace to orthogonal subspaces, then, there is a measurement to determine which subspace (thus which error has occurred), and the error can be reverted.

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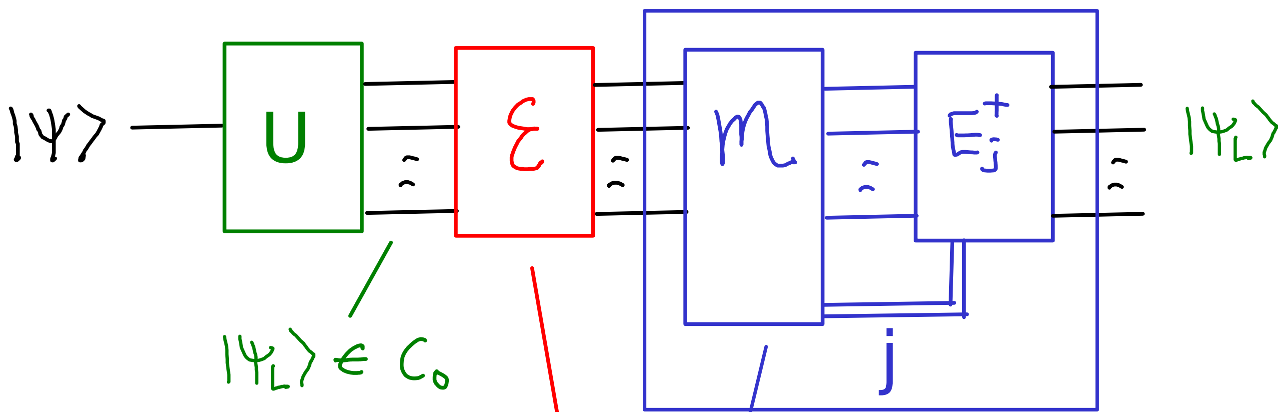

measuring which $E_j C_0$

$$\mathcal{E}(\sigma) = \sum_i E_i \sigma E_i^\dagger,$$

$E_i C_0$'s mutually orthogonal

Useful general observations

4. Discretization of continuous set of errors

If the error is neither I nor X, but $e^{-i\theta X} \otimes I \otimes I$

where θ is arbitrary.

iClicker question: can the quantum 3-bit code correct this error? (a) yes, (b) no

Useful general observations

4. Discretization of continuous set of errors

For any arbitrary unknown θ, $e^{-i\theta X} \otimes I \otimes I$ can be corrected by the 3-bit code.

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Idea: $e^{-i\theta X} \otimes I \otimes I \, (a|000\rangle + b|111\rangle)$

$= (\cos\theta \, I - i\sin\theta \, X) \otimes I \otimes I \, (a|000\rangle + b|111\rangle)$

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$$= \cos\theta I \otimes I \otimes I \, (a|000\rangle + b|111\rangle)$$
$$- i\sin\theta X \otimes I \otimes I \, (a|000\rangle + b|111\rangle)$$

The syndrome measurement collapses the above to one term, turning the error into an I or an X, and we revert I or X (without knowing θ).

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$$= \cos\theta \, I \otimes I \otimes I \, (a|000\rangle + b|111\rangle) - i\sin\theta \, X \otimes I \otimes I \, (a|000\rangle + b|111\rangle)$$

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The continuous parameter is transformed from being part of "what error" to part of the "prob of error."

We will extend these general observations later ...

First we extend the 3-bit code.

What about Z errors?

What happens if we have unknown Z errors instead? (These can be very harmful, say, during computation with phase kick-back.)

For example, we can have some adversarial noise process:

$$\mathcal{A}_z(\rho) = p_0\rho + p_1 ZII\rho ZII + p_2 IZI\rho IZI + p_3 IIZ\rho IIZ$$

with unknown probabilities $p_{0,1,2,3}$, or probabilistic noise $\mathcal{N}^{\otimes n}$ where $\mathcal{N}(\rho) = (1-p)\rho + pZ\rho Z$.

There are two (related) methods to derive an error correcting code for Z errors.

- by analogy with X errors
  pick codeword such that error spaces are orthogonal
- by "modifying the noise"

Both methods give the same QECC.

Quantum 3-bit repetition code for Z errors

By analogy with QECC for X errors:

Encoding: $|0\rangle \rightarrow |0_L\rangle = |+\rangle^{\otimes 3}$, $|1\rangle \rightarrow |1_L\rangle = |-\rangle^{\otimes 3}$

$|\psi\rangle = a|0\rangle + b|1\rangle \rightarrow |\psi_L\rangle = a|0_L\rangle + b|1_L\rangle = a|+\rangle^{\otimes 3} + b|-\rangle^{\otimes 3}$.

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Noise process: The 4 errors, III, ZII, IZI, IIZ (A_k's) take $|\psi_L\rangle$ to:

$a|{+}{+}{+}\rangle + b|{-}{-}{-}\rangle$, $a|{-}{+}{+}\rangle + b|{+}{-}{-}\rangle$, $a|{+}{-}{+}\rangle + b|{-}{+}{-}\rangle$, $a|{+}{+}{-}\rangle + b|{-}{-}{+}\rangle$

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Decoding: apply syndrome measurement with projectors (B_j's)

$$M_\emptyset = |{+}{+}{+}\rangle\langle{+}{+}{+}| + |{-}{-}{-}\rangle\langle{-}{-}{-}|$$
$$M_1 = |{-}{+}{+}\rangle\langle{-}{+}{+}| + |{+}{-}{-}\rangle\langle{+}{-}{-}|$$
$$M_2 = |{+}{-}{+}\rangle\langle{+}{-}{+}| + |{-}{+}{-}\rangle\langle{-}{+}{-}|$$
$$M_3 = |{+}{+}{-}\rangle\langle{+}{+}{-}| + |{-}{-}{+}\rangle\langle{-}{-}{+}|$$

The syndrome measurement can be implemented as two binary measurements, projecting onto the ± 1 eigenspaces of XXI, IXX.

Quantum 3-bit repetition code for Z errors

Recall from A2 that $HZH = X$ where H is the Hadamard gate.

QECC by modifying the noise: If we apply H before and after the noise process, the Z errors are turned into X errors!

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The evolution by $H^{\otimes 3}$, followed by $\mathcal{A}_z$, followed by $H^{\otimes 3}$ is:

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$$= p_0\rho + p_1 XII\rho XII + p_2 IXI\rho IXI + p_3 IIX\rho IIX = \mathcal{A}_x(\rho)$$

Quantum 3-bit repetition code for Z errors

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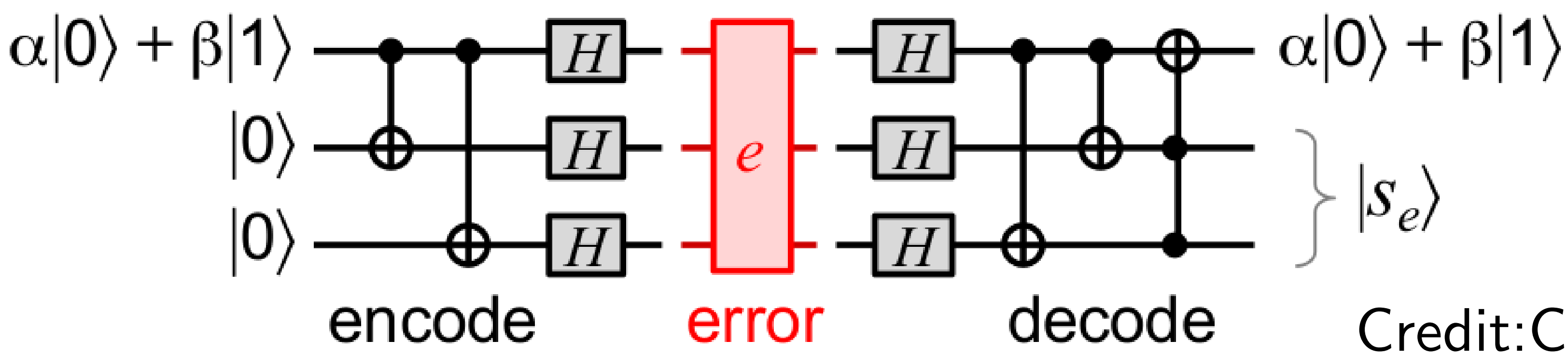
So, if we encode $a|0\rangle + b|1\rangle \to a|000\rangle + b|111\rangle$, apply $H^{\otimes 3}$, let $\mathcal{A}_z$ (the Z errors) occur, apply $H^{\otimes 3}$, decode for the 3-qubit X error code (measure ZZI, IZZ), we correct for the error!



Credit:Cleve

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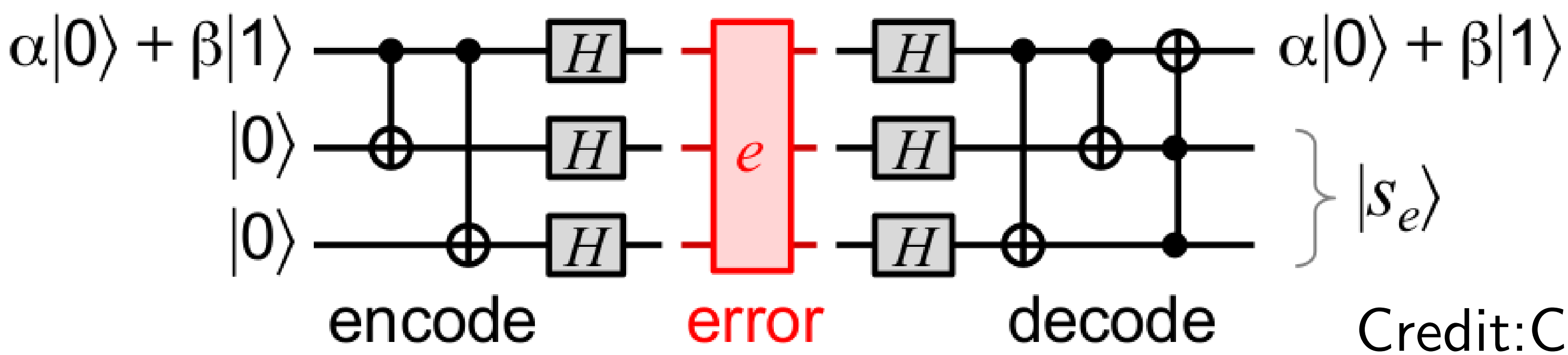

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But the encoder for the X-error correcting code followed by $H^{\otimes 3}$ is the encoder for the Z-error correcting code $a|0\rangle + b|1\rangle \rightarrow a|{+}{+}{+}\rangle + |{-}{-}{-}\rangle$, the $H^{\otimes 3}$ followed by ZZI, IZZ eigen-space measurements measures the eigenspace of XXI, IXX.

The 9-bit Shor code for one X, Y, or Z error

Protect both X and Z errors simultaneously?
Consider the encoding:

$$|0\rangle \rightarrow |0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
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for a "concatenated code":
Step 1: $a|0\rangle + b|1\rangle \to a|{+}{+}{+}\rangle + b|{-}{-}{-}\rangle$ (the Z error code)
Step 2: each $|\pm\rangle \to |000\rangle \pm |111\rangle$ (the X error code)

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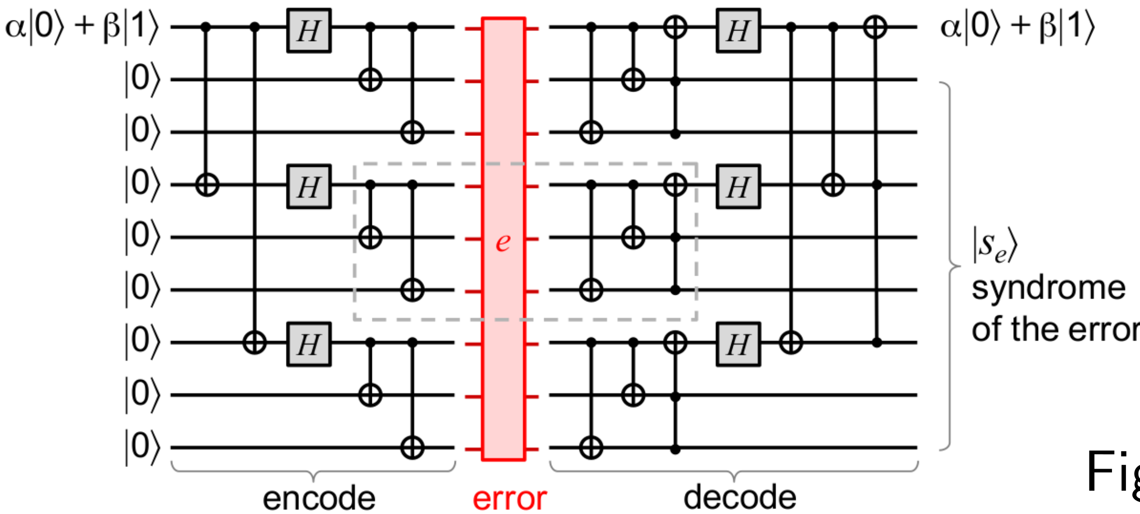

Figure credit:Cleve

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Goal: identify and correct any of: I, $X_{1,2,\cdots,9}$, $Z_{1,2,\cdots,9}$,
where I acts on 9 qubits, X_i, Z_i are X, Z errors on the ith qubit.

e.g. $X_7|0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|100\rangle + |011\rangle)$
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For X errors:
Each block is in a 3-qubit X error code; checking parities in each block identifies which of I and $X_{1,2,\cdots,9}$ has happened.

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i.e., we measure the ±1 eigenvalues of
$ZZIIIIIII$, $IZZIIIIII$, $IIIZZIIII$, $IIIIZZIII$, $IIIIIIZZI$, $IIIIIIIZZ$.

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e.g., the syndrome $+++-++$ means $++$, $+-$, $++$ for the 1st, 2nd, and 3rd block, so, 2nd block has error, which is X_6.

iClicker question:
e.g., which error if the syndrome is $++++-+$? (a) X_5, (b) X_7.

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e.g., which error if the syndrome is $+ + + + -+$? (a) X_5, (b) X_7.

Note syndrome measurement does not measure the code space.
Ex: check I and $X_{1,\cdots,9}$ have different syndromes.

The 9-bit Shor code for one X, Y, or Z error

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$$|1\rangle \rightarrow |1_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

For Z errors:
a Z error on any one qubit within a block acts identically on the codespace. e.g., for $k = 1, 2, 3$:

$$Z_k|0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
$$Z_k|1_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

The 9-bit Shor code for one X, Y, or Z error

$$|0\rangle \rightarrow |0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
$$|1\rangle \rightarrow |1_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

For Z errors:
a Z error on any one qubit within a block acts identically on the codespace. e.g., for $k = 1, 2, 3$:

$$Z_k|0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
$$Z_k|1_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

Interestingly, we don't need to distinguish which of Z_1, Z_2, Z_3 has occured. All 3 can be reversed by applying Z_1 :)

Similarly, $Z_{4,5,6}$ act identically on the codespace, and separately, $Z_{7,8,9}$ act identically on the codespace.

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$$|1\rangle \rightarrow |1_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

For Z errors: we want to discriminate the 4 possibilities:
no Z error, or 1 Z in block $1, 2$ or 3.

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For Z errors: we want to discriminate the 4 possibilities: no Z error, or 1 Z in block $1, 2$ or 3.

Method 1: transform $|000\rangle \pm |111\rangle$ to $|\pm\rangle$ (using CNOTs from qubit 1 to 2, and from qubit 1 to 3, remove the $|0\rangle$ states in qubits 2, 3 afterwards, and repeat for all blocks). Then we measure XXI and IXX of the remaining 3 qubits. (e.g., see the circuit).

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For Z errors: we want to discriminate the 4 possibilities:
no Z error, or 1 Z in block $1, 2$ or 3.

Method 2: $|000\rangle \pm |111\rangle$ is a ± 1 eigenstate of XXX. The codespace C_0 is a simultaneous $+1$ eigenspace of $XXXXXXIII$ and $IIIXXXXXX$. A Z error on the 1st or 2nd block turns C_0 to a -1 eigenspace of $XXXXXXIII$ and a Z error on the 2nd or 3rd block turns C_0 to a -1 eigenspace of $IIIXXXXXX$

The 9-bit Shor code for one X, Y, or Z error

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Method 2: $|000\rangle \pm |111\rangle$ is a ± 1 eigenstate of XXX. The codespace C_0 is a simultaneous $+1$ eigenspace of $XXXXXXIII$ and $IIIXXXXXX$. A Z error on the 1st or 2nd block turns C_0 to a -1 eigenspace of $XXXXXXIII$ and a Z error on the 2nd or 3rd block turns C_0 to a -1 eigenspace of $IIIXXXXXX$

The syndromes $++$, $-+$, $--$, $+-$ identifies no Z error, Z error in the 1st, 2nd, and 3rd block resp.

The 9-bit Shor code for one X, Y, or Z error

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We saw how to identify an X error using measurements of $ZZIIIIIII$, $IZZIIIIII$, $IIIZZIIII$, $IIIIZZIII$, $IIIIIIZZI$, $IIIIIIIZZ$, and how to identify a Z error using measurements of $XXXXXXIII$, $IIIXXXXXX$.

If we measure all 8 operators, can we identity an error chosen from I, X_k, and Z_k for $k = 1, \cdots, 9$?

e.g. $X_7|0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|100\rangle + |011\rangle)$

$X_7|1_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|100\rangle - |011\rangle)$

The 9-bit Shor code for one X, Y, or Z error

$$|0\rangle \to |0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
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$$X_7|1_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|100\rangle - |011\rangle)$$

Still $+1$ eigenstates of $XXXXXXIII$, $IIIXXXXXX$!

The 9-bit Shor code for one X, Y, or Z error

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We saw how to identify an X error using measurements of $ZZIIIIIII$, $IZZIIIIII$, $IIIZZIIII$, $IIIIZZIII$, $IIIIIIZZI$, $IIIIIIIZZ$, and how to identify a Z error using measurements of $XXXXXXIII$, $IIIXXXXXX$.

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We saw how to identify an X error using measurements of $ZZIIIIIII$, $IZZIIIIII$, $IIIZZIIII$, $IIIIZZIII$, $IIIIIIZZI$, $IIIIIIIZZ$, and how to identify a Z error using measurements of $XXXXXXIII$, $IIIXXXXXX$.

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$$Z_7|0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)$$
$$Z_7|1_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)$$

Still $+$ eigenstates of $ZZIIIIIII$, $IZZIIIIII$, $IIIZZIIII$, $IIIIZZIII$, $IIIIIIZZI$, $IIIIIIIZZ$.

The 9-bit Shor code for one X, Y, or Z error

$$|0\rangle \to |0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
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We saw how to identify an X error using measurements of $ZZIIIIIII$, $IZZIIIIII$, $IIIZZIIII$, $IIIIZZIII$, $IIIIIIZZI$, $IIIIIIIZZ$, and how to identify a Z error using measurements of $XXXXXXIII$, $IIIXXXXXX$.

If we measure all 8 operators, can we identity an error chosen from I, X_k, and Z_k for $k = 1, \cdots, 9$?

The X errors have trivial last 2 bits of syndrome, and the Z errors have trivial first 6 bits of syndrome. So the identifications of the X and Z errors are independent when measuring these 8 operators!

The 9-bit Shor code for one X, Y, or Z error

$$|0\rangle \to |0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
$$|1\rangle \to |1_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

Theorem: if one of I or $X_{1,2,\cdots,9}$, $Z_{1,2,\cdots,9}$, $Y_{1,2,\cdots,9}$ occurs to $a|0_L\rangle + b|1_L\rangle$, we can recover the state.

The 9-bit Shor code for one X, Y, or Z error

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Proof: from the 8-tuple of outcomes (each $\pm$) when measuring $ZZIIIIIII$, $IZZIIIIII$, $IIIZZIIII$, $IIIIZZIII$, $IIIIIIZZI$, $IIIIIIIZZ$, $XXXXXXIII$, $IIIXXXXXX$, determine (a) if the first 6 bits are all $+$'s, and (b) if the last 2 bits are all $+$'s.

The 9-bit Shor code for one X, Y, or Z error

$$|0\rangle \rightarrow |0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
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Proof: from the 8-tuple of outcomes (each $\pm$) when measuring $ZZIIIIIII$, $IZZIIIIII$, $IIIZZIIII$, $IIIIZZIII$, $IIIIIIZZI$, $IIIIIIIZZ$, $XXXXXXIII$, $IIIXXXXXX$, determine (a) if the first 6 bits are all $+$'s, and (b) if the last 2 bits are all $+$'s.

- Yes for both (a), (b): I occurred.

The 9-bit Shor code for one X, Y, or Z error

$$|0\rangle \to |0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
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Proof: from the 8-tuple of outcomes (each $\pm$) when measuring $ZZIIIIIII$, $IZZIIIIII$, $IIIZZIIII$, $IIIIZZIII$, $IIIIIIZZI$, $IIIIIIIZZ$, $XXXXXXIII$, $IIIXXXXXX$, determine (a) if the first 6 bits are all $+$'s, and (b) if the last 2 bits are all $+$'s.

- Yes for both (a), (b): I occurred.
- Yes for (a), no for (b): Z error occurred, determine which Z_k.

The 9-bit Shor code for one X, Y, or Z error

$$|0\rangle \to |0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
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Theorem: if one of I or $X_{1,2,\cdots,9}$, $Z_{1,2,\cdots,9}$, $Y_{1,2,\cdots,9}$ occurs to $a|0_L\rangle + b|1_L\rangle$, we can recover the state.

Proof: from the 8-tuple of outcomes (each $\pm$) when measuring $ZZIIIIIII$, $IZZIIIIII$, $IIIZZIIII$, $IIIIZZIII$, $IIIIIIZZI$, $IIIIIIIZZ$, $XXXXXXIII$, $IIIXXXXXX$, determine (a) if the first 6 bits are all $+$'s, and (b) if the last 2 bits are all $+$'s.

- Yes for both (a), (b): I occurred.
- Yes for (a), no for (b): Z error occurred, determine which Z_k.
- No for (a), yes for (b): X error occurred, determine which X_k.

The 9-bit Shor code for one X, Y, or Z error

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Theorem: if one of I or $X_{1,2,\cdots,9}$, $Z_{1,2,\cdots,9}$, $Y_{1,2,\cdots,9}$ occurs to $a|0_L\rangle + b|1_L\rangle$, we can recover the state.

Proof: from the 8-tuple of outcomes (each $\pm$) when measuring $ZZIIIIIII$, $IZZIIIIII$, $IIIZZIIII$, $IIIIZZIII$, $IIIIIIZZI$, $IIIIIIIZZ$, $XXXXXXIII$, $IIIXXXXXX$, determine (a) if the first 6 bits are all $+$'s, and (b) if the last 2 bits are all $+$'s.

- Yes for both (a), (b): I occurred.
- Yes for (a), no for (b): Z error occurred, determine which Z_k.
- No for (a), yes for (b): X error occurred, determine which X_k.
- No for both (a), (b): a $Y = iZX$ error must have occurred. Find which X_k, then check if Z_k consistent with last 2 bits of syndrome.

The 9-bit Shor code for one X, Y, or Z error

$$|0\rangle \to |0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
$$|1\rangle \to |1_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

Theorem: if one of I or $X_{1,2,\cdots,9}$, $Z_{1,2,\cdots,9}$, $Y_{1,2,\cdots,9}$ occurs to $a|0_L\rangle + b|1_L\rangle$, we can recover the state.

Proof (ctd): after determining which of I, X_k, Z_k, Y_k has occurred, revert the error.

The 9-bit Shor code for one X, Y, or Z error

$$|0\rangle \rightarrow |0_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
$$|1\rangle \rightarrow |1_L\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

iClicker question: if one of I or $X_{1,2,\cdots,9}$, $Z_{1,2,\cdots,9}$, $Y_{1,2,\cdots,9}$ occurs to $a|0_L\rangle + b|1_L\rangle$, and the measurement outcome for $ZZIIIIIII$, $IZZIIIIII$, $IIIZZIIII$, $IIIIZZIII$, $IIIIIIZZI$, $IIIIIIIZZ$, $XXXXXXIII$, $IIIXXXXXX$ is $+-++++-+$, what is the error?

(a) X_2, (b) X_3, (c) Z_8, (d) Y_3, (e) Y_7