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(NC 10.5-10.6)

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Fault-tolerant quantum computation

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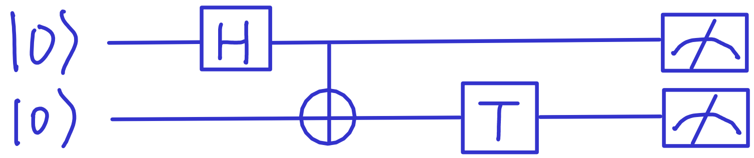
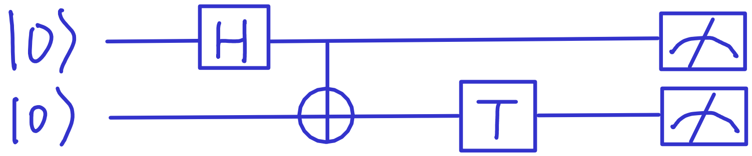
Goal of FTQC:

Given a perfect circuit of the above form, design a new circuit using noisy initial states, gates, & measurement that gives similar final measurement outcomes.

Our approach: concatenation of QECC



Wanted: circuit C0 (level 0)



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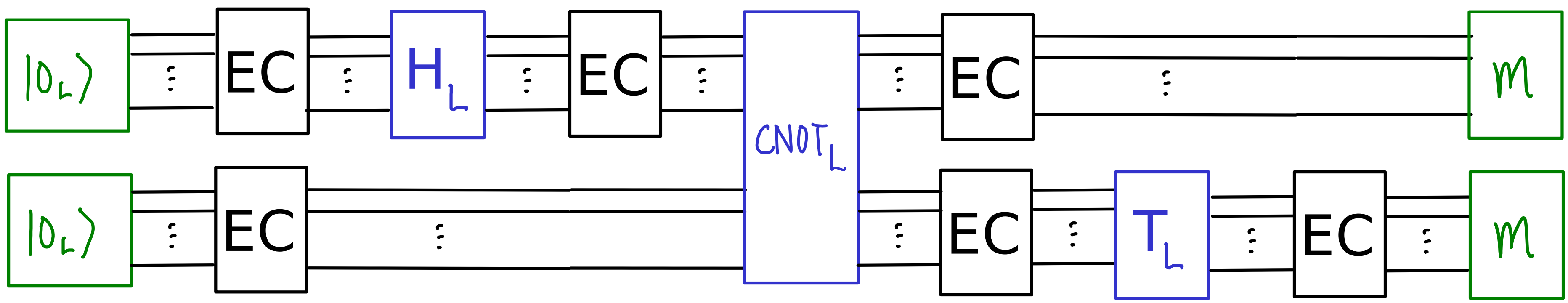

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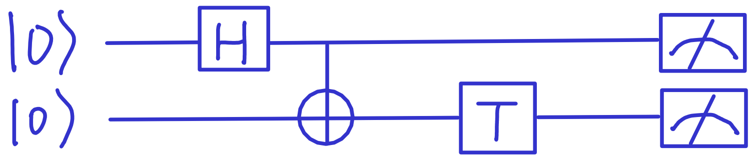

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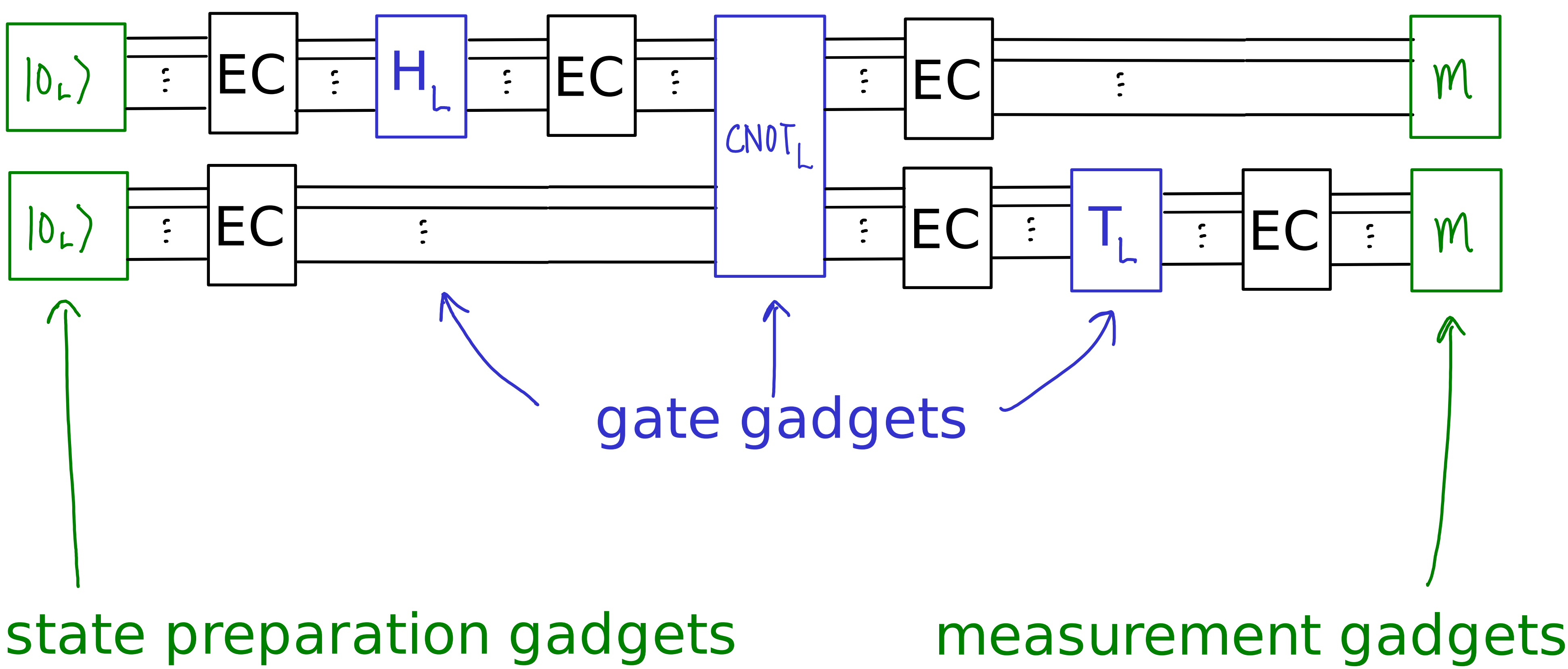
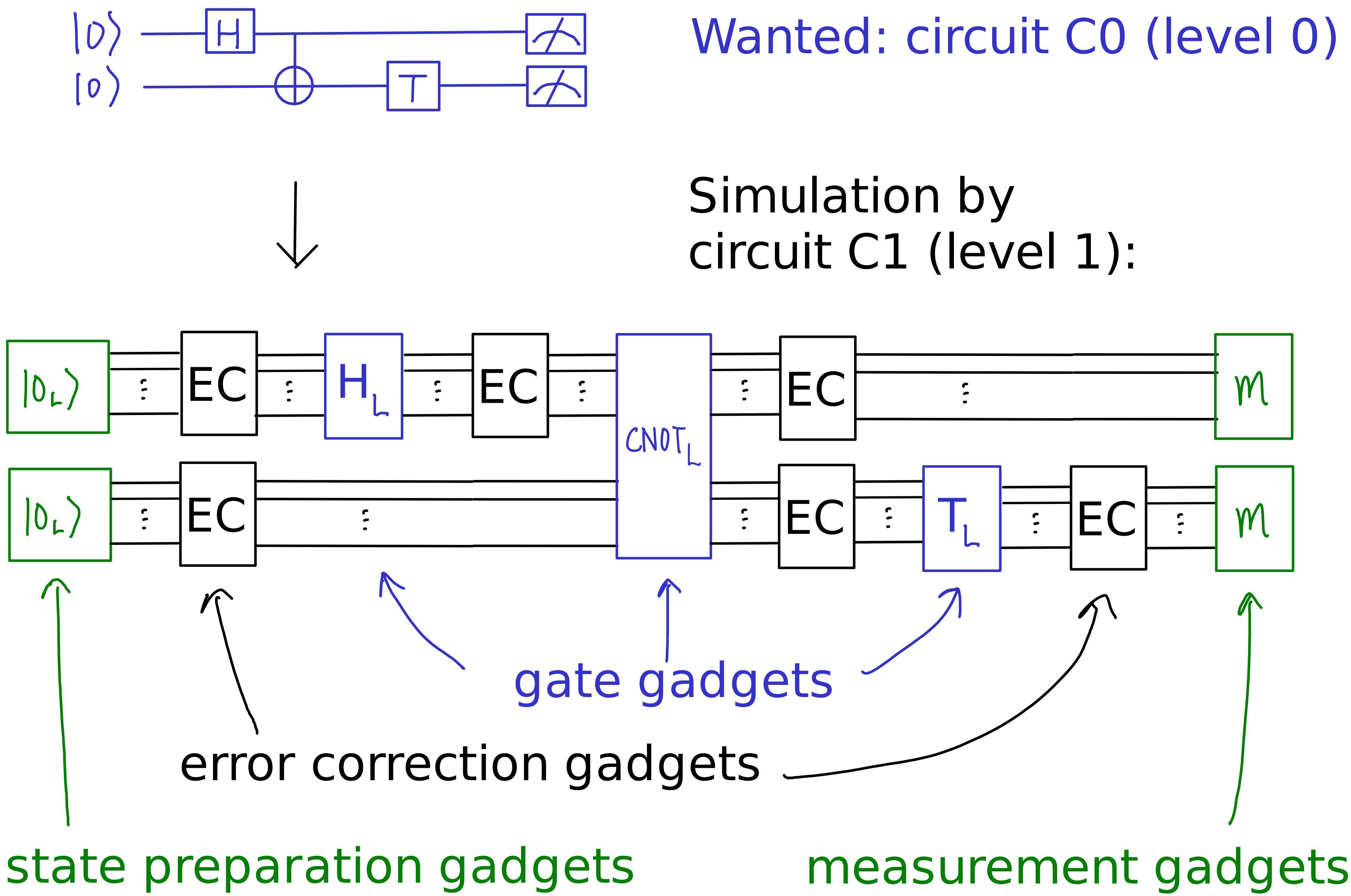
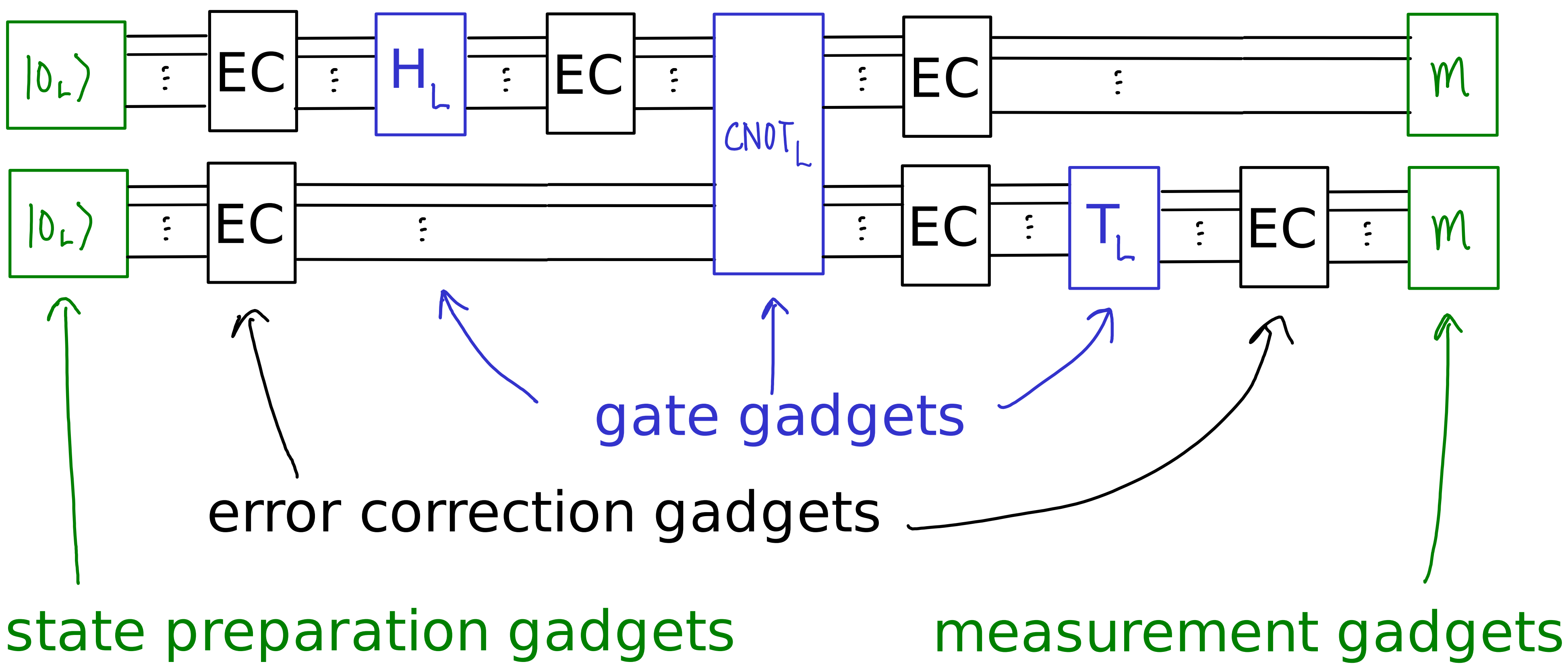




Idea: for very low noise, QECC is still useful ...

Gadget: circuit of noisy components (state prep, gates, storage, measurements); each fails with some prob.



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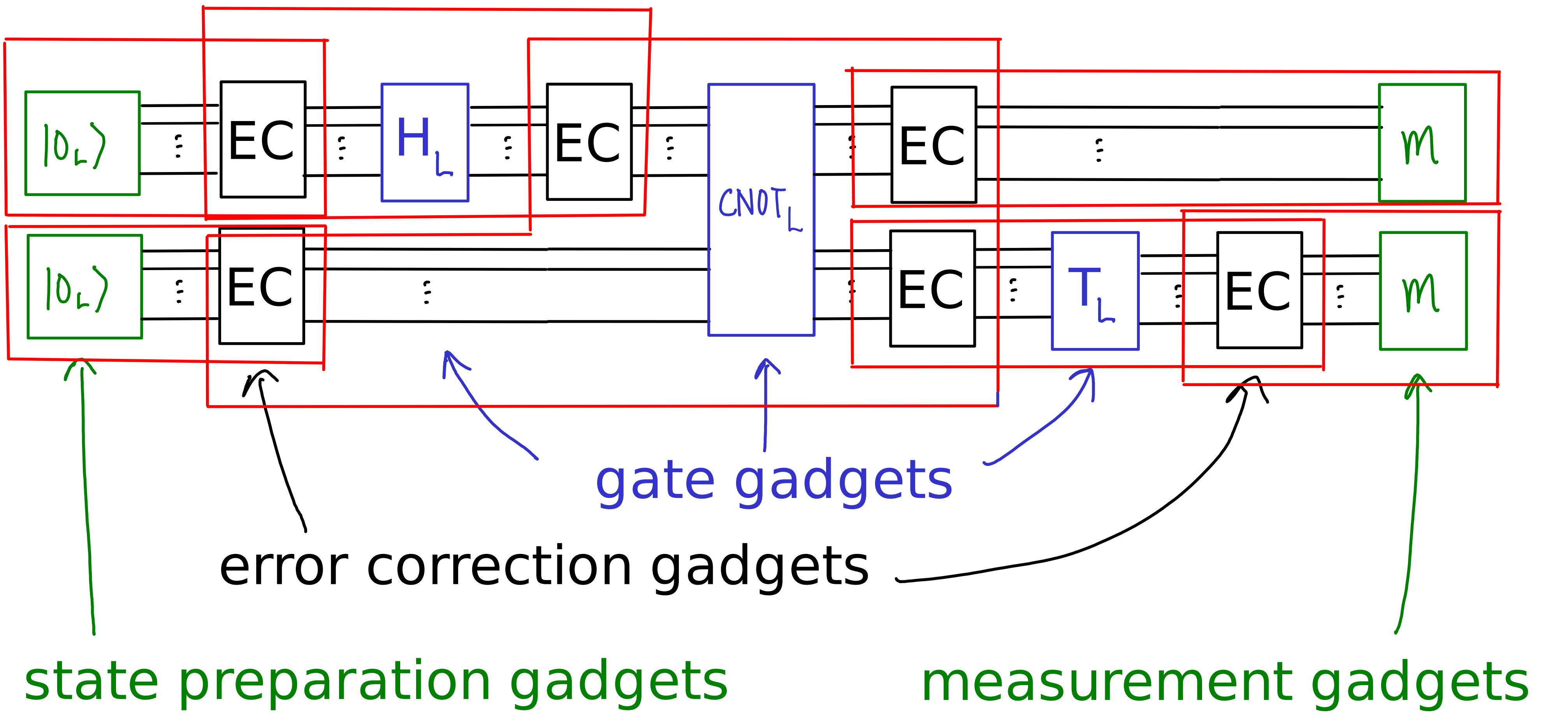
Gadget: circuit of noisy components (state prep, gates, storage, measurements); each fails with some prob.

Goal: each region (rectangle) in C1 simulates a circuit component (state prep, ..., meas) in C0 (level 0) "correctly", unless there are at least 2 faults.



Simulation by circuit C1 (level 1)

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Each component in C2 is physical, fails with prob p. Each component of C1 (level-1) is simulated in C2 "correctly" unless 2 or more faults appear in the simulating region. So, simulated C1 (level-1) components is faulty with prob $O(p^2)$.

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The simulated C1 in turns simulates C0 (level-0); each C0 component now fails with prob $O((p^2)^2)$.

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The recursive circuit simulation with property 2 above is called a "fault-tolerant" simulation of a circuit.

1. How does the error change with # levels?

2. How does the circuit size change with # levels?

3. How to achieve arbitrarily accurate simulation without arbitrarily accurate physical components?

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Physical error rate is $p_0 = p$

With 1 level, error rate for simulating C0 components

$p_1 \leq \binom{s}{2} p_0^2$ where s = size of the largest region

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Note: $r_1 = \frac{p_1}{p^*} \leq \frac{\binom{s}{2} p_0^2}{p^*} = \left(\frac{p_0}{p^*}\right)^2 = r_0^2$.

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The error decreases doubly exponentially with k!

$$p_k = r_k p^* \leq r_0^{2^k} p^* = \left(\frac{p}{p^*}\right)^{2^k} p^* .$$

2. How big is a level-k circuit?

For an original circuit C0 of size N (width * depth, roughly # components), an overall error of simulating C0 to be $\leq \varepsilon$, set component-wise error rate $< \varepsilon/N$.

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$$2^k \log r_0 \approx \log \varepsilon/N$$

log

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$$k \approx \log\left(\frac{\log N/\varepsilon}{\log 1/r_0}\right)$$

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Good news (complexity): polynomial speed up by QC (such as Grover's algorithm) is roughly preserved.

Not so good news (practical): overhead is large.

For example: encoding 1 qubit in 7,

- N = 10000 (small circuit)
- e = 1% (reasonable accuracy)
- S = size of largest region, dominated by EC gadget

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Overhead: $$\left(\frac{\log N/\varepsilon}{\log 1/r_0}\right)^{\log S} \approx \left(\frac{20}{\log 1/r_0}\right)^{8.8} \approx \begin{cases} 3 \times 10^{11} & \text{if } r_0 = \frac{1}{2} \\ 7 \times 10^{6} & \text{if } r_0 = 1/10 \end{cases}$$

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Meanwhile, a not so gentle reminder:
Present expts: 20-160 qubits, error: 0.1-5%
(These numbers did not even exist around year 2000, when the threshold result was proved, progress!!)

We should be optimistic, but only very cautiously.

2019 writting
Use other screenshots for this part of the discussion.
The qubit count and error have not changed too much!

The threshold theorem: if p < p*, then, we can simulate any circuit (C0) of size N with arbitrarily small error ε by a circuit (Ck) of size $N \left(\frac{\log N/\varepsilon}{\log 1/r_0} \right)^{\log S}$,

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- each component err independently
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Goal: design these gadgets, with S as small as possible.

W25: start topic 10 here ... return to first half of this set of slides later.

Fault-tolerant quantum computation

QECC : assume ancilla preparation, encoding, syndrome measurement, and decoding are perfect

Can QECC still work if these operations are imperfect?

To quantum-compute more & more reliably, do we need higher & higher precision in our elementary operations?

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Intuition: if noise is low enough, QECC "kind of work" so, we take additional care to not spread errors ...

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Holy grail:

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Elaborating what is good enough:

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Principles for Fault Tolerant QC:

(1) Use a discrete universal set of gates at each level

A slight deviation from an intended gate should lead to an illegitimate gate if we were to detect the deviation.

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We use the universal set of gates {H,T,CNOT}.

We add $R = T^2$ to the set for reasons that'll be clear later.

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Cannot correct error on unencoded quantum data. Also, an error during encoding of an unknown quantum state into a QECC can introduce a logical error that can never be caught.

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NB: threshold theorem only considers circuits with no unknown quantum input / output. The simulation takes the descriptipn of a quantum circuit C0, and outputs measurement outcomes. Size of C0 is arbitrary, but it has to be known before the simulation.)

Principles for Fault Tolerant QC:

(3) Encoded operations should not spread errors.

(4) EC between gadgets for state preparation and and encoded operation.

(5) Syndrome measurements themselves should not spread errors.

We will elaborate on (3)-(5) the rest of this segment of the course.

Encoded operations for stabilizer codes:

Recall that a logical operation of a stabilizer code is a unitary U that permutes the stabilizer group elements by conjugaton.

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How do we find logical H or CNOT ?

Note that the logical X, Z should be chosen so that the logical H and CNOT are easy to implement.

Before discussing encoded operations, we first see characterizations of H and CNOT (unencoded).

Lemma: If a unitary U on 2-dim satisfies the conditions

$$U X U^\dagger = Z \,, \qquad U Z U^\dagger = X$$

then, U = H (the Hadamard gate).

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The converse holds, that H and CNOT conjugate the Pauli matrices as described above. (Proof: A2 / Ex)

These identities also states how H and CNOT propagate Pauli errors (more later).

Proof of lemma:

Suppose $UXU^\dagger = Z$, $UZU^\dagger = X$.

We can find $U|0\rangle$, $U|1\rangle$ up to an overall phase each:

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We can find $U|0\rangle$, $U|1\rangle$ up to an overall phase each:

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$$\therefore\ U|0\rangle = a|+\rangle,\quad U|1\rangle = b|-\rangle,\qquad |a| = |b| = 1.$$

To relate a and b, use 2nd commutation relation:

$$U |+\rangle\langle +| U^\dagger = U \tfrac{1}{2}(I + X) U^\dagger = \tfrac{1}{2}(I + Z) = |0\rangle\langle 0|$$

$$\therefore\ U|+\rangle \propto |0\rangle$$

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$$\therefore U|0\rangle = a|+\rangle, \quad U|1\rangle = a|-\rangle$$

$\therefore U = H$ up to an overall phase a.

The proof is similar for U = CNOT, left as exercise with answer below.

Given: $U XI U^\dagger = XX, \quad U ZI U^\dagger = ZI$

$$U IX U^\dagger = IX, \quad U IZ U^\dagger = ZZ$$

$$\begin{aligned}
\text{So } U|00\rangle\langle 00|U^\dagger &= \frac{1}{4} U(I+Z)\otimes(I+Z)U^\dagger \\
&= \frac{1}{4} U(II + ZI + IZ + ZZ)U^\dagger \\
&= \frac{1}{4}(II + ZI + ZZ + IZ) = |00\rangle\langle 00|
\end{aligned}$$

$$\begin{aligned}
U|01\rangle\langle 01|U^\dagger &= \frac{1}{4} U(I+Z)\otimes(I-Z)U^\dagger \\
&= \frac{1}{4} U(II + ZI - IZ - ZZ)U^\dagger \\
&= \frac{1}{4}(II + ZI - ZZ - IZ) = |01\rangle\langle 01|
\end{aligned}$$

$$U|10\rangle\langle 10|U^\dagger = \frac{1}{4} U(I-Z)\otimes(I+Z)U^\dagger$$

$$= \frac{1}{4} U(II - ZI + (IZ - ZZ))U^\dagger$$

$$= \frac{1}{4}(II - ZI + ZZ - IZ)$$

$$= \frac{1}{4}(I-Z)\otimes(I-Z) = |11\rangle\langle 11|$$

$$U|11\rangle\langle 11|U^\dagger = \frac{1}{4} U(I-Z)\otimes(I-Z)U^\dagger$$

$$= \frac{1}{4} U(II - ZI - (IZ + ZZ))U^\dagger$$

$$= \frac{1}{4}(II - ZI - ZZ + IZ)$$

$$= \frac{1}{4}(I-Z)\otimes(I+Z) = |1\rangle\langle 1| \otimes |0\rangle\langle 0| = |10\rangle\langle 10|$$

So $U|00\rangle = a|00\rangle$, $\quad U|01\rangle = b|01\rangle$

$U|10\rangle = c|11\rangle$, $\quad U|11\rangle = d|10\rangle$

We now show that a,b,c,d are equal. Of many correct methods, an easy way connects 2 variable for each constraint. e.g.,

From the commutation relations on ZI, IZ:

$$U|+\rangle|0\rangle = (a|00\rangle + c|11\rangle)/\sqrt{2}$$

Meanwhile:

$$\begin{aligned} U|+\rangle|0\rangle\langle +|\langle 0|U^\dagger &= U\, |+\rangle\langle +| \otimes |0\rangle\langle 0|\, U^\dagger \\ &= \tfrac{1}{4} U\,(I+X)(I+Z)\,U^\dagger \\ &= \tfrac{1}{4} U\,(II + XI + IZ + XZ)\,U^\dagger \\ &= \tfrac{1}{4}\,(II + XX + ZZ - YY) \\ &= \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\,\tfrac{1}{\sqrt{2}}(\langle 00| + \langle 11|) \end{aligned}$$

$$\begin{aligned} & UXZU^\dagger \\ &= U(XI)(IZ)U^\dagger \\ &= UXIU^\dagger \cdot UIZU^\dagger \\ &= XX \cdot ZZ \\ &= -YY \quad (Y = iXZ) \end{aligned}$$

So, a = c.

$$U|0\rangle|+\rangle = (a|00\rangle + b|01\rangle)/\sqrt{2}$$

Meanwhile:

$$U|0\rangle|+\rangle\langle 0|\langle +|U^\dagger = U\,(|0\rangle\langle 0| \otimes |+\rangle\langle +|)\,U^\dagger$$

$$= \frac{1}{4} U\,(I+Z)(I+X)\,U^\dagger$$

$$= \frac{1}{4} U\,(II + ZI + IX + ZX)\,U^\dagger$$

$$= \frac{1}{4}\,(II + ZI + IX + ZX) = |0\rangle|+\rangle\langle 0|\langle +|$$

So, a = b.

With a=b=c, superpose all computational basis input:

$$U|+\rangle|+\rangle = U\frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle) = \frac{1}{2}(a|00\rangle + b|01\rangle + c|11\rangle + d|10\rangle)$$

$$= \frac{1}{2}(a|00\rangle + a|01\rangle + a|11\rangle + d|10\rangle)$$

$$U|++\rangle\langle ++|U^\dagger = \frac{1}{4}U(I+X)(I+X)U^\dagger = \frac{1}{4}U\,(II + IX + XI + XX)\,U^\dagger$$

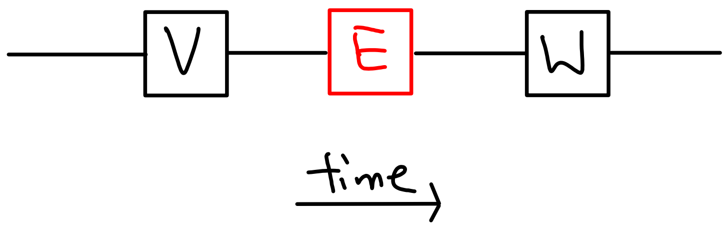
$$= \frac{1}{4}\,(II + IX + XX + XI) = |++\rangle\langle ++|$$

So, a = d. So, U = a * CNOT.

$\square$

How H and CNOT propagate Pauli errors:

Suppose a Pauli error E occurs right before a single-qubit gate W:



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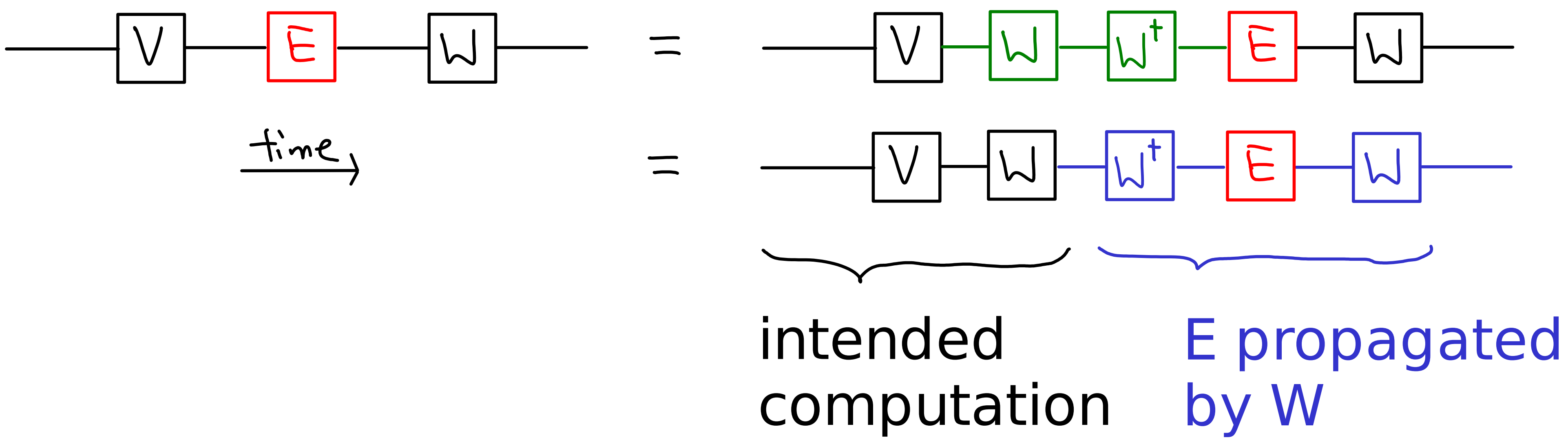

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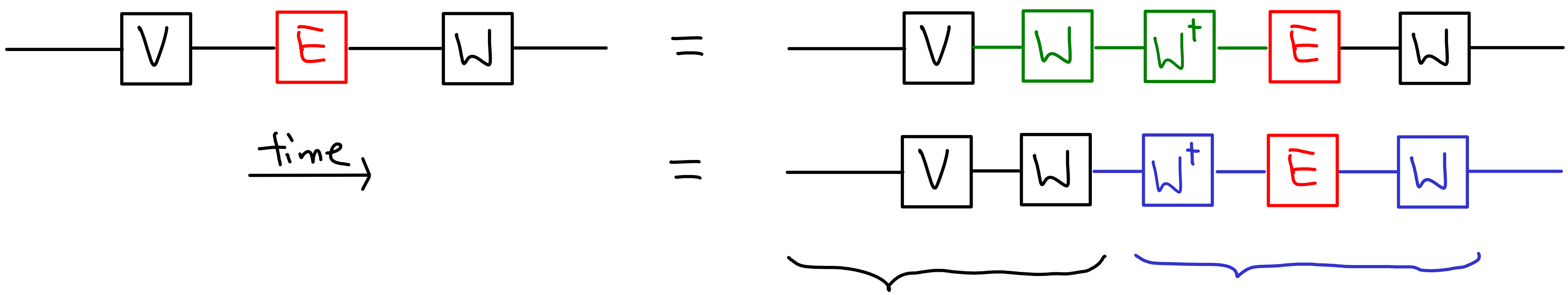

intended computation

E propagated by W

So, we can "propagate" the error to the right (later in time), E becomes $WEW^\dagger$.

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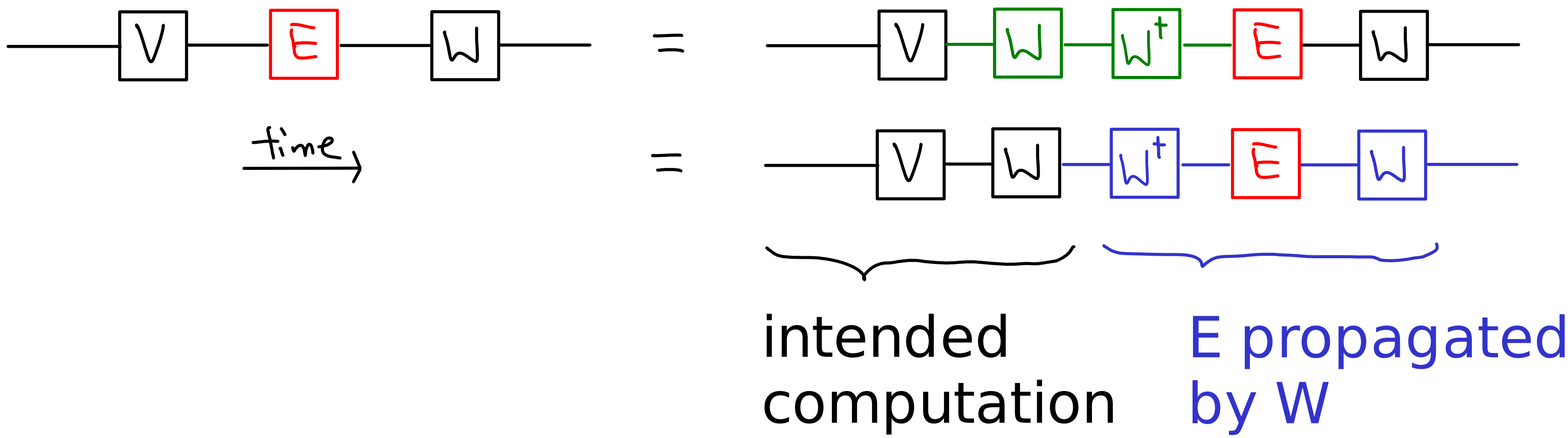

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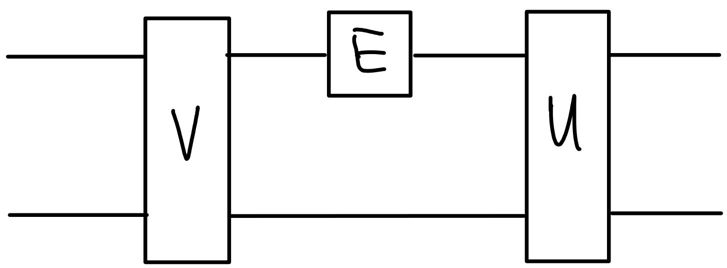
Note a single qubit gate propagates a Pauli error (E) to another single-qubit error ($WEW^\dagger$) on the same qubit.

Trickier for U = CNOT:

$U XI U^\dagger = XX$, $U IX U^\dagger = IX$, $U ZI U^\dagger = ZI$, $U IZ U^\dagger = ZZ$



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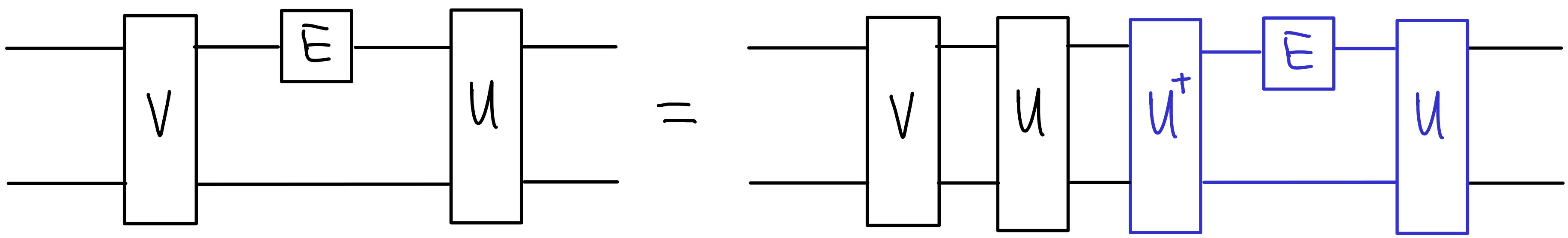

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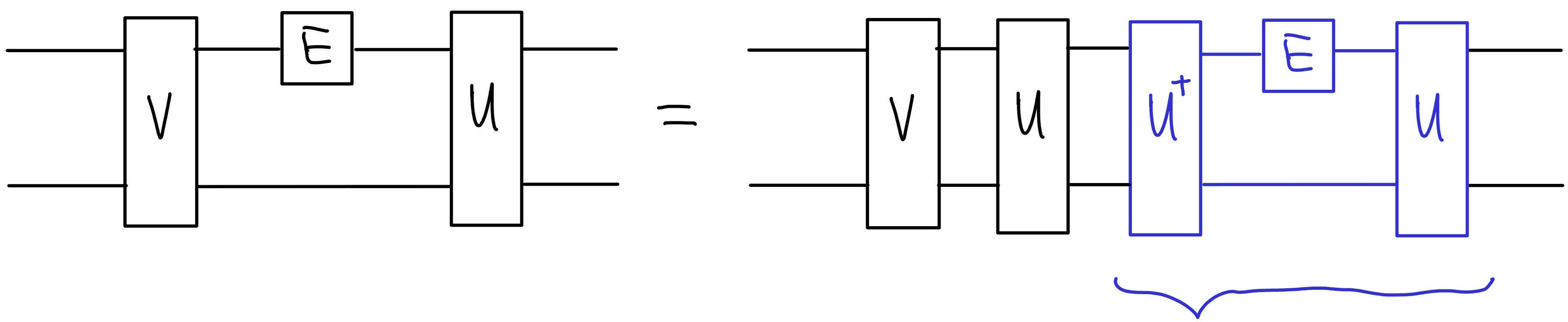

If E = X:
XI is propagated to XX,
1 error is turned into 2.
We call this "forward"
propagation (along the
direction of the CNOT).
This happens classically
too.

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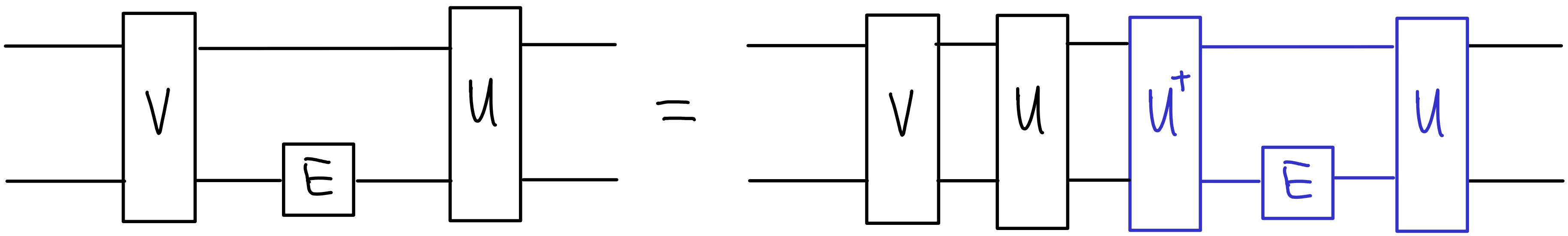

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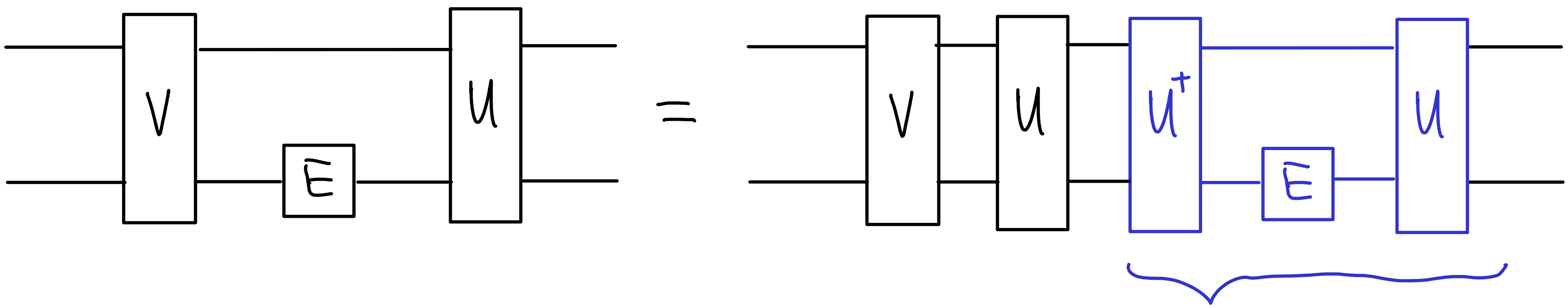

If E = Z:
IZ is propagated to ZZ
1 error turned into 2.
This is called "back action"
or "backward" propagation
of error (against the direction
of CNOT). That the error can
reach the controlled qubit is
uniquely quantum !

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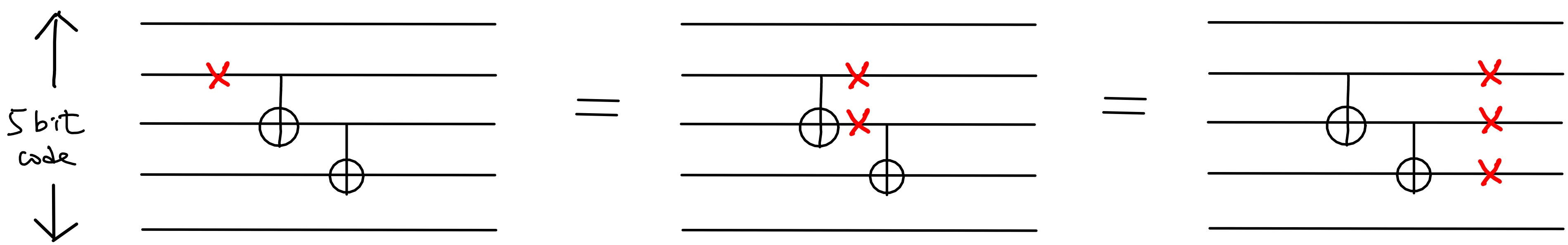
If a circuit propagate 1 error to multiple errors within the code block (before the next error correction step), the new errors may not be correctible, resulting in a logical error. e.g.,



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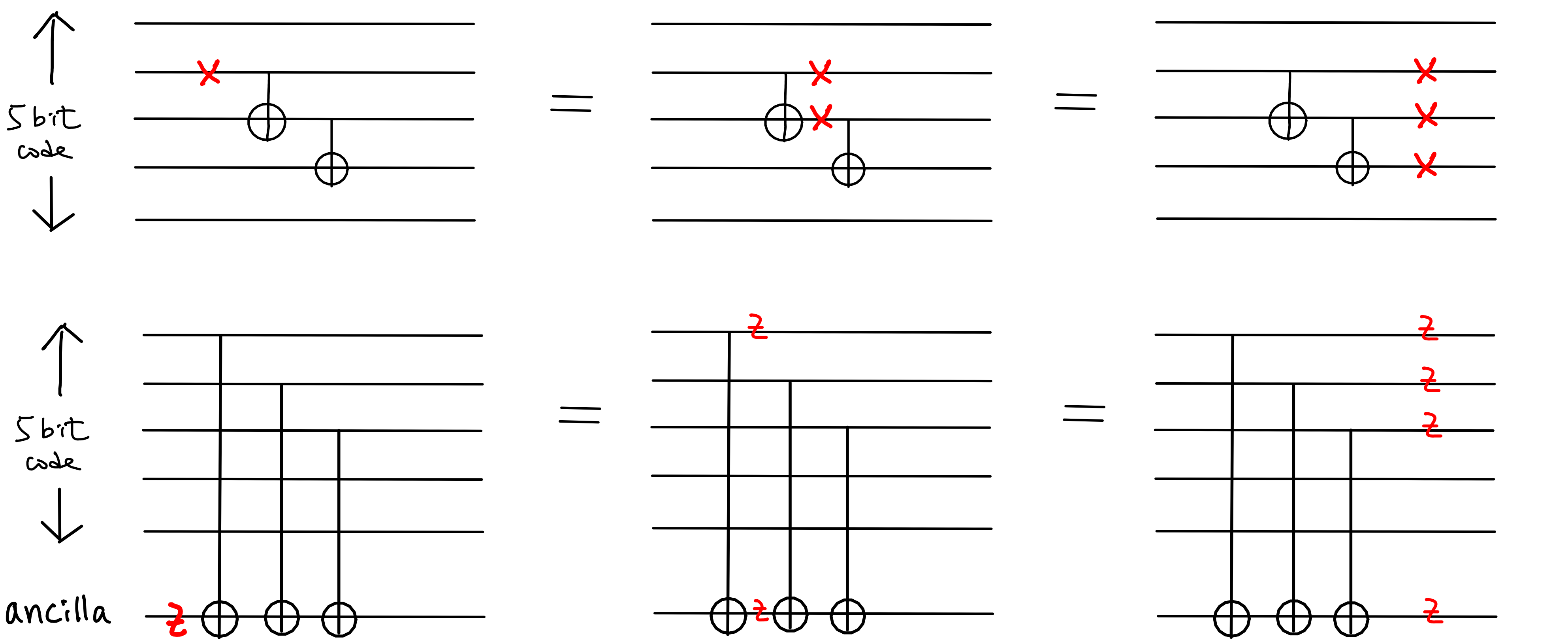
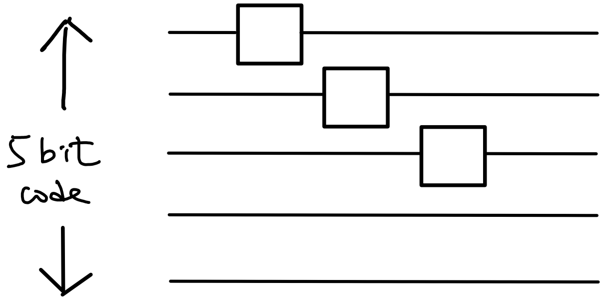


In contrast, the following circuits do not propagate 1 error to multiple errors within the same code-block:



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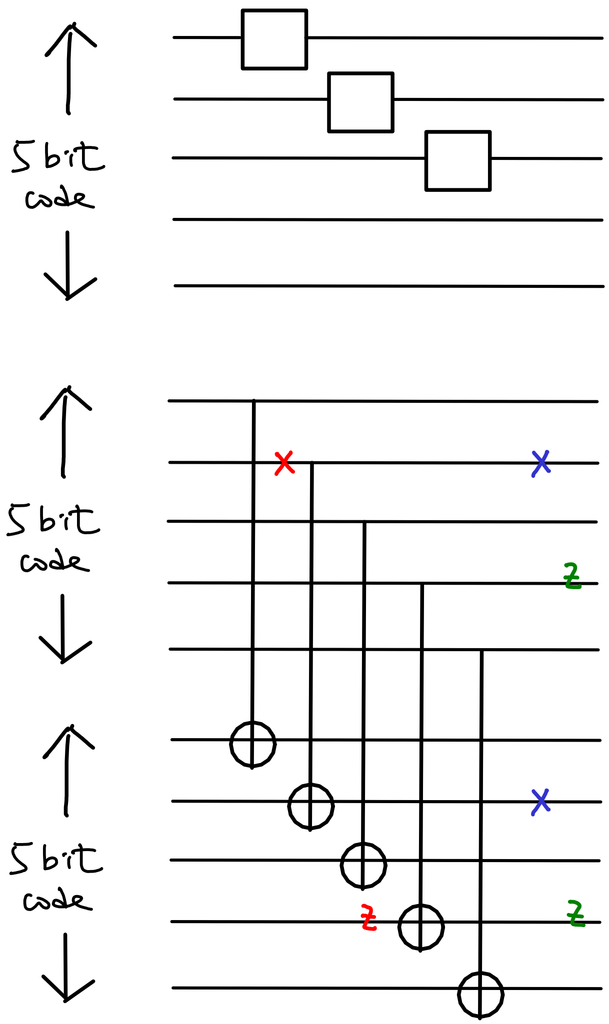

e.g., 1 red X spreads to 2 blue X's, one in each code block.

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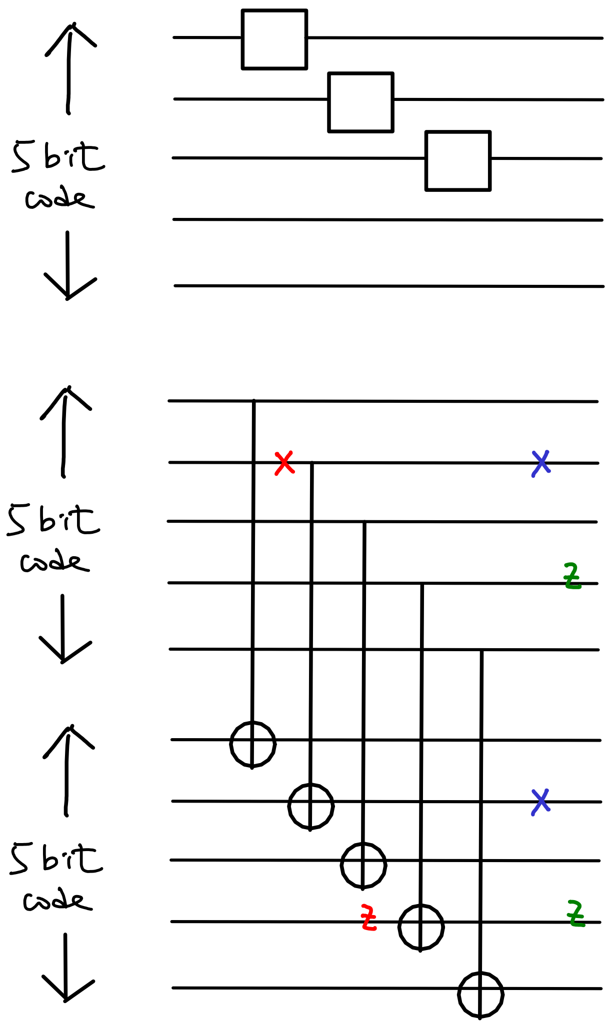

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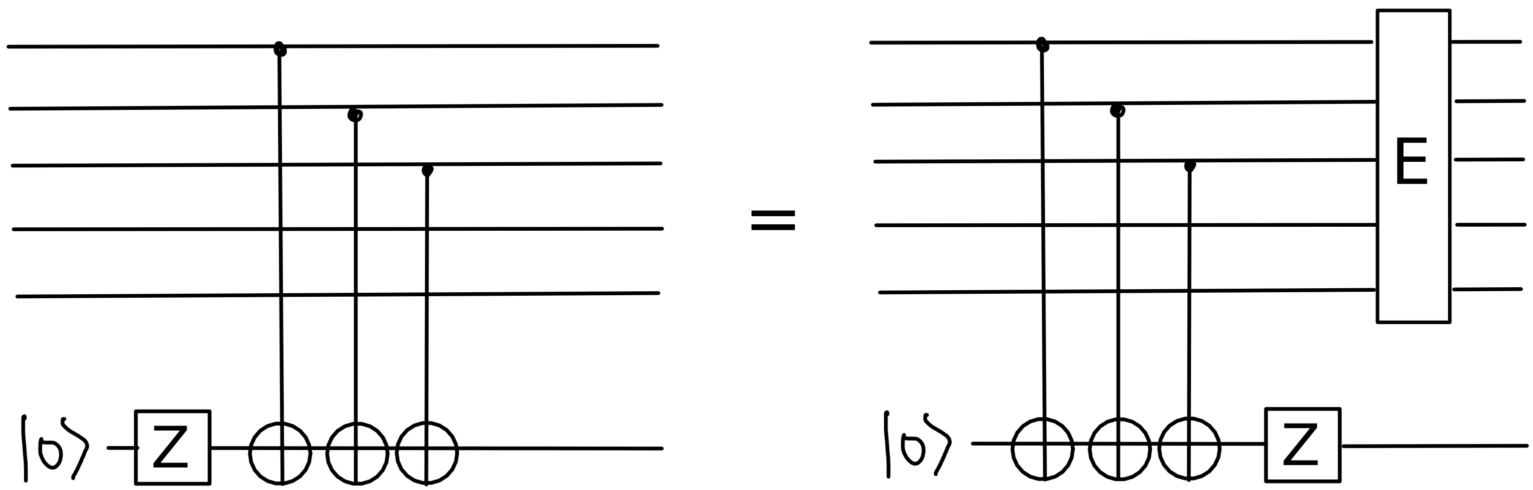
e.g., 1 red Z spreads to 2 green Z's, one in each code block.

Question: How many Z errors are in E?



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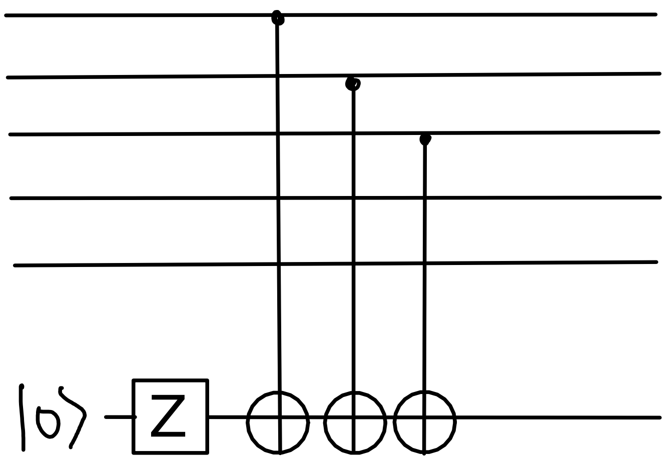

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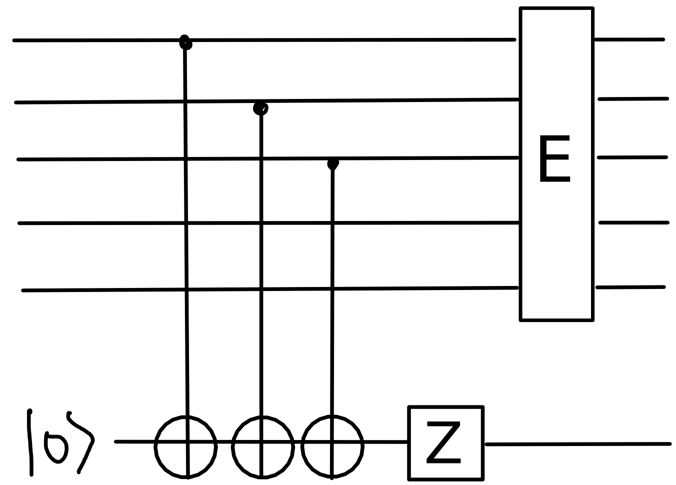
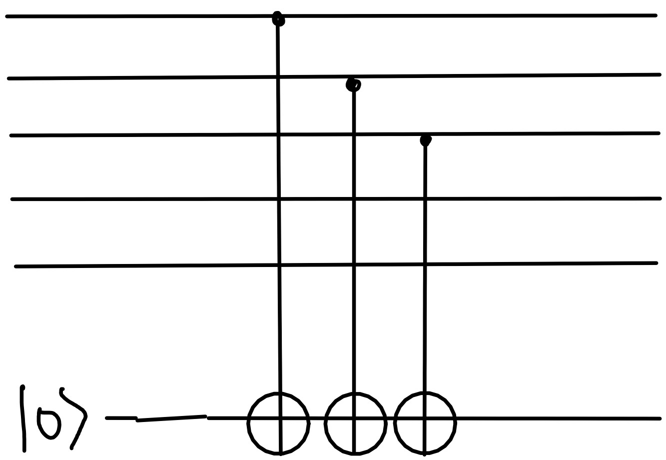


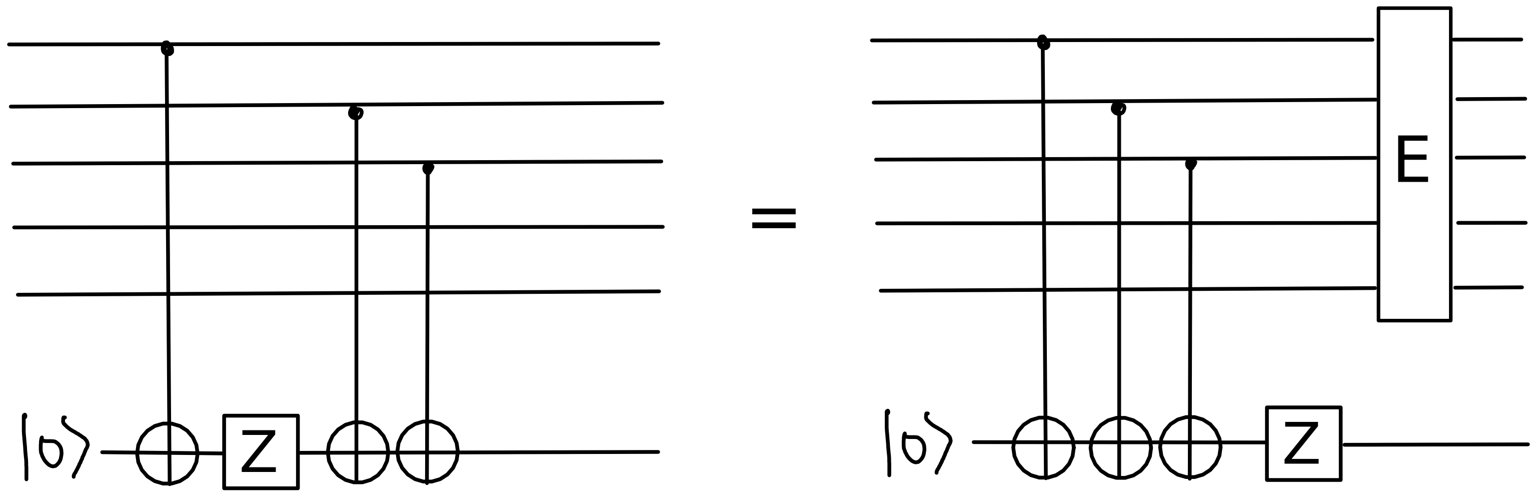
Answer: 3, E = ZZZII

||



But really no error ...

A better question:



E=IZZII here!

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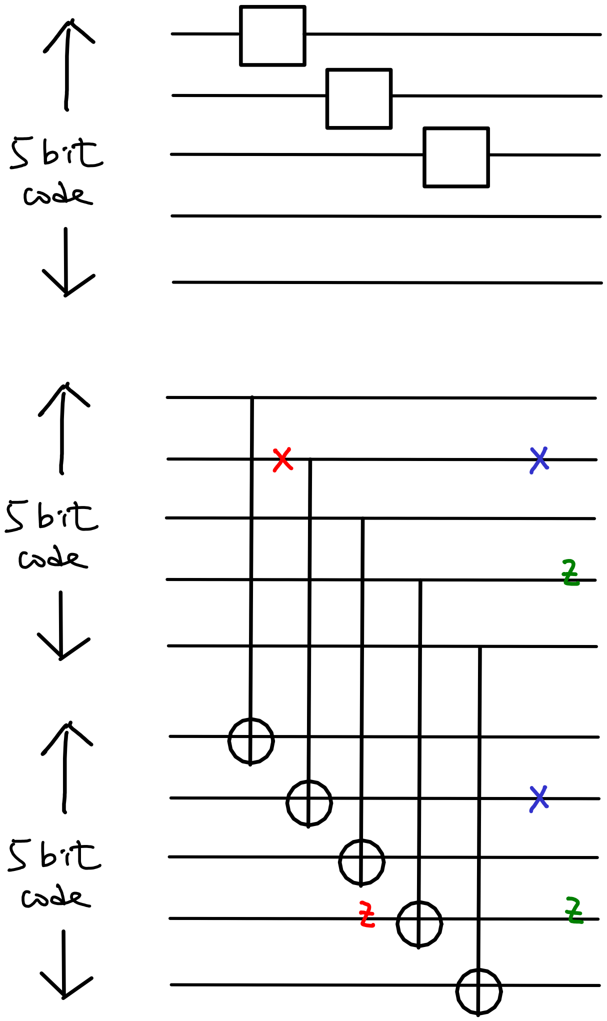

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These circuits are called "transversal".

Goal: find transversal encoded operations.