Topics 11-13: Quantum cryptography in the presence of noise and adversaries

1. Quantum money
2. Quantum bit commitment (M 6.3)
3. Quantum key distribution (NC 12.6, M 6.2)
   - Encryption
   - Classical one-time pad
   - Key distribution problem
   - QKD through a noiseless insecure channel (BB84, E92, and their relation)
   - QKD through noisy insecure channels

Encryption:

Reasons for communication with privacy:
- national secret, wartime commands
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The new proposals that may be quantum safe still relies on computational assumptions (e.g. $P \neq PSPACE$).

Are there encryption schemes that do not rely on computational assumptions?

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It turns out, if the sender and the receiver have prior contact and share a secret key, they can instead use private key cryptosystem, some has information theoretic security.

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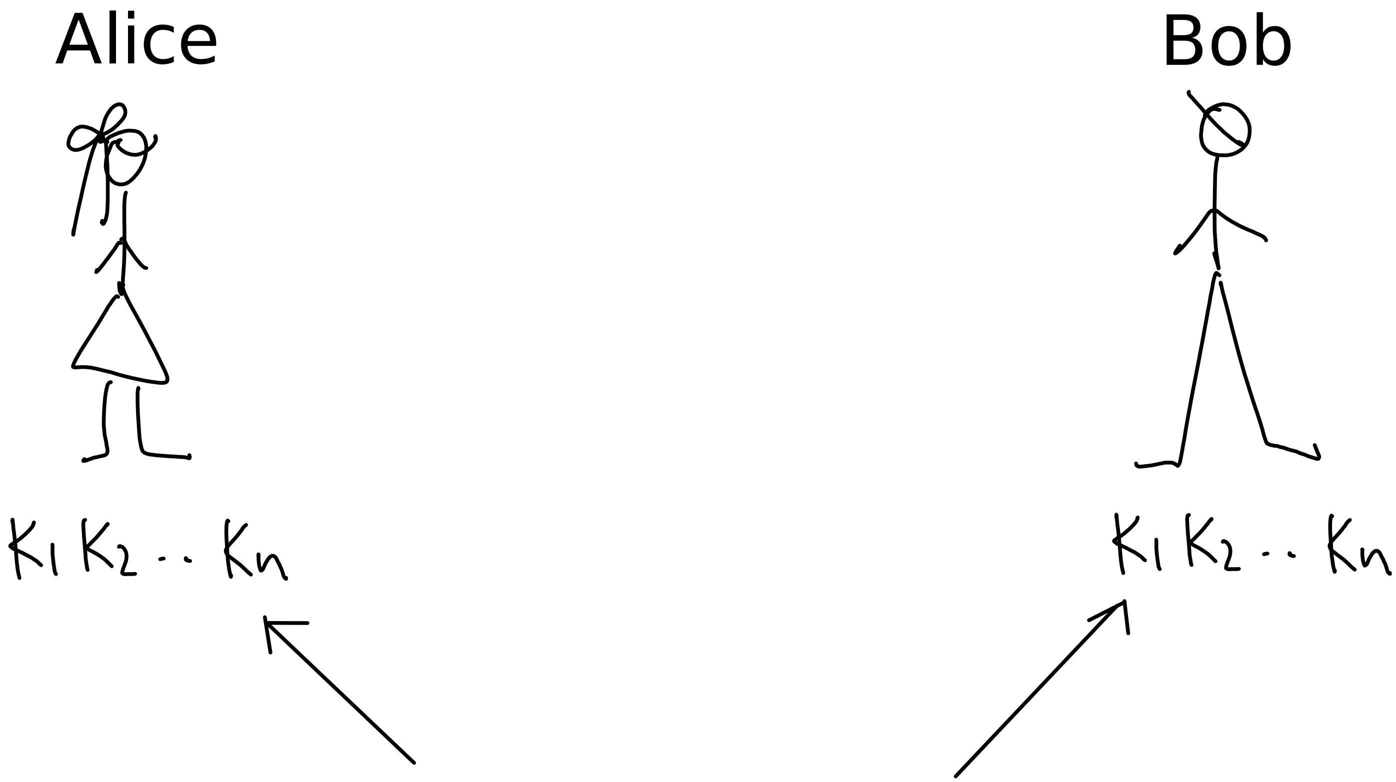

0. n-bit key shared
before the communication

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1. Alice wants to communicate n bits to Bob $m_1 m_2 \dots m_n$

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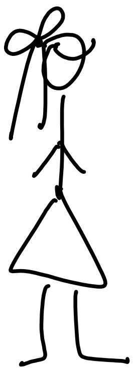
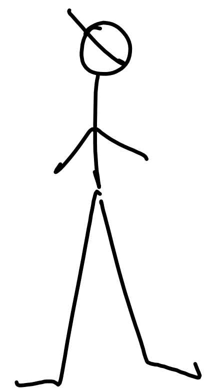


2. she sends to Bob the ciphertext:

$k_1 \oplus m_1, k_2 \oplus m_2, \dots, k_n \oplus m_n$

Bob



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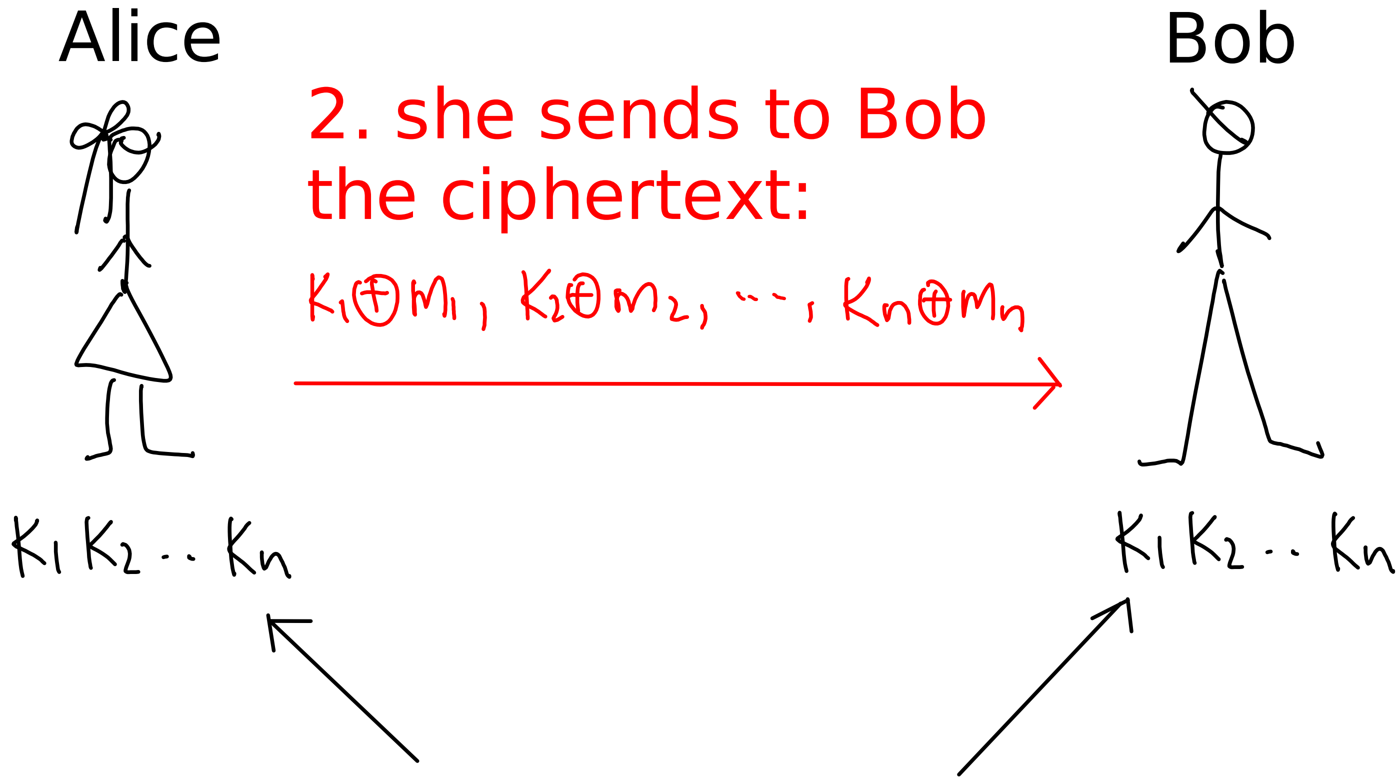

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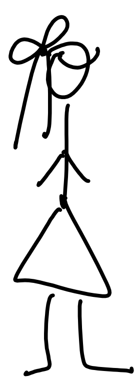

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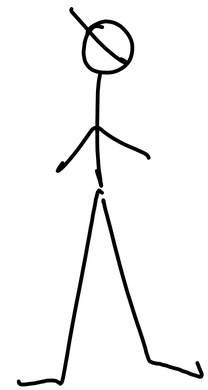

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Any eavesdropper Eve, without info on the key, and only seeing the ciphertext, will have no information on the message (Shannon 49)!

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Problem: how do Alice and Bob share this key?

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The second requirement is often met with a classical message authentication scheme that in turns requires a key. Goal: use a small authentication key and QKD to obtain a larger key -- QKD achieves key EXPANSION.

QKD schemes that are secure given an insecure noiseless channel --

In other words, without an eavesdropping the channel should be noiseless.

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1. Alice picks 2n random bits: $c_1 c_2 \cdots c_n,\ b_1 b_2 \cdots b_n$

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VERY CRUCIAL STEP !!
(WITHOUT THIS THE PROTOCOL IS INSECURE)

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Alice tells Bob all basis info & discloses a random half of the ci's.

7. Bob measures Bi in the {|0>,|1>} basis if bi = 0, in the {|+>,|->} basis if bi = 1. Let the outcomes be d1 d2 ... dn.

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If they are always the same, he concludes that the channel is secure. He tells Alice the protocol is successful. Alice take the ci's for which ri = 0 as the key; Bob takes the di's.

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Alice outputs c1 c2 c4 c7 c8 c10 = 101100,
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Most generally, Eve applies a quantum operation jointly on all n qubits. She keeps the environment output system of the Stinespring dilation (what she gains from eavesdropping) and gives the output (n qubits) to Bob. By discretization of errors (writing the Kraus operations as a linear combination of Pauli errors) we can focus on Pauli errors. QKD is a Pauli error detecting code.

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QKD is insecure only if Alice and Bob generate a compromised key. (OK if they abort the protocol ...)

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The benefit:

Bob does not need quantum storage (really hard for photons which is widely used). BB84 is called a "prepare-measure" scheme.

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f,g can be quite small (e.g., 1%) if n is very large (e.g., 10^9) while maintaining security.
(Small pe can be handled by methods to be described.)

Now comes the real, big, problem ...

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But a convincing security proof was elusive (84-96).
e.g., can Eve mask the error she induces as noise?
e.g., what if Eve does not find the key, but keep the quantum state from tampering with BB84 and use it for future attack when Alice and Bob use their key?

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Shor & Preskill 00: show Lo-Chau proof of security of E91 can be turned into the Mayers proof of BB84!

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Lo & Chau 97: easy security proof of E91
(based on BDSW96 -- Bennett, DiVincenzo, Smolin, Wootters on entanglement purification and QECC)

Shor & Preskill 00: show Lo-Chau proof of security of E91 can be turned into the Mayers proof of BB84!

E91 is hard to implement experimentally but easy to analyse, BB84 is opposite. Relating the two gives the best of the 2 worlds ...

Plan:

1. Ekert's QKD scheme E91
2. Lo-Chau security proof for E91
   General Eve, noiseless channel, then add noise.
3. Relate E91 and BB84 (noiseless case)
4. Relating security of E91 to BB84 (Shor-Preskill)
   (brief ideas)

E91 (Ekert 91) (version 1)

If Alice and Bob share: $|\Phi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

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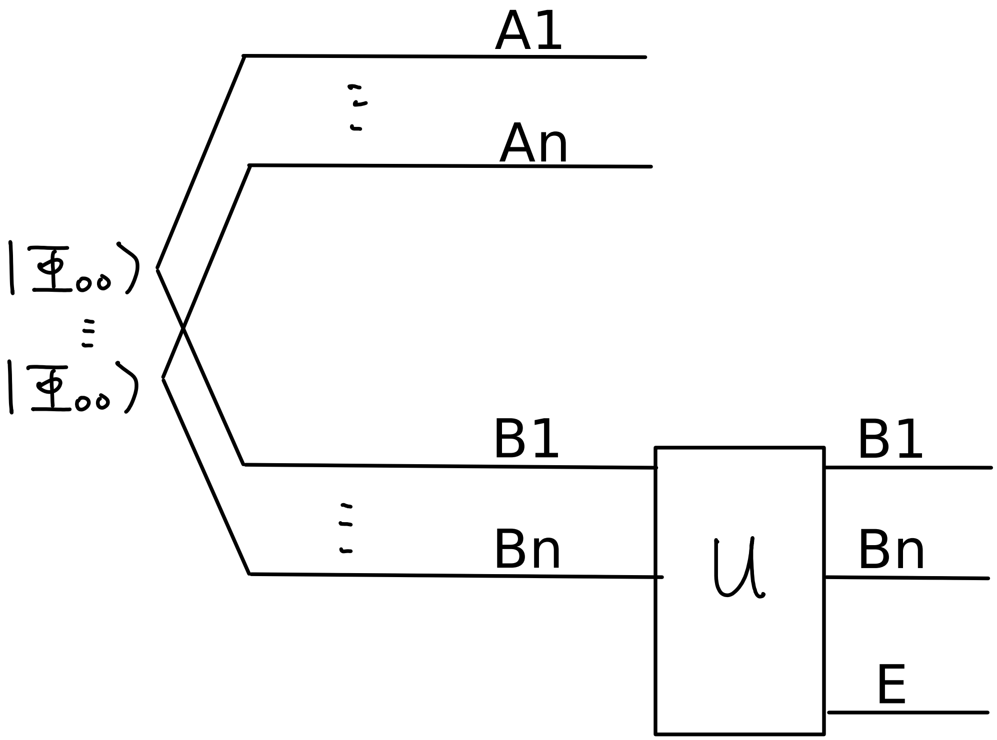
Solution: Alice prepare many copies of $|\Phi_{00}\rangle$.

Let the i-th copy live on systems Ai Bi. Alice sends B1 ... Bn to Bob via the insecure quantum channel .

Security of E91 via insecure noisy channel:



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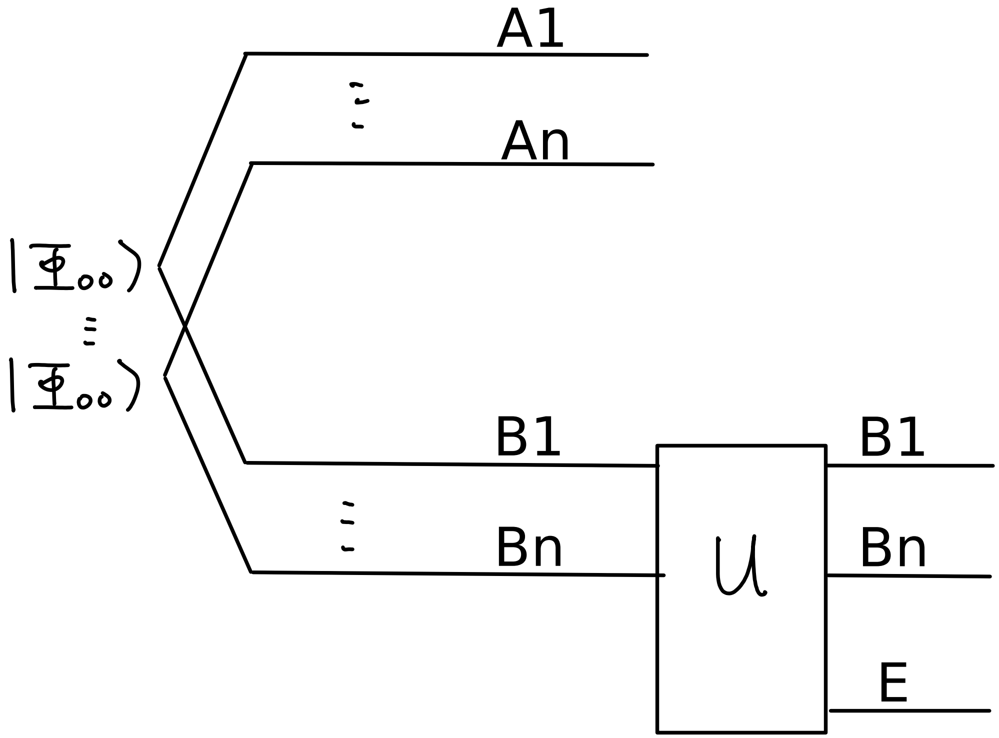

Claim 1: the most general error reduces to Pauli errors !

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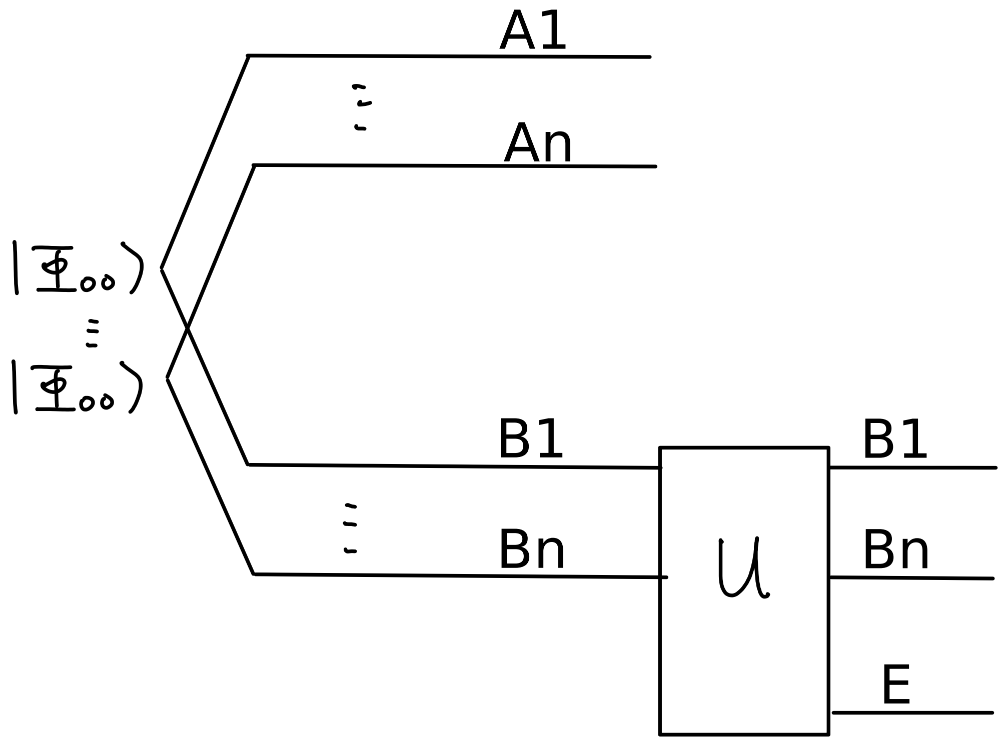

Claim 1: the most general error reduces to Pauli errors !

(based on discretization of error for sharing $|\Phi_{00}\rangle$)

Claim 2: it is possible to identify & correct Pauli errors
if there are not too many of them ...

Detecting and identifying Pauli error on Bell states:

Label each of the 4 Bell states with 2 bits

$$|\Phi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = I \otimes I\,|\Phi_{00}\rangle \quad \longleftrightarrow \quad 00$$

$$|\Phi_{10}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = I \otimes X\,|\Phi_{00}\rangle \quad \longleftrightarrow \quad 10$$

$$|\Phi_{01}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = I \otimes Z\,|\Phi_{00}\rangle \quad \longleftrightarrow \quad 01$$

$$|\Phi_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = I \otimes Y\,|\Phi_{00}\rangle \quad \longleftrightarrow \quad 11$$

↑ up to an overall phase

$$|\Phi_{ab}\rangle = I \otimes X^a Z^b\,|\Phi_{00}\rangle \quad \longleftrightarrow \quad ab$$

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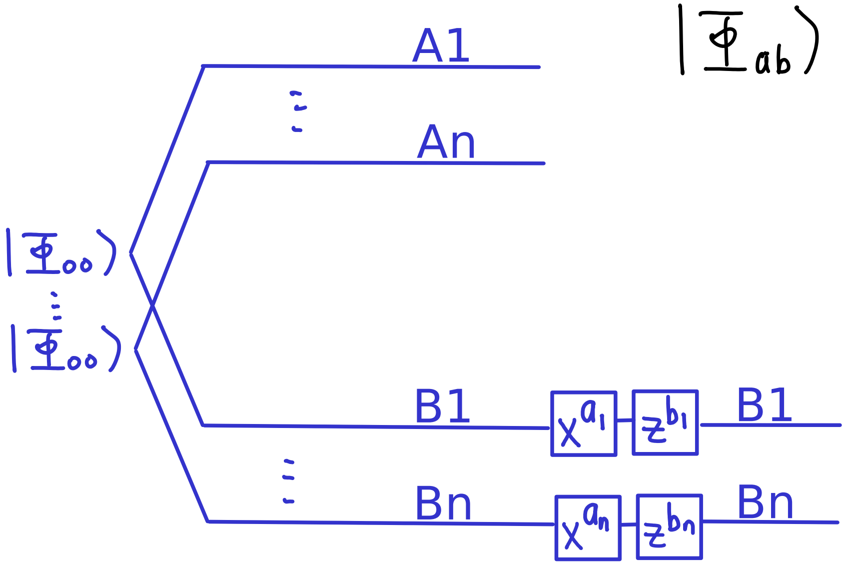
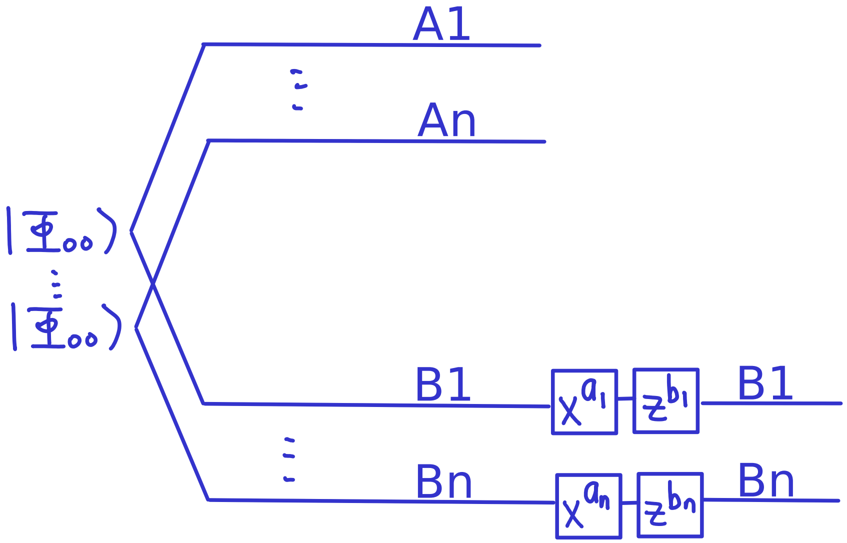


$$\longleftrightarrow \quad a_1 b_1 a_2 b_2 \dots a_n b_n$$

Theorem: suppose Alice and Bob share the following:



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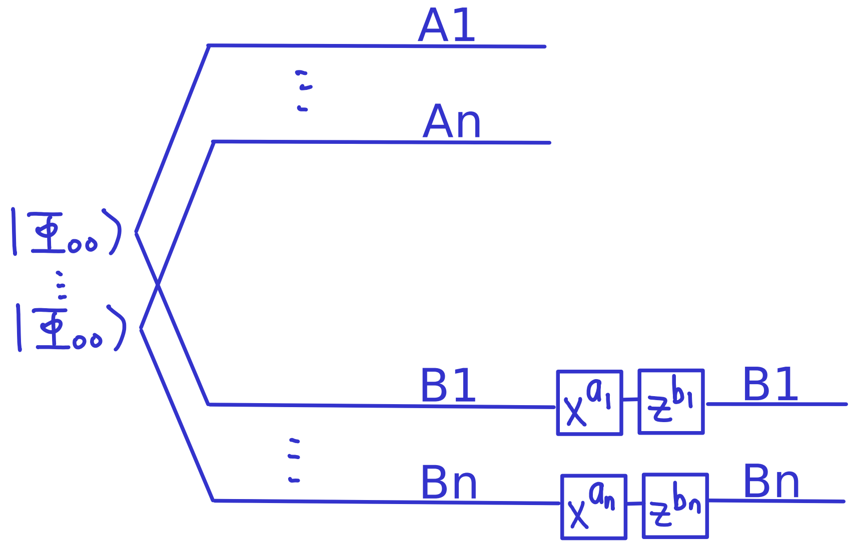

$\longleftrightarrow \quad a_1 b_1 a_2 b_2 \cdots a_n b_n$

There is a test that consumes k copies of $|\Phi_{00}\rangle$ uses classical communication (CC), without changing the above state, such that

$$\Pr\left(a_1 b_1 \cdots a_n b_n \neq 00 \cdots 00 \text{ and test passes}\right) \leq \frac{1}{2^k}$$

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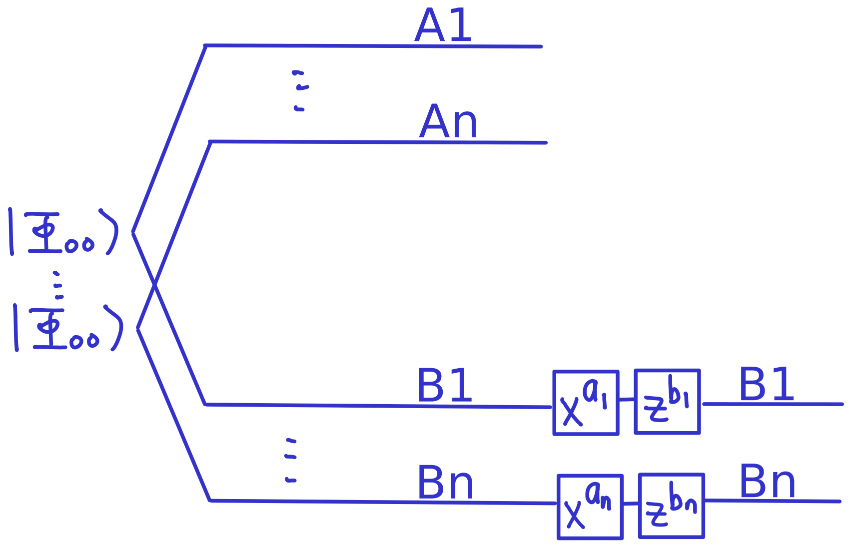

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not a chicken-n-egg problem, if k < n and can be borrowed and return (more later)

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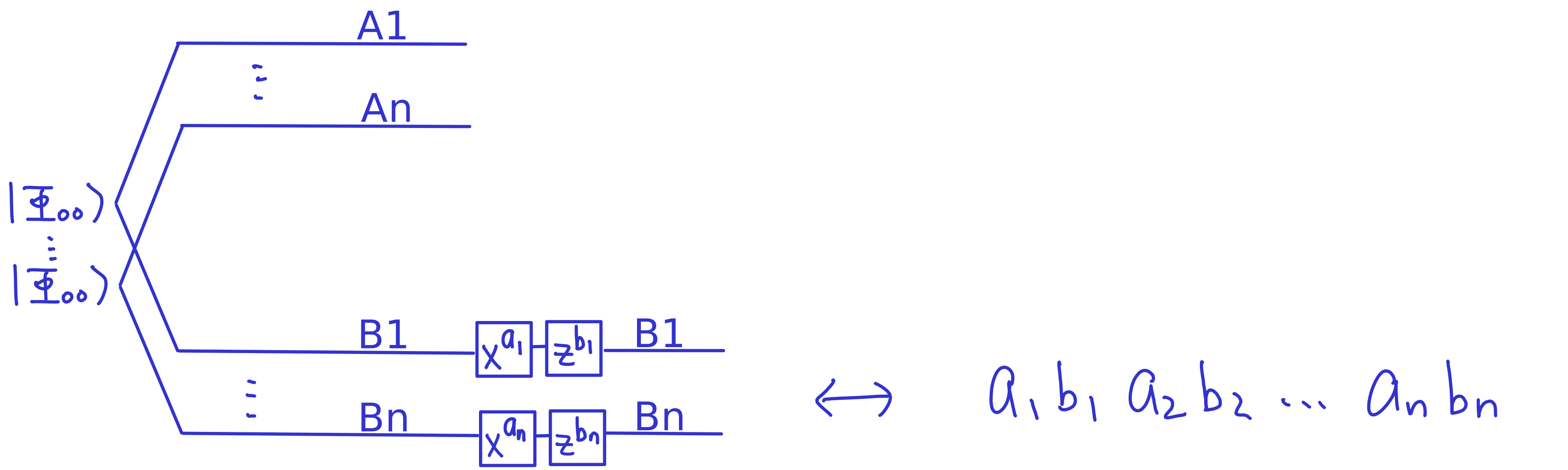

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Lemma: with 1 copy of $|\Phi_{00}\rangle$ & CC, Alice and Bob can learn the parity of any subset in $a_1 b_1 a_2 b_2 \cdots a_n b_n$.

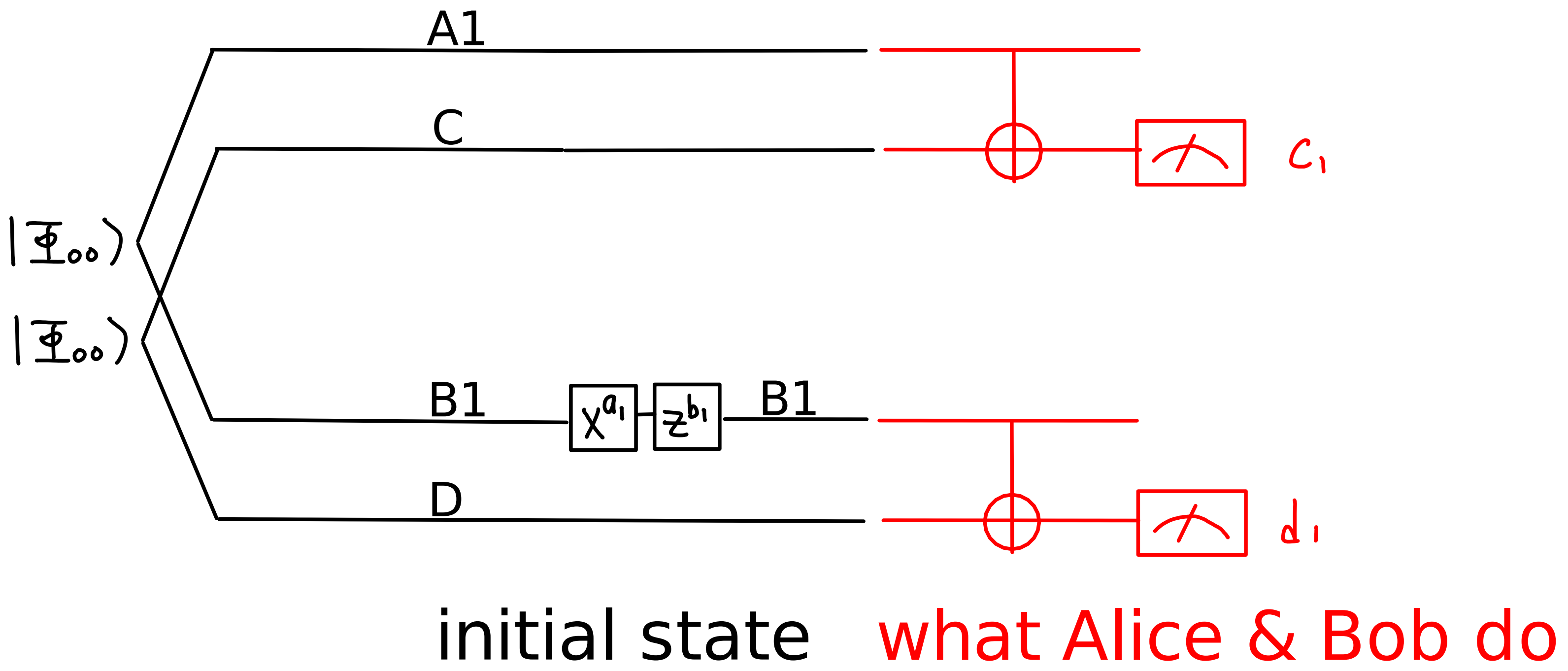
NB The parity of a subset of bits is called a "hash".

Switch to continuous view ...

Why lemma holds (via examples):

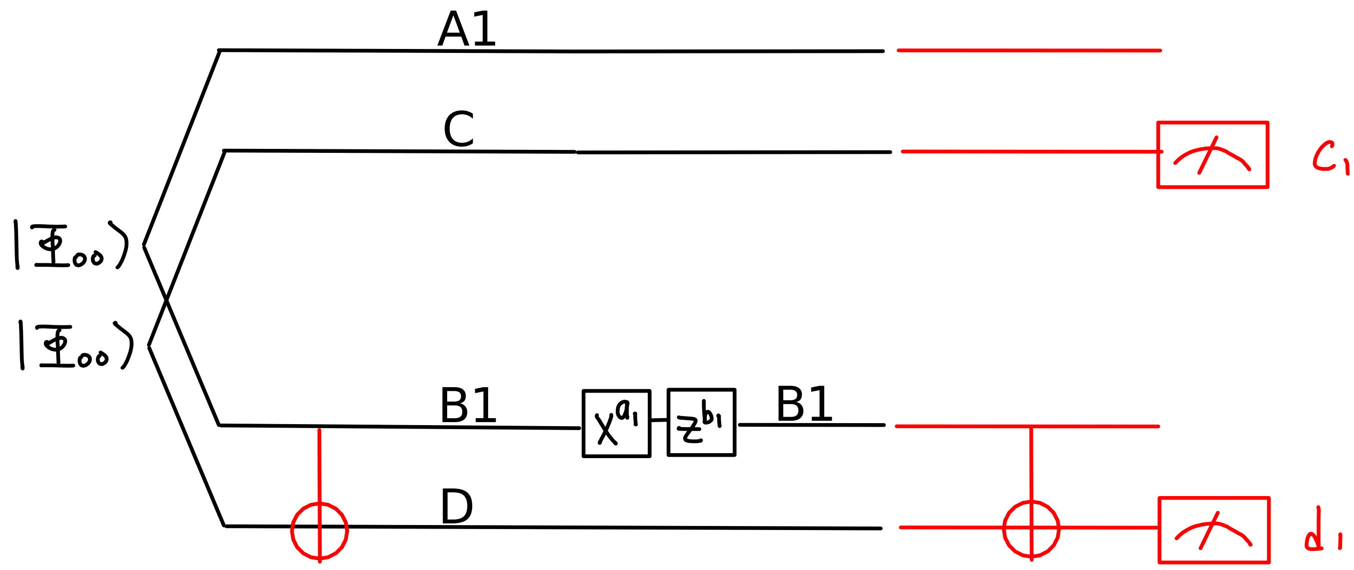
Let the noiseless copy of $|\Phi_{00}\rangle$ live on CD.

Circuit to learn a1:



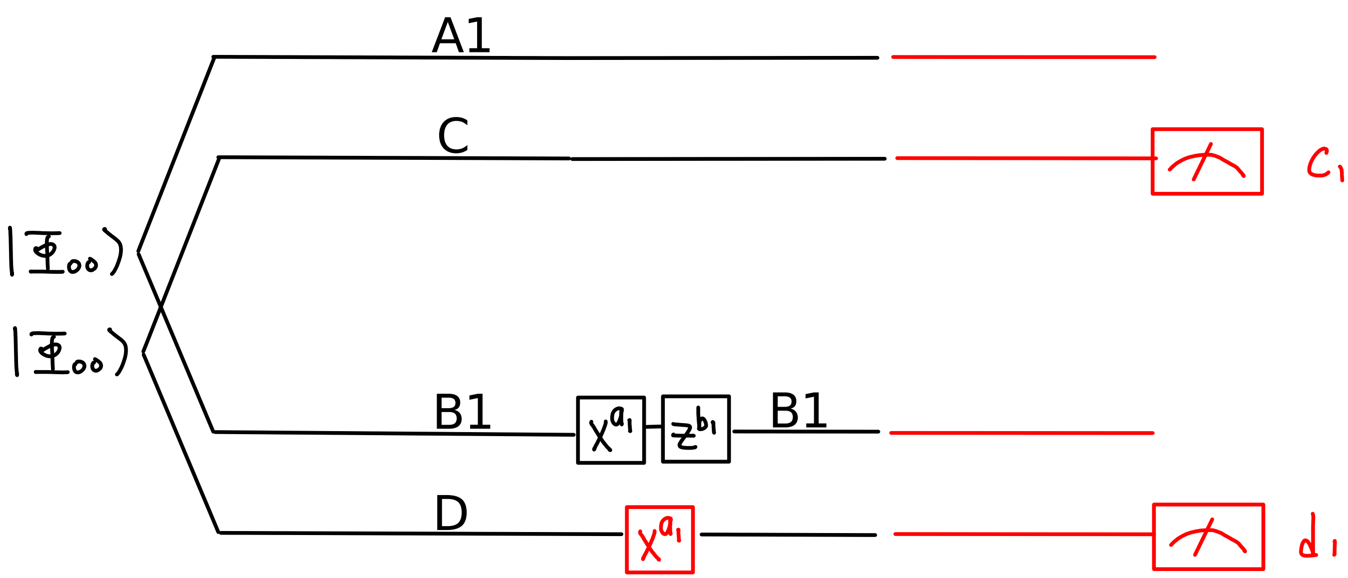
Claim: a1 = c1 + d1 mod 2

|| circuit giving the same output



A1 C and B1 D are in a maximally entangled state, use transpose trick (from assignment 1)

||



CNOT X1 CNOT = XX

CNOT Z1 CNOT = Z1

So, A1 B1 unchanged but CD is now $X^{a_1} \otimes I \, |\Phi_{00}\rangle$.

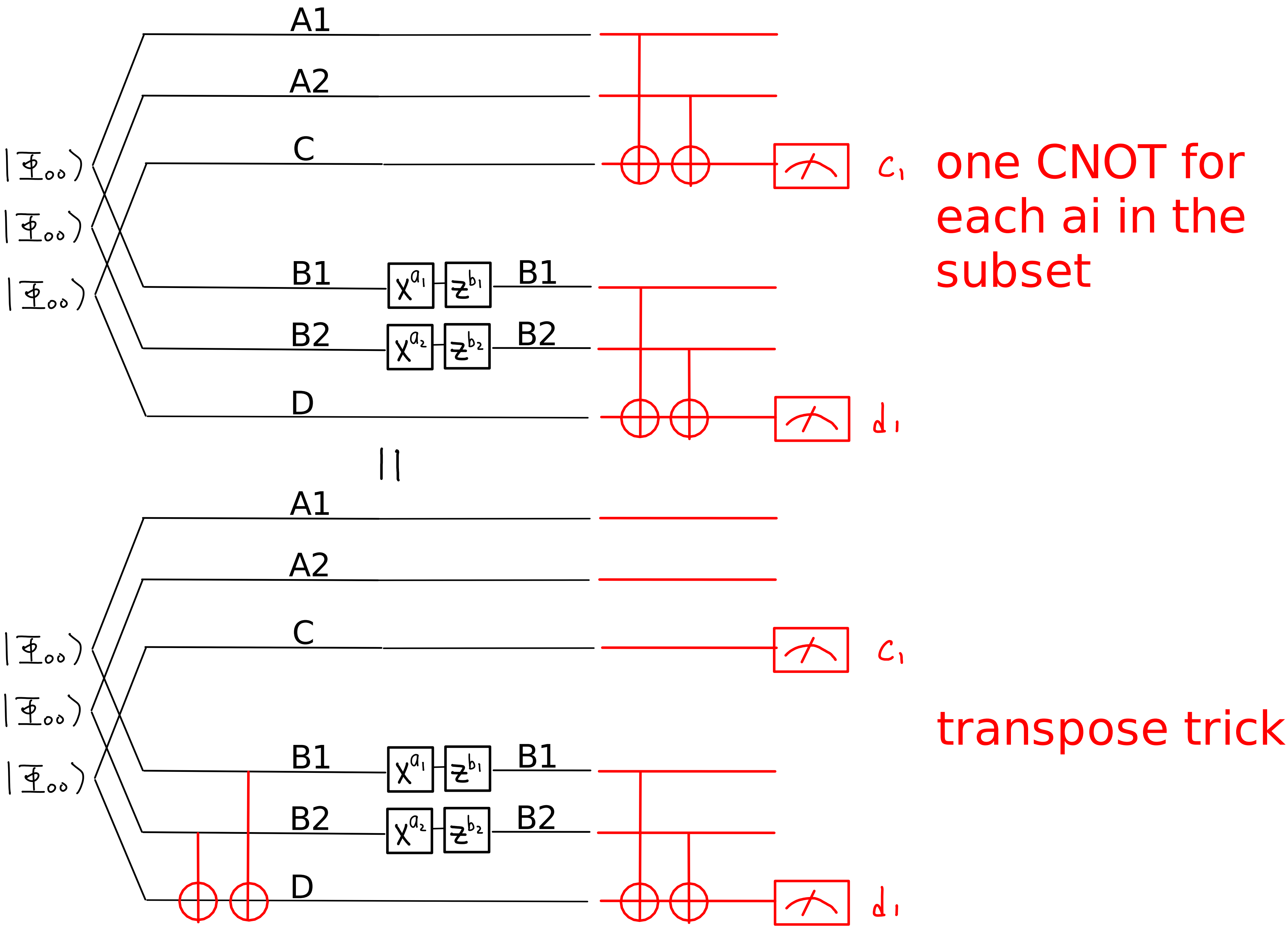
If a1 = 0, meas outcomes satisfy c1 = d1.

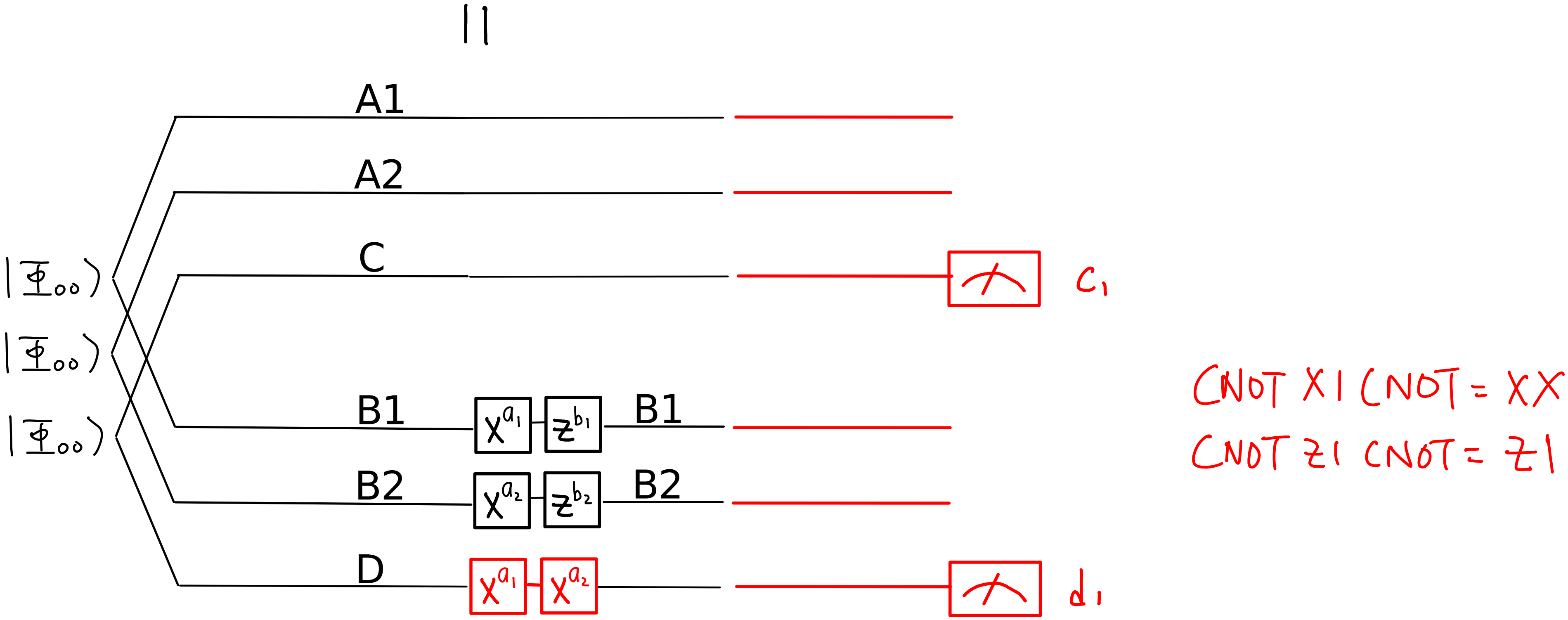
If a1 = 1, state is $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ so c1 + d1 mod 2 = 1.

$\therefore$ a1 = c1 + d1 mod 2

Using the authenticated classical channel, Alice and Bob compare c1, d1 to find a1.

Circuit to learn a1+a2 mod 2:

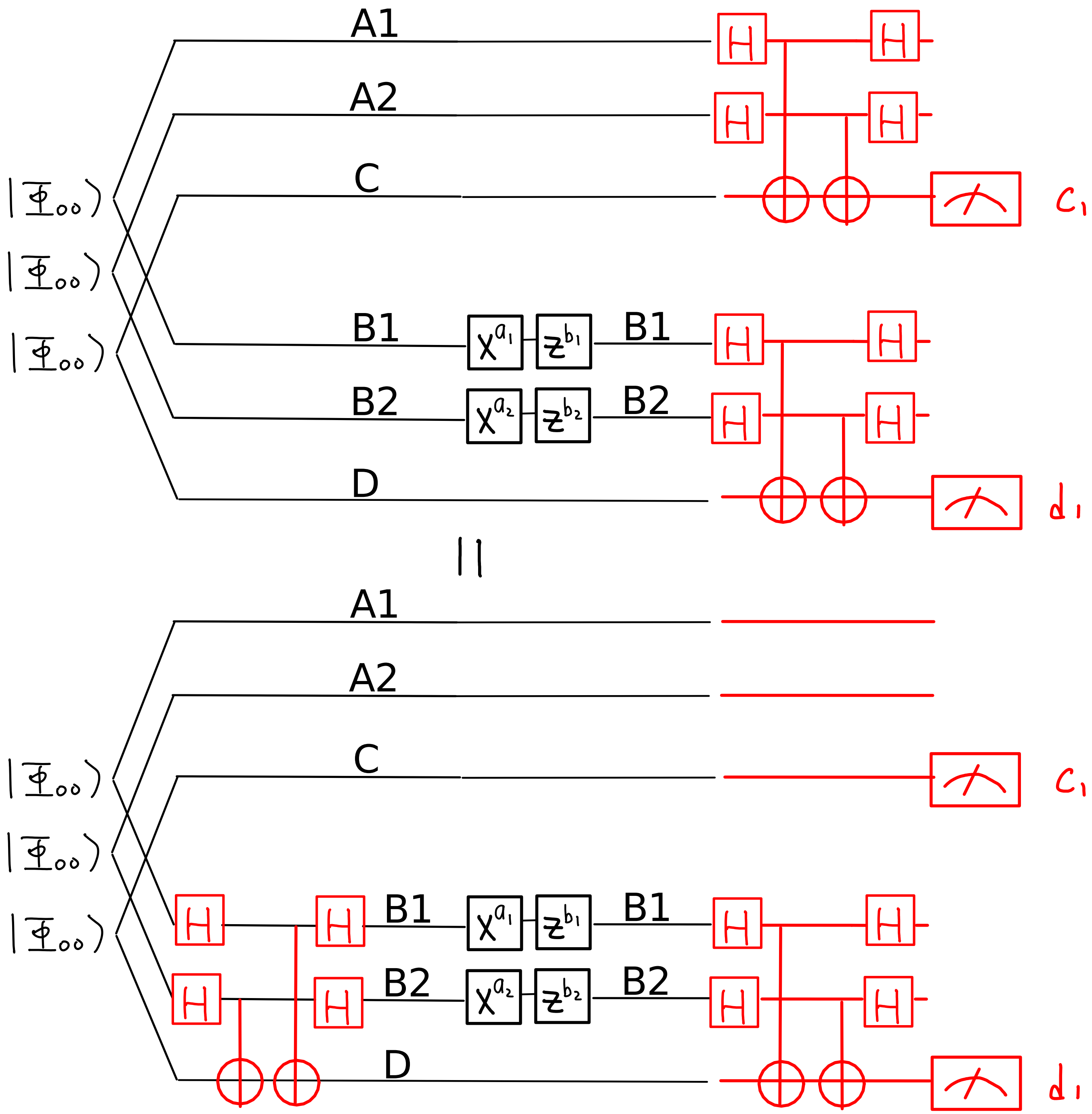




So, c1 + d1 = a1 + a2 mod 2.

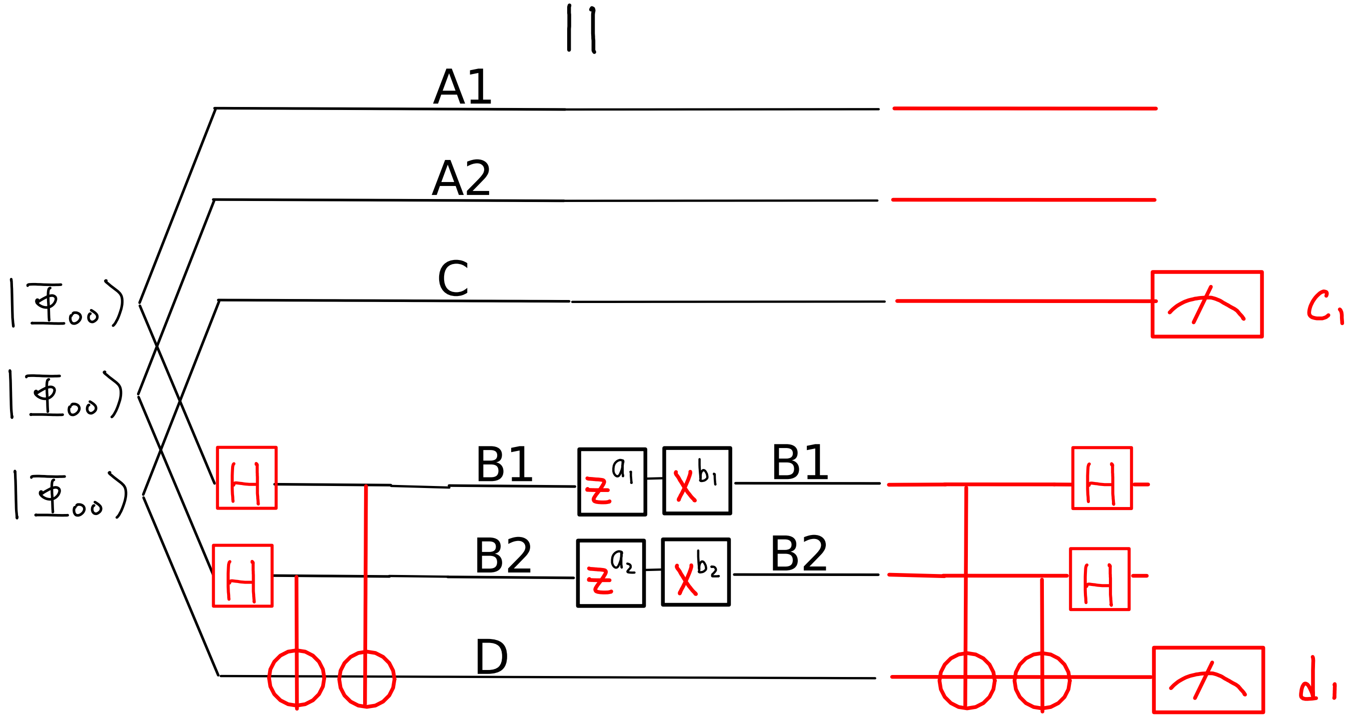
Method applies to the sum of any subset S of the ai's:
if aj in S, Alice applies CNOT from Aj to C,
Bob applies CNOT from Bj to D.

To learn b1+b2 mod 2:

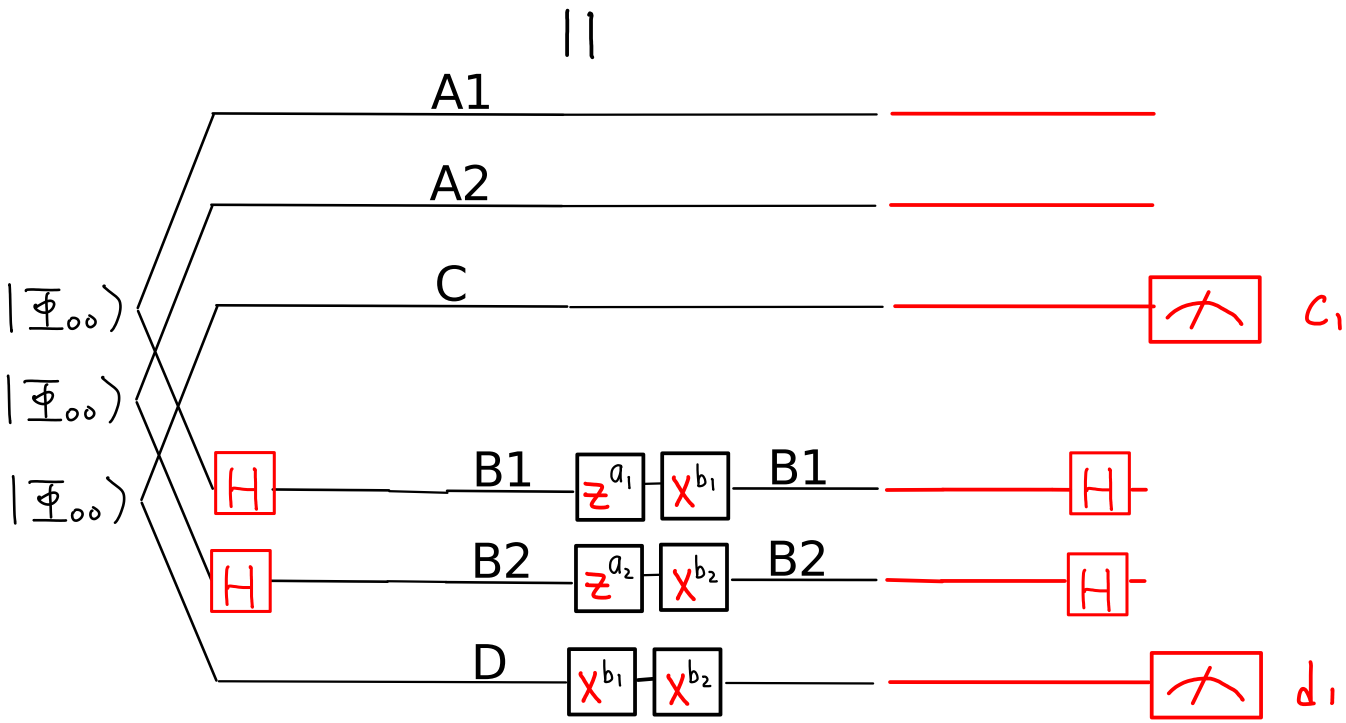


H's added to the control qubits before and after the CNOTs.

transpose trick



$$HXH = Z$$
$$HZH = X$$



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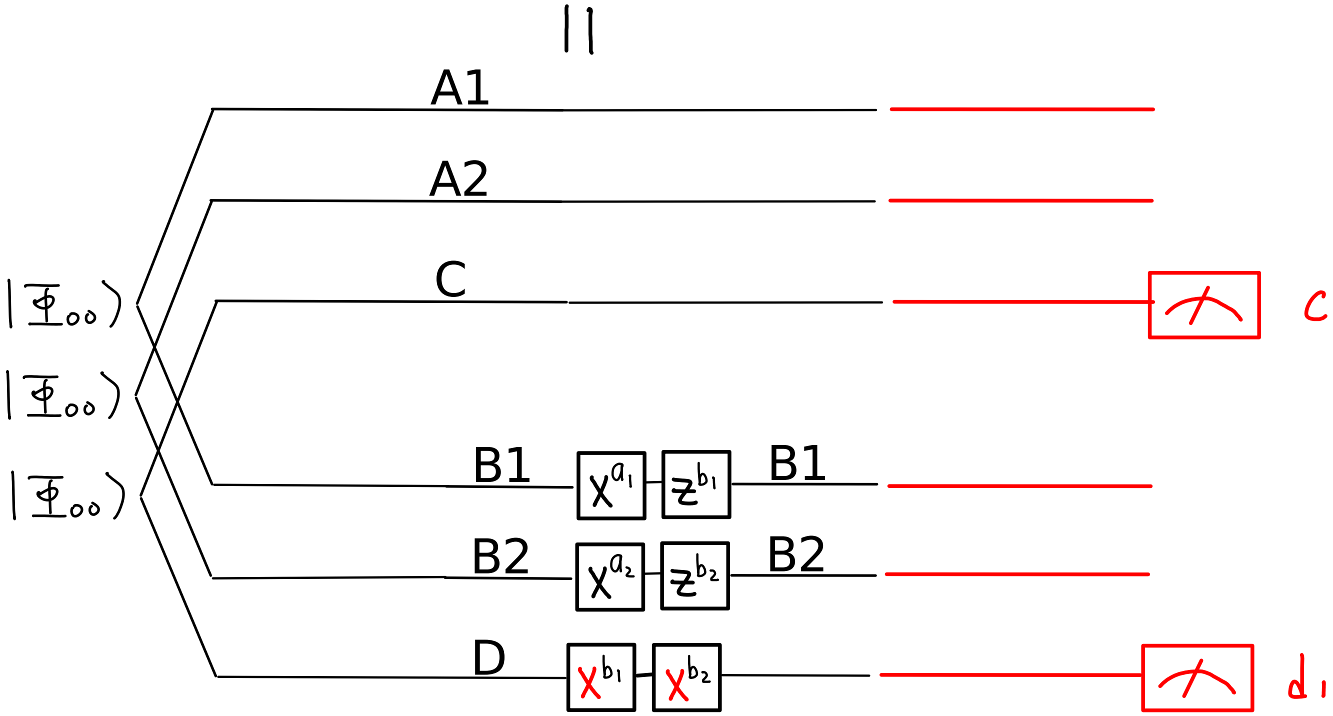

HXH=Z
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So, c1+d1 = b1+b2 mod 2.
The state on A1 A2 B1 B2 is unchanged by the meas.

To learn a1+b2 mod 2:

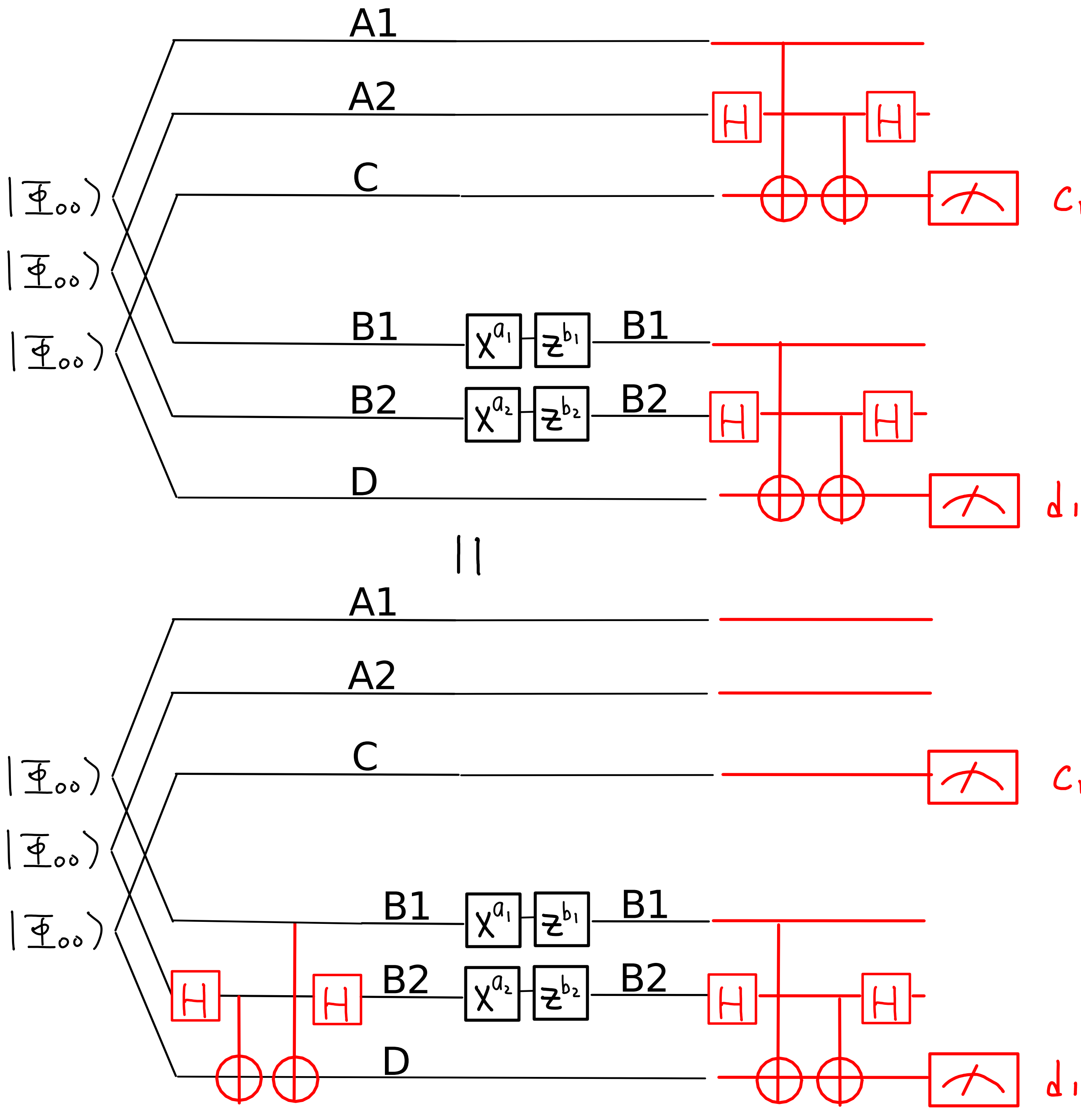
If in the exam, for parts (a), (b) you're guided to find

a1+a2 mod 2

b1+b2 mod 2

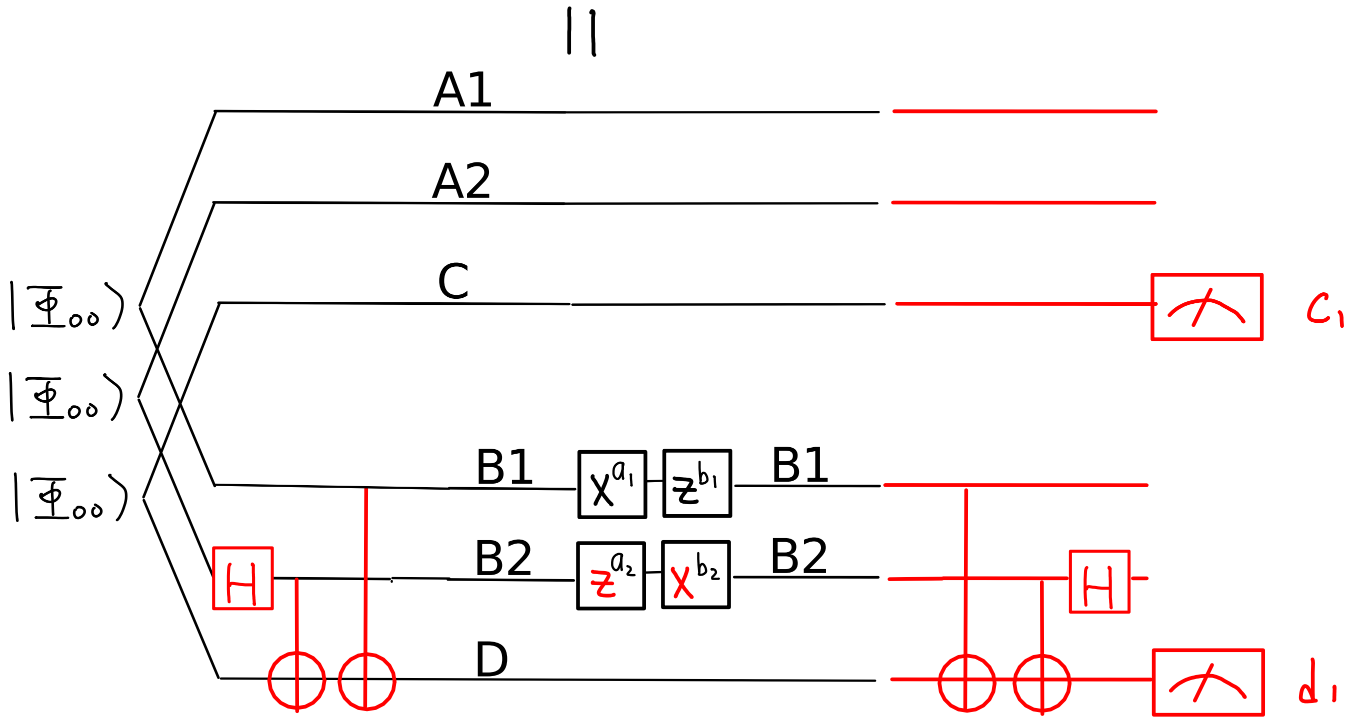
what would you propose for to learn a1+b2 mod 2?

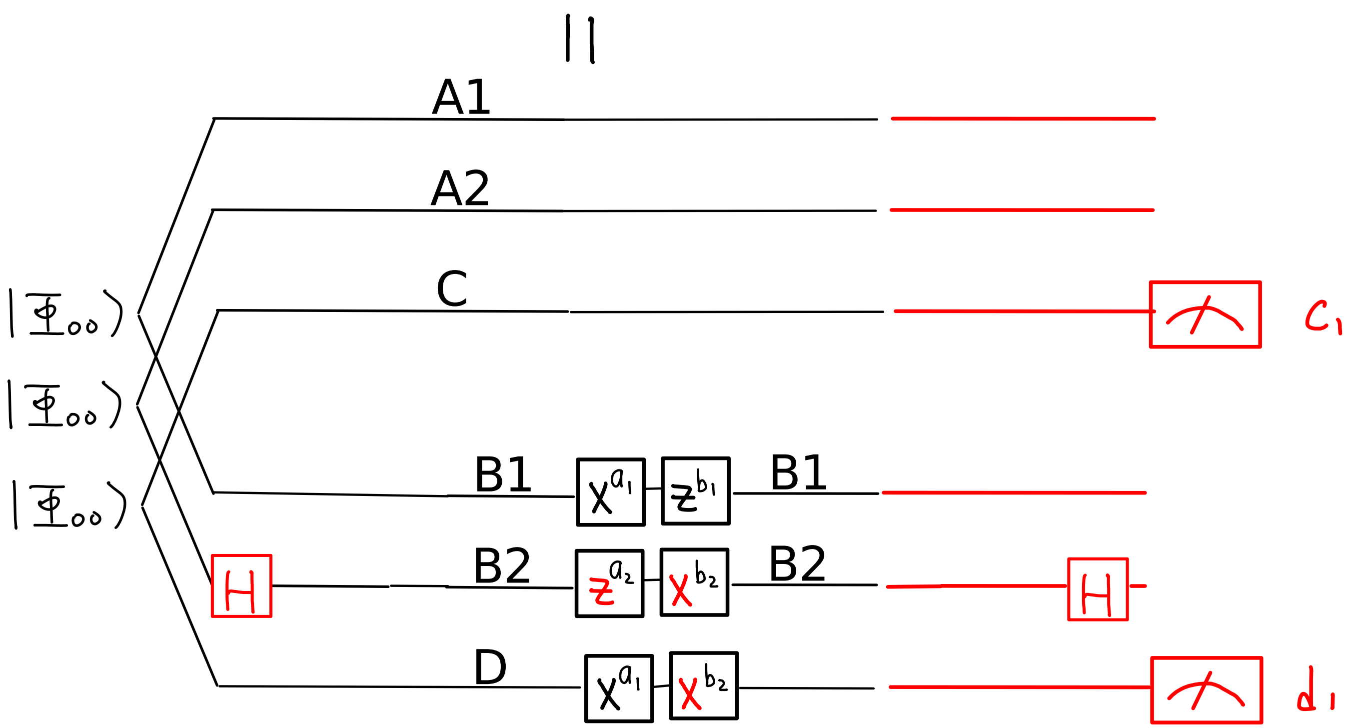
To learn a1+b2 mod 2:



combining how to copy ai and bi (as X's) to CD

transpose trick


A1
A2
C
B1
B2
D
H
B1
B2
c1
d1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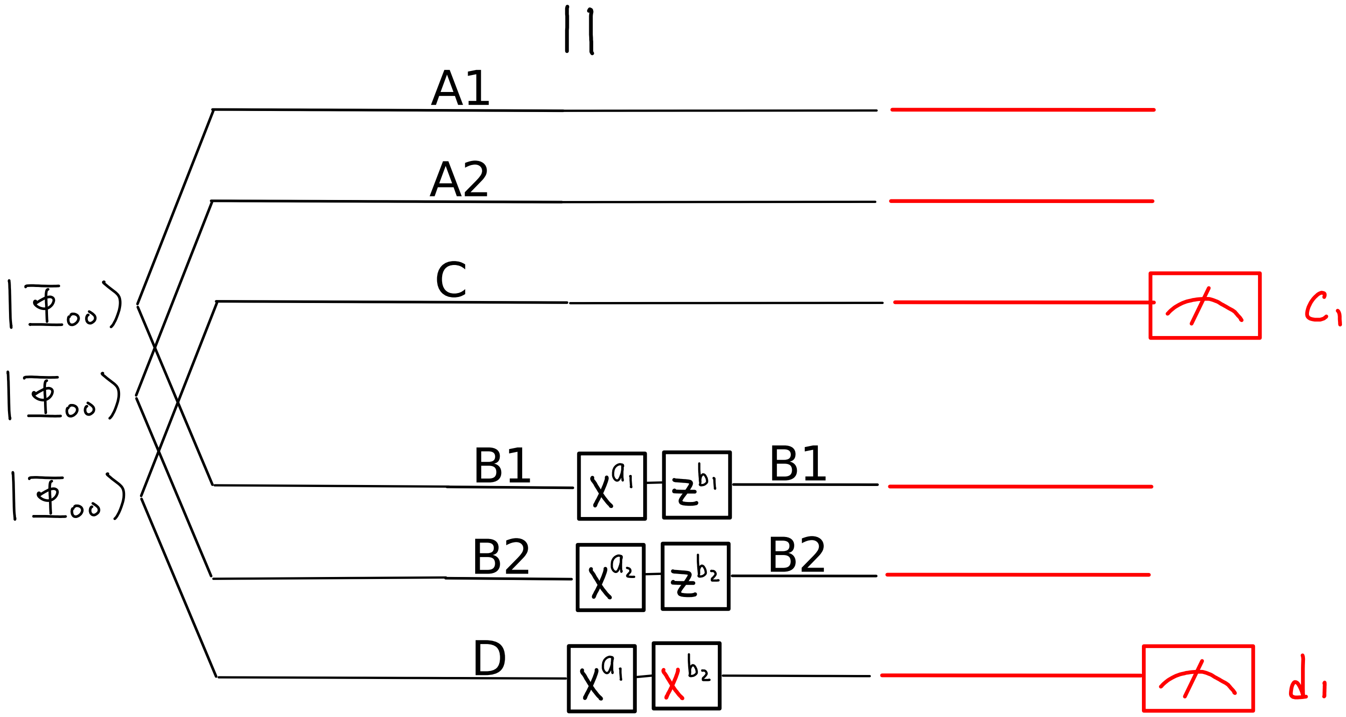

Again, A1B1A2B2 unchanged, c1+d1=a1+b2 mod 2.

Circuit to learn a1+b1:

What circuit should we use?
Both a1, b1 come from the same EPR pair ...

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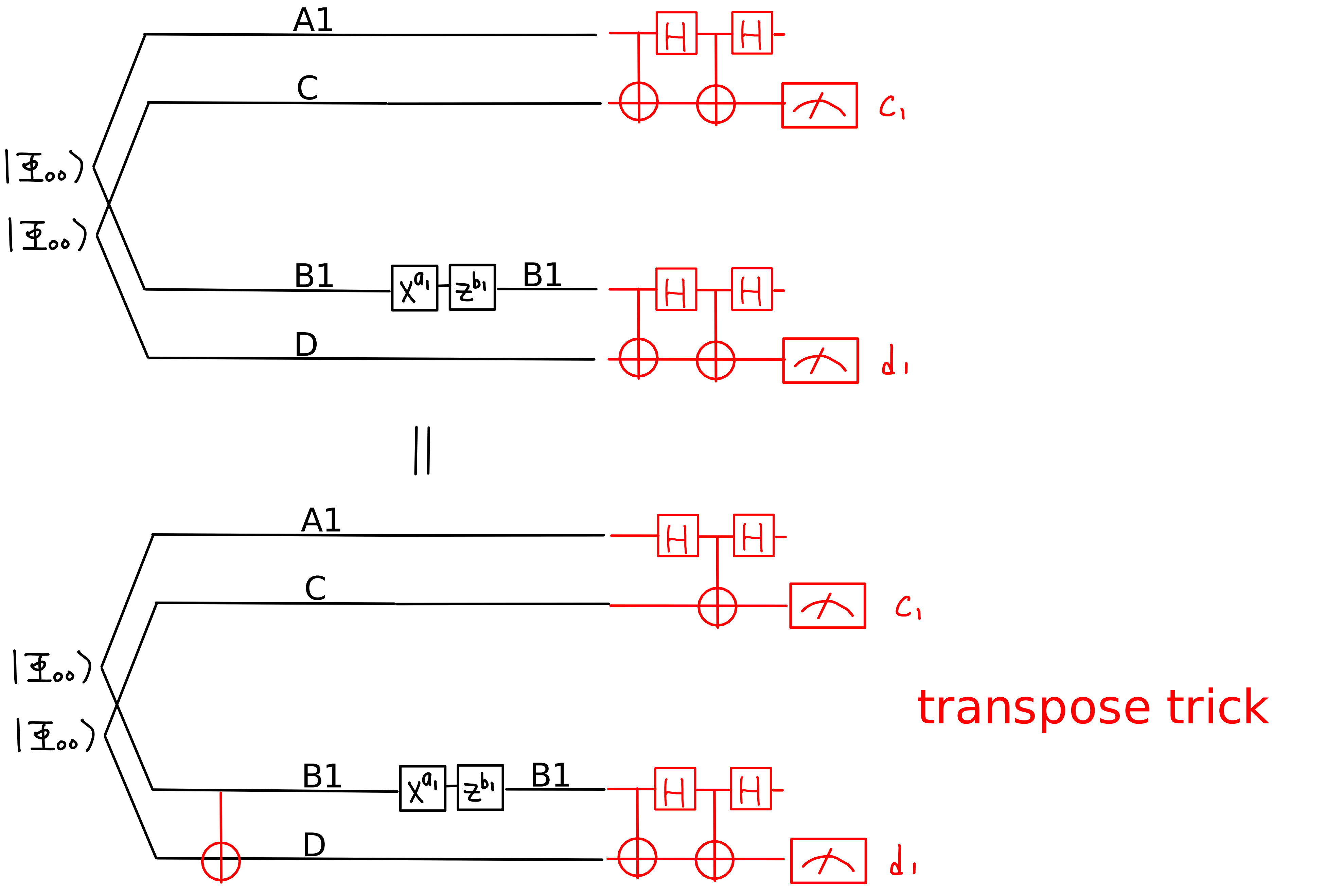

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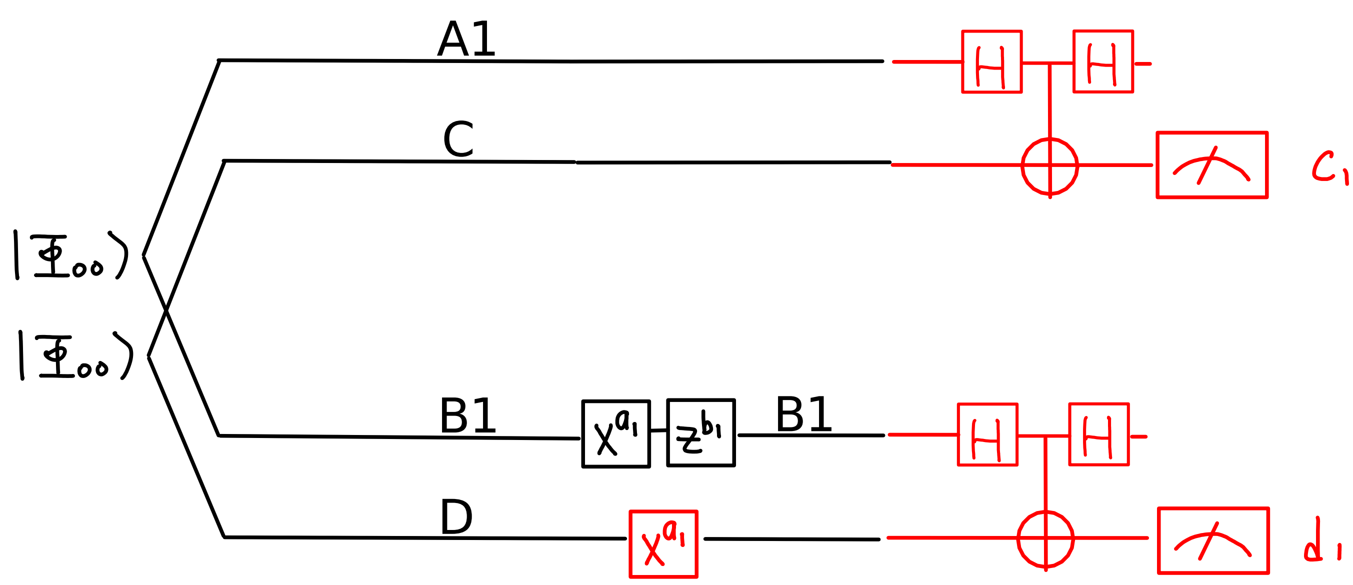

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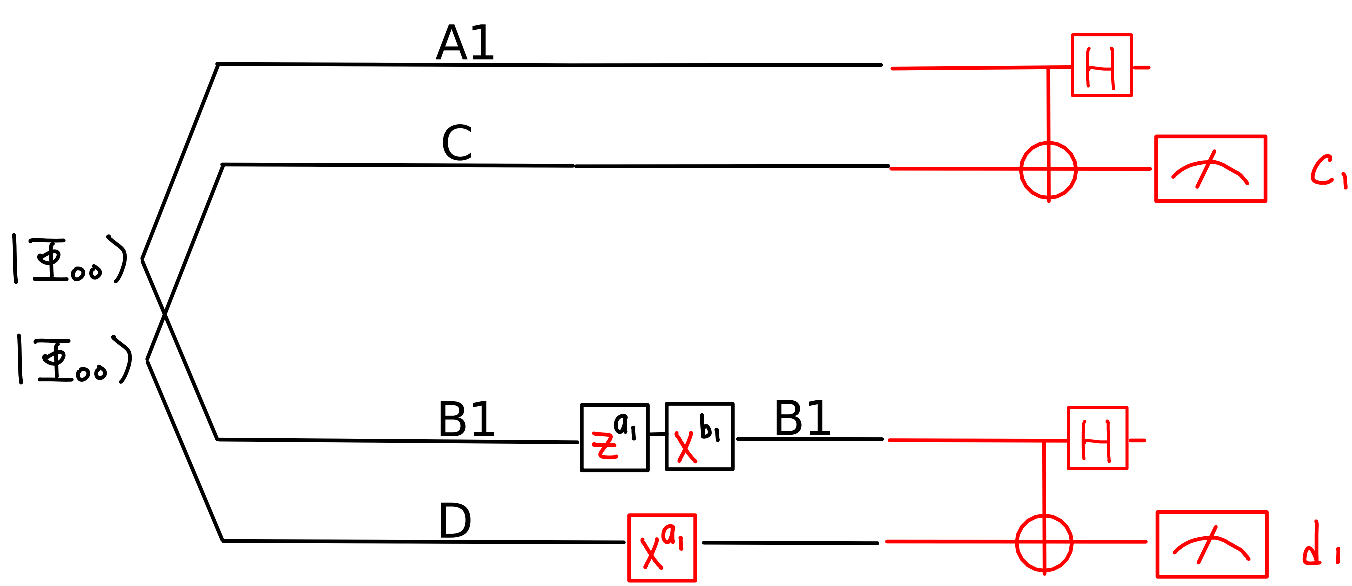

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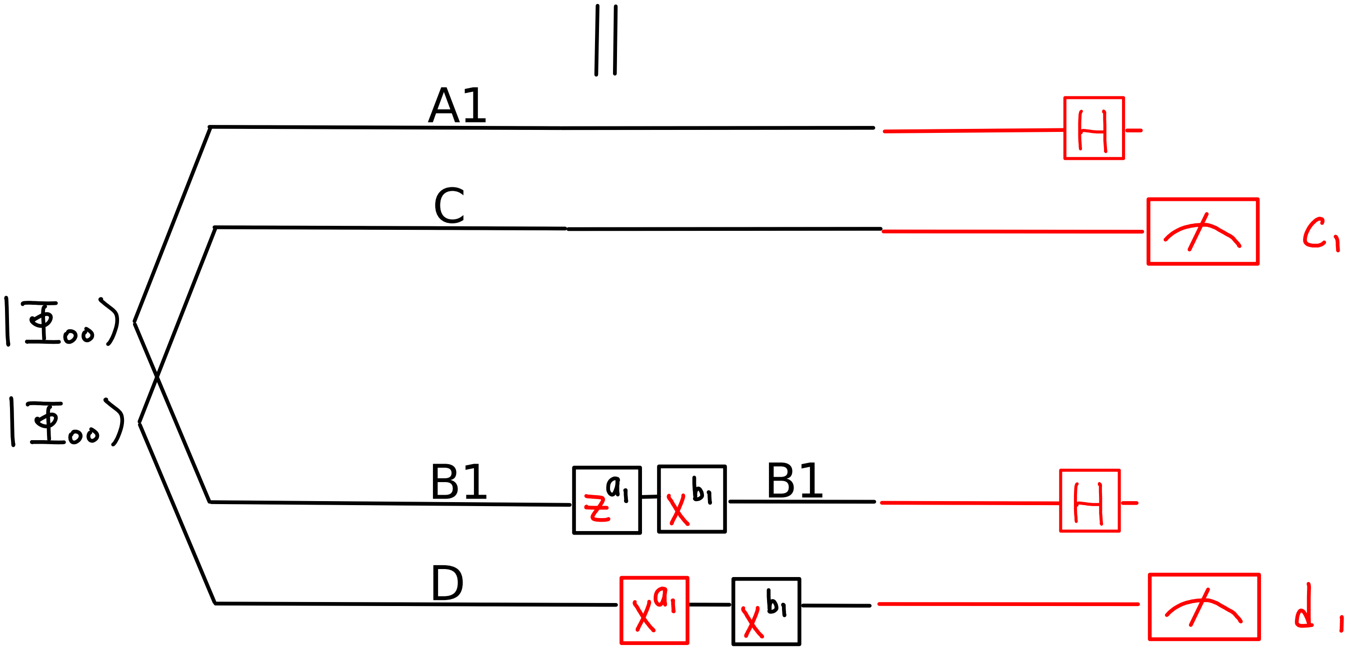

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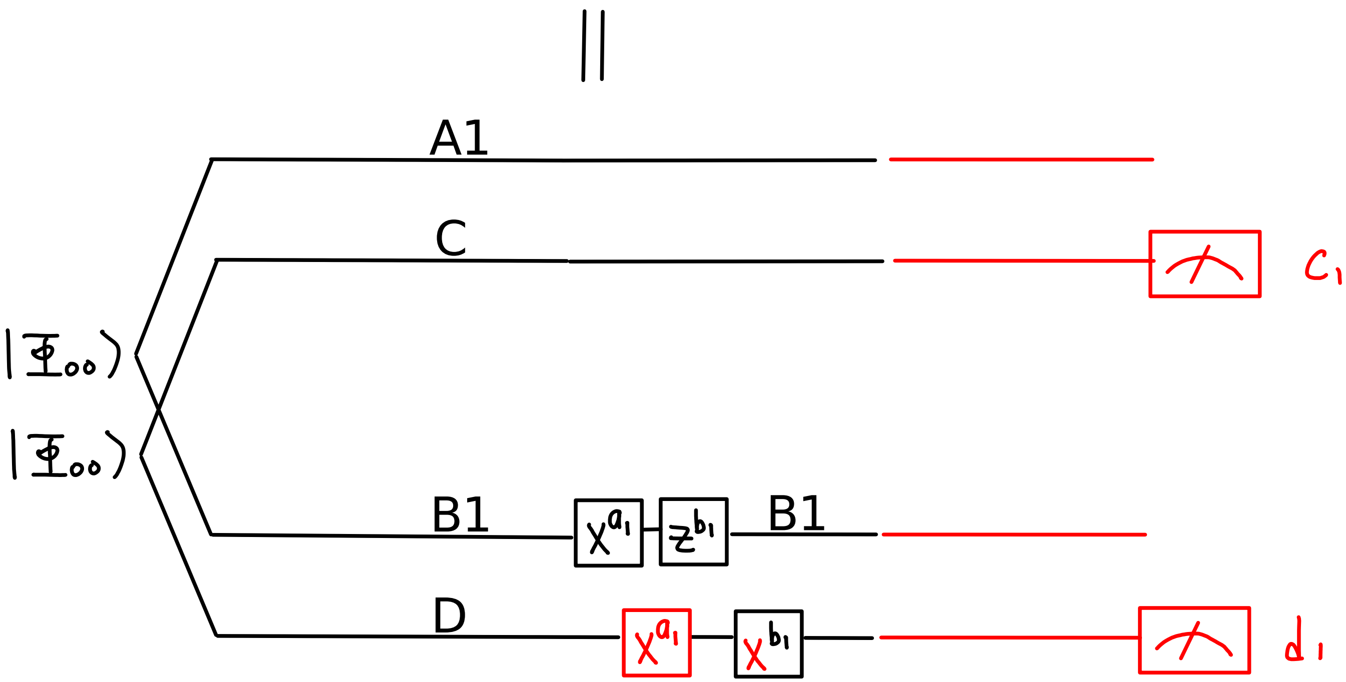

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∴ $a1 + b1 = c1 + d1 \mod 2$

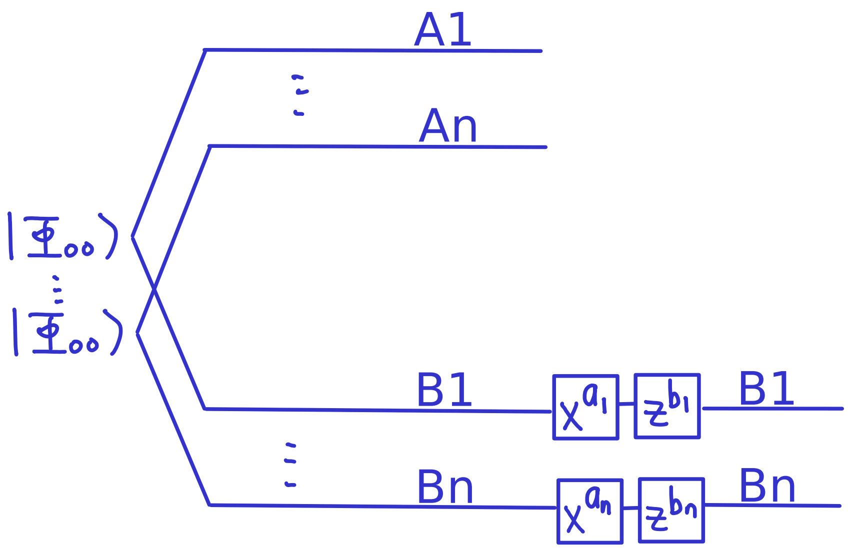
Generalizing these examples give an algorithmic proof for the lemma:

Lemma: with 1 copy of $|\Phi_{00}\rangle$ & CC, Alice and Bob can learn the parity of any subset in $a_1 b_1 a_2 b_2 \ldots a_n b_n$.

We can return to the theorem:

Stop continuous view ...

Theorem: suppose Alice and Bob share the following:



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There is a test that consumes k copies of $|\Phi_{00}\rangle$ uses classical communication, without changing the above state, such that

$$\Pr(a_1 b_1 \cdots a_n b_n \neq 00 \cdots 00 \text{ and test passes}) \leq \frac{1}{2^k}$$

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For all u, wp 1/2, parity(S) = 1.

Thus a random subset parity will detect any nontrivial Pauli error on the n EPR pairs wp 1/2.

Can amplify this detection ability by repeating ...

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Alice and Bob pick k random subsets S1, S2, ... , Sk, independently, and find their parities. They pass the test iff all the subset parities are even.

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$$\Pr(a_1 b_1 \cdots a_n b_n \neq 00 \cdots 00 \text{ and test passes})$$
$$\leq \Pr(\text{test passes} \mid a_1 b_1 \cdots a_n b_n \neq 00 \cdots 00) = \frac{1}{2^k}$$

If all the subset parities are even, Alice and Bob conclude that they hold n noiseless EPR pairs $|\Phi_{00}\rangle$.
They return k pairs to "the bank" and harvest n-k pairs which can be measured to give them n-k bits of key.

This gives an information theoretic security proof of E91 using an insecure noiseless quantum channel.

On borrowing and return k EPR pairs from the bank -- this is not necessary. Also, an evil Eve can jam the quantum channel and "bankrupt" Alice and Bob causing them to be in serious entanglement debt :)

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2. Repeat for n-1 EPR pairs
3. Repeat for n-2 EPR pairs
...
until n-k pairs remaining.
The state changes, more messier algebra ... but it works.

Will keep our discussion clean with the catalytic approach.

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Fact: noiseless copies of $|\Phi_{00}\rangle$ are necessarily secure; doesn't matter how the noise get there, once we get rid of it.

Correct intuition: if the EPR pairs are noiseless, then they're in a pure state so no one else has correlations with the state so measuring will give a private key!

We reduce the security of E91 using noisy channels to the ability to correct Pauli errors (an upgrade from the security of E91 using noiseless channel via the ability to detect Pauli errors).

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If Alice and Bob have characterized the channel,
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So, there are $\approx \binom{n}{n p_x} \binom{n}{n p_z} \approx 2^{n h(p_x)} \, 2^{n h(p_z)}$ such $a_1 b_1 \cdots a_n b_n$.

$h(p_x) = -p_x \log p_x - (1-p_x) \log (1-p_x)$

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So, there are $\approx \binom{n}{np_x}\binom{n}{np_z} \approx 2^{nh(p_x)}\, 2^{nh(p_z)}$ such $a_1 b_1 \cdots a_n b_n$.

$h(p_x) = -p_x \log p_x - (1-p_x)\log(1-p_x)$

Each random subset parity gives roughly 1 bit of info.

$\therefore\ nh(p_x) + nh(p_z)$ parities (+ a little more) identify the error.

(Syndrome of a random stabilizer code !)

Requires a bit more analysis, but learning subset parities is an efficient way to learning bit-strings.

Finally, Alice and Bob correct the identified error, and run the k parity checks as in the noiseless case, s.t.:

$$\Pr\left(\text{output } Q\,|\Phi_{00}\rangle^{\otimes n(1-h(p_x)-h(p_z))} \text{ and test passes}\right) \leq 2^{-k}.$$

any nontrivial Pauli error

key rate

Entanglement is awesome!

It can be reliably tested by two remote parties.

Many testing methods have been developed.

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e.g., self-testing means even the operations by Alice and Bob to test EPR pairs need not be trusted! Recall nonlocal games from topic 4. Some are "rigid": if the observed correlation is close to max allowed by QM, the shared state and local operations *must be of a certain form* and some games (CHSH, magic square game) can be verified to be like E91 with measurement outcomes giving secure key.

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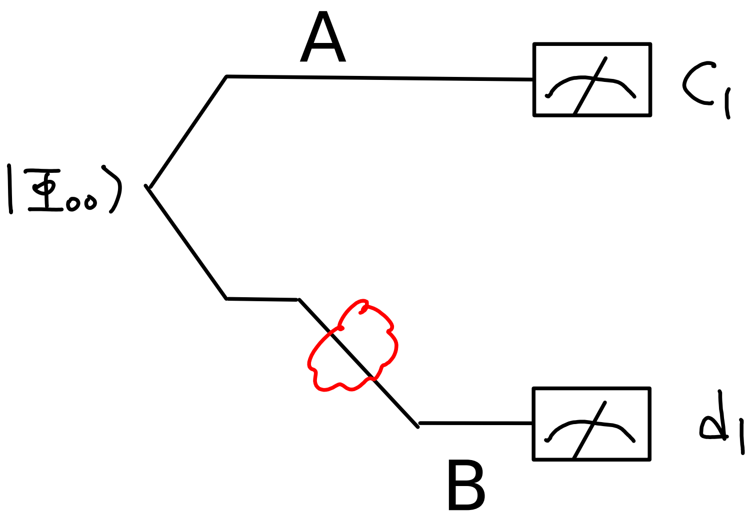
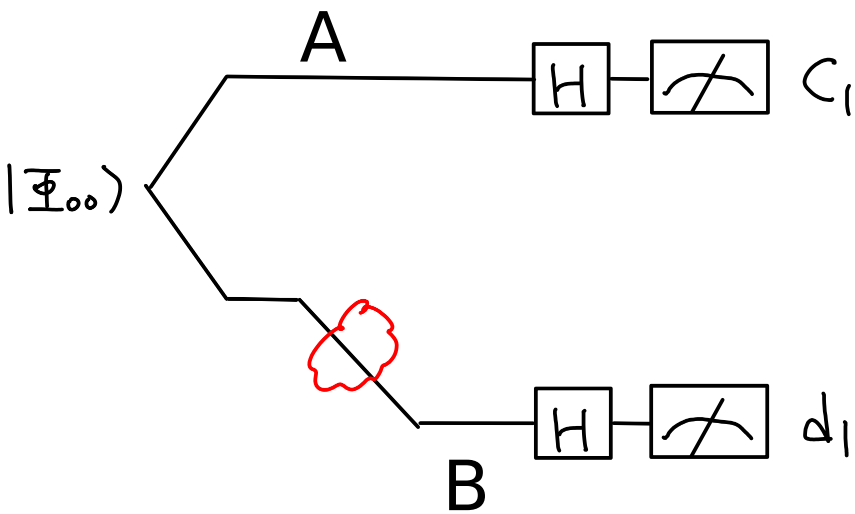
BB84 is much easier to implement.

How do we relate these 2 protocols?

Recall bit commitment, how Alice can measure entangled state to create random bits in {|0>,|1>} or {|+>,|->} basis.

E91 : wp 1/2 do each of the following





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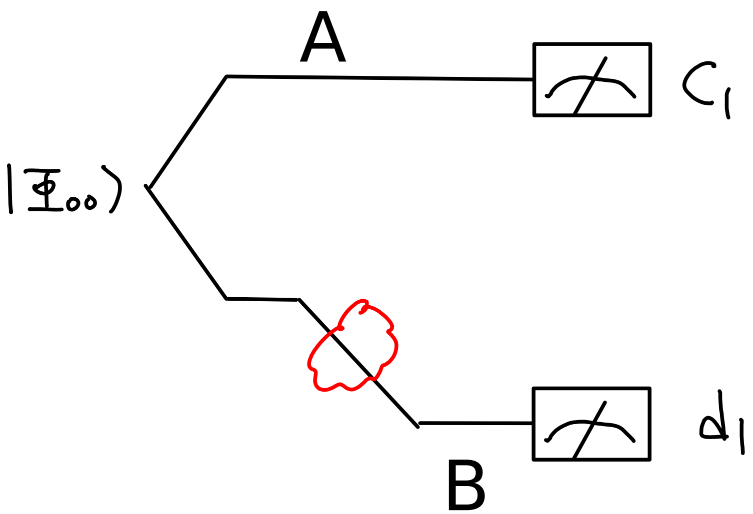
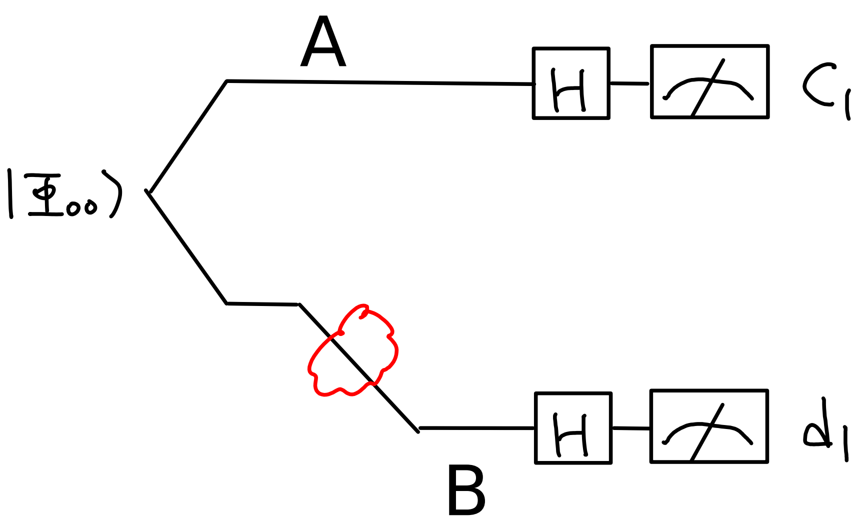
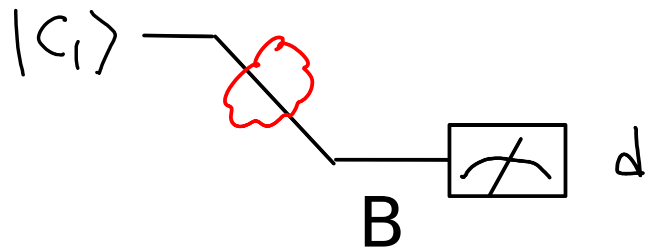




Operations on A commute with the sending. If Alice measures first, there is no observable diff, & we get:



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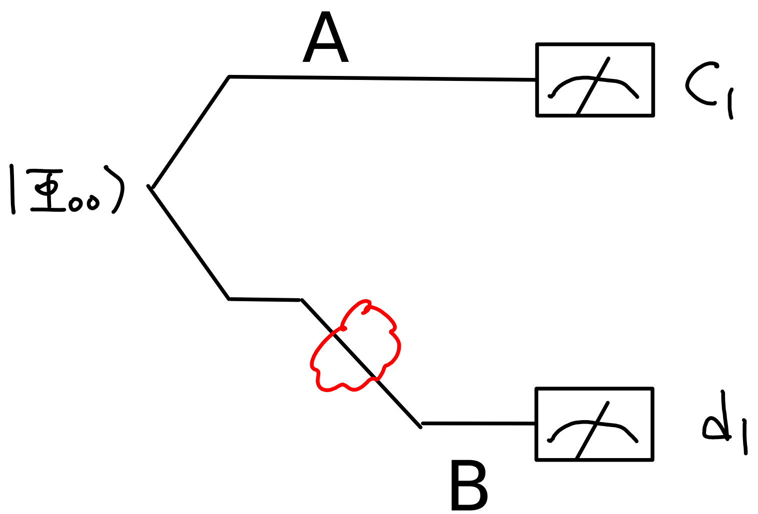

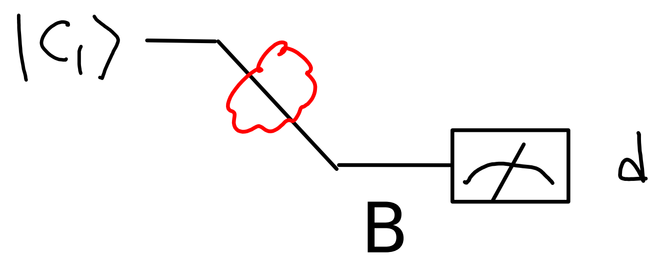

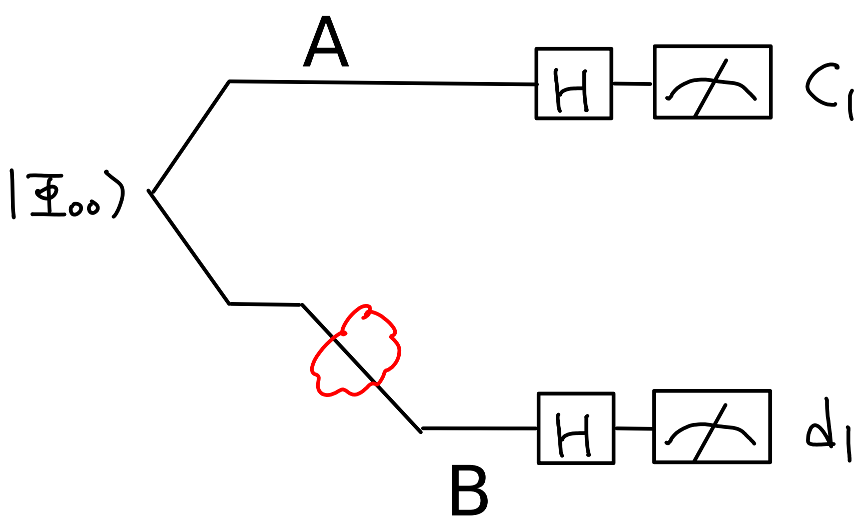

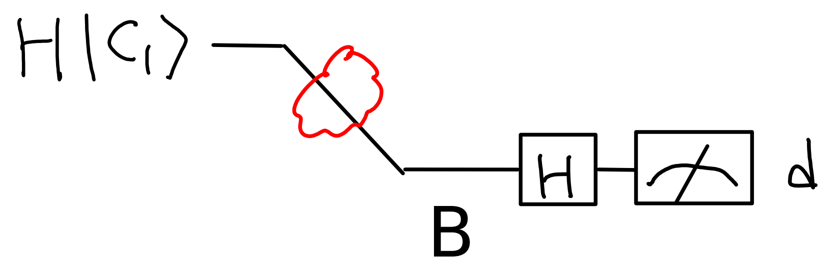

BB84!

Harder to transform the security proof from E91 (with random subset parity checks) to BB84 (with error correction and privacy amplification).

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* X error correction (by Z generators) corresponds to classical error correction, and Z error correction (by X generators) corresponds to privacy amplification.

Prof. Thomas Jennewein's experiment has Alice riding on a plane and Bob sitting on the ground. Each pass generates 100 k bits of key!

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It's nice some version of QKD has been realized and we do not need to wait for 20 or 30 years ...