CO481/CS467/PHYS467 Linear algebra test

Due Tuesday January 14, 2025, 8:30am

Instructions:

Please submit your solutions to Crowdmark by the due date and time. Take special care to place the answer to each question in the right place. Show your steps clearly, give brief justifications if appropriate. This test is self-administered, open-book, and has no time limit. It is also a test run for using Crowdmark. You can typeset or legibly hand-write your solutions.

The goal is for the students to assess their readiness for the course. The questions are *representative* of the type of linear algebra required for the course; they were taken from the W2018 lectures. The solutions also indicate the level of detail and rigor required for your assignments and exams. Your test score will not contribute to your course total. We grade your test to help with your self-assessment.

Question 1. Basis change for a Hamiltonian evolution [4 marks]

(a) [1 mark] Write down the power series expansion for the exponentiation of a $d \times d$ matrix.

(b) [3 marks] Let H be a (bounded) $d \times d$ hermitian matrix, U a $d \times d$ unitary matrix, and $t \in \mathbb{R}$. Show that $e^{-iUHU^\dagger t} = Ue^{-iHt}U^\dagger$.

(This question means that, if your laboratory has the ability to evolve a system according to the Hamiltanian H, and to perform $U, U^\dagger$, it also has the ability to evolve a system according to another Hamiltonian $K = UHU^\dagger$, which is the original Hamiltonian H in a different basis.)

Question 2. The Bloch vector [6 marks]

Consider an arbitrary vector in the form

$$\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \in \mathbb{C}^2.$$

Recall that this can be written in the Dirac notation as $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$.

(a) [2 mark] Express $|\psi\rangle\langle\psi|$ explicitly as a 2×2 matrix with complex entries, and show that it is hermitian.

(b) [4 marks] Recall that the Pauli matrices are given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find the parameters a, b, c, d (in terms of ϕ, θ) such that $|\psi\rangle\langle\psi| = aI + bX + cY + dZ$. Note that $a, b, c, d \in \mathbb{R}$.

You can use without proof the facts that:
1. $\forall M, N \in \{I, X, Y, Z\}$, $M \neq N$, it holds that $\mathrm{tr}(MN) = 0$,
2. $\forall M \in \{I, X, Y, Z\}$, $\mathrm{tr}(MM) = 2$.
That is, $\{I, X, Y, Z\}$ is a rescaled basis for 2×2 hermitian matrices.

Question 3. Eigenspaces of tensor products of Pauli matrices [5 marks]

The Pauli matrices I, X and Z are as defined in Question 2.

(a) [1 mark] Let $e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the respective eigenvectors of Z corresponding to the eigenvalues $+1$ and -1. Denote the $+1$ and -1 eigenvectors of X as f_0 and f_1. Express f_0 and f_1 in terms of e_0 and e_1.

(b) [2 marks] Write down a basis for the $+1$ eigenspace of $X \otimes Z \otimes X$ (where the tensor product of matrices is as defined in class). Your answer should be in terms of e_0, e_1, f_0, f_1.

(c) [2 marks] Write down a basis for the nullspace of $I \otimes I \otimes I + X \otimes Z \otimes X$. Your answer should be in terms of e_0, e_1, f_0, f_1.