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COMPLEX BASES AND FRACTAL SIMILARITY*

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Résumé

Les représentations des nombres complexes de base complexe produisent plusieurs examples d'ensembles fractals, tel la courbe jumelle du dragon. Ces ensembles comprennent les nombres complexes dont la partie entière est égale à zéro dans une base donnée. Les frontières de ces ensembles consistent en des nombres complexes ayant deux expansions différentes dans la base donnée. Les ensembles eux-mêmes possèdent une homothétie interne, mais non leur frontière; cependant les frontières possèdent la propriété d'être partiellement semblables à elles-mêmes. Le concept d'autosimilitude partielle est défini en termes généraux et plusieurs exemples sont donnés. On peut mieux formuler cette autosimilitude partielle au moyen d'un graphe orienté. Nous montrons comment construire certains de ces graphes à partir de bases complexes. On peut obtenir, grâce à l'ordinateur, des approximations des frontières des ensembles dérivés à partir de ces base en utilisant des chemins dans le graphe correspondant. La matrice de transition du graphe permet de calculer la dimension d'homothétie de la frontière.

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AMS Subject Classifications: 11A63, 11K55, 28A75, 58F11.

Introduction

Radix representations of complex numbers, using complex bases, yield many examples of fractal sets, such as the space-filling twin dragon. These sets consist of the complex numbers with zero integer part in a given base. The boundaries of these sets consist of complex numbers having two different radix expansions in the given base. The sets themselves are strictly self-similar, but their boundaries are not; however the boundaries have a kind of partial self-similarity. The concept of partial self-similarity is defined in general, and various examples are given. This partial self-similarity can best be expressed in terms of a directed graph. It is shown how to construct some of these directed graphs from complex bases. Approximations to the boundary of the sets derived from complex bases can easily be drawn by a computer, using paths in the corresponding directed graph. The similarity dimension of the boundary can be calculated from the transition matrix of the directed graph.

1. Complex bases

The complex number z is said to be *represented in the base* b using the *digit set* $\mathcal{D}$ if

$$z = \sum_{r=-\infty}^{N} a_r b^r \quad \text{where} \quad a_r \in \mathcal{D}.$$

This is denoted, in radix notation, by $z = (a_N \ldots a_1 a_0 \cdot a_{-1} a_{-2} \ldots)_b$. The number $(a_N \ldots a_1 a_0)_b$ is called the *integer part* of the representation. The set $\mathcal{D}$ is called a *feasible digit set* for the Gaussian integer base b, if every complex number can be represented in the base b using the digit set $\mathcal{D}$, and if every Gaussian integer has a unique representation using only the integer part.

For example, $\{0,1\}$ is a feasible digit set for the base $-1+i$ and so provides a binary representation of all the complex numbers. In this representation $-4-i = (10111)_{-1+i}$ and $(-7+4i)/5 = (1001.\overline{01})_{-1+i}$, where the digits under the bar are to be repeated indefinitely. The base $-3+i$ gives a decimal representation of all the complex numbers using the usual digit set $\{0,1,2,\ldots,9\}$. These are examples where the digits are natural numbers. The only bases for which there exist feasible digit sets of natural numbers are of the form $-n\pm i$, for positive n. (See [9] and also [5].)

There are also many more examples when the digits are allowed to be complex. Some criteria for such digit sets are given in [1]. The digit set must be a complete residue system modulo the base, and the number of digits must be the square of the modulus of the base. For example, $\mathcal{D} = \{0,\pm 1,\pm i\}$ is a feasible digit set for the base $2+i$.

For any base b with feasible digit set $\mathcal{D}$, consider the set S of complex numbers, whose integer part is zero. This set often has a fractal boundary. For example, the set derived from the binary representation using base $-1+i$ is the space-filling twin dragon curve (see [5]). However, with the base 3 and digit set $\{0,\pm 1,\pm i,\pm 1\pm i\}$, the set S, of numbers with zero integer part, is just the unit square centred at the origin.

In general, this set S, of numbers with zero integer part in the base b, is a closed set of unit area in the complex plane. The set consisting of the complex numbers with integer part Q in the base b is a translate of S by the Gaussian integer Q. Since every complex number is representable, the translates of S by the Gaussian integers tile the whole of the complex plane (see [5; Figure 7]). Points on the boundary ∂S of S are complex numbers that have two representations, with different integer parts in the given base, one integer part being zero [6]. For example, we shall see that $(2+i)/10$ has the two representations $(0.00\overline{001})_{-1+i}$ and $(1.11\overline{010})_{-1+i}$ and so lies on the boundary ∂S for the base $-1+i$.

2. Partial self-similarity

The set S, of complex numbers with zero integer part, is self-similar because it is the union of m similar copies of itself, where $m = \mathrm{Norm}(b) = \#\mathcal{D}$. The m similitudes are given by $f_a(z) = (z+a)/b$, for $a \in \mathcal{D}$.

However the boundary ∂S is not self-similar but does have a certain kind of similarity. For example, the boundaries considered in [7] can be built up from three parts, each of which is a union of various contracted copies of these parts.

We call a subset K of R^n *partially self-similar* if there are sets $K_1,\ldots,K_t$ such that

$$K = \bigcup_{i=1}^{u} K_i$$

and, for each K_i, there are contraction maps φ_{ijk} (for $i = 1,\ldots,t$; $j = 1,\ldots,t$; $k = 1,\ldots,w(i,j)$ with $w(i,j) > 0$) such that

$$K_i = \bigcup_{j,k} \varphi_{ijk} K_j .$$

This definition of partial self-similarity is related to the recurrent sets of Dekking [2], the partially self-similar curves of Dubuc [4] and the transfer matrix of the fractal of Mandelbrot, Gefen, Aharony and Peyrière [10].

This partial self-similarity can be represented by means of a directed graph, such as that shown in Figure 1. The nodes of the graph correspond to the sets $K_1,\ldots,K_t$. Each K_i is a union of contracted copies of the K_j's and there is one directed edge from K_i to K_j for each contraction map φ_{ijk}. Note that the contraction φ_{ijk} maps K_j to K_i but that the arrow goes from the node K_i to the node K_j.

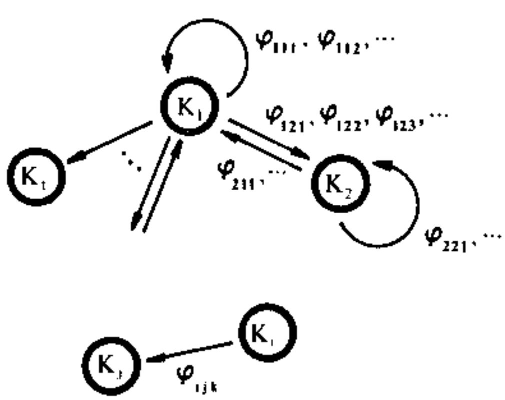


Figure 1. The directed graph for a partial self-similarity.

If the directed graph has only one node then the set is strictly self-similar.

Suppose that each contraction map φ_{ijk}, in a partial self-similarity, is a similitude with ratio r_{ijk}. Suppose also that the set of similitudes satisfies an open set condition, such as in [8] and [11], so that the subsets K_i do not overlap too much. Then, as in [3; p. 9], a similarity argument shows that the similarity dimension of the set K is the largest value of x such that the $t \times t$ matrix, whose (i,j)th element is $\sum_k r^x_{ijk}$, has 1 as an eigenvalue.

If all the similitudes have the same ratio, so that $r_{ijk} = r$ for all i, j and k, then this similarity dimension is $\log \lambda/\log(1/r)$, where λ is the dominant eigenvalue of the matrix whose (i,j)th element is N_{ij}, the number of edges from the node K_i to the node K_j. This matrix (N_{ij}) is the transition matrix of the directed graph and is called the transfer matrix of the fractal in [10].

Hutchinson [8] shows that a closed and bounded self-similar set is uniquely determined by its set of contraction maps. Is it true that a closed and bounded partially self-similar set K is uniquely determined by its directed graph and the contraction maps φ_{ijk} that correspond to the edges? Furthermore, is each subset K_i the closure of the fixed points of the compositions of contraction maps that correspond to closed paths in the directed graph that start and end at the node K_i?

A simple example of partial self-similarity is the generalized Cantor set shown in Figure 2. This is a subset of the unit interval. Its directed graph and the corresponding contraction maps that are used in its construction are shown in Figure 3. The edges labelled 0 and 1 in the directed graph correspond to the similitudes $\psi_q(x) = (x+q)/2$, for $q = 0$ and 1, that contract the interval onto its left and right halfs respectively. Hence $\varphi_{111} = \psi_o$ and $\varphi_{121} = \varphi_{211} = \psi_1$.

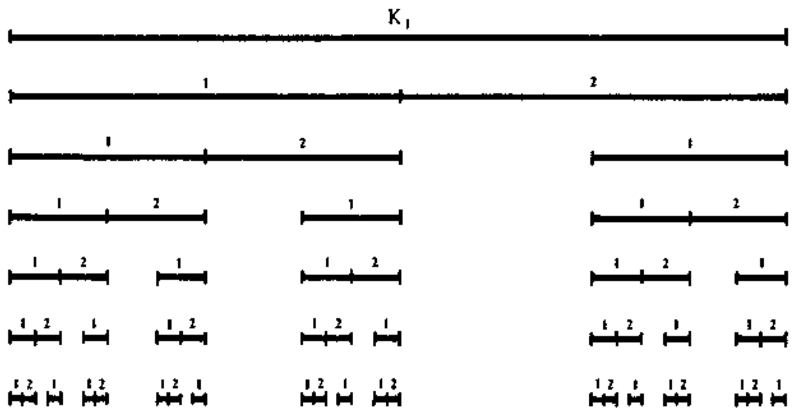


Figure 2. A generalized Cantor set.

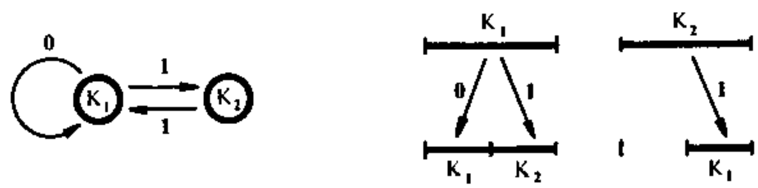


Figure 3
The directed graph and contraction maps of the generalized Cantor set.

Any point in the limit set K has a binary expansion $(0.a_1a_2\ldots)_2$, where $a_1a_2\ldots$ is a sequence of digits from $\{0,1\}$ that correspond to an infinite path in the directed graph, starting from the initial node K_1.

The transition matrix of the graph is $\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$, with eigenvalues $(1\pm\sqrt{5})/2$. All the similitudes have ratio $\frac{1}{2}$ so the similarity dimension of K is $\log \tau/\log 2 \approx 0.6942$, where $\tau = (1+\sqrt{5})/2$.

This example is actually strictly self-similar because

$$K = K_1 = \psi_0(K_1) \cup \psi_1(K_2) = \psi_0(K_1) \cup \psi_1^2(K_1).$$

The similitudes ψ_0 and ψ_1^2 have ratios $\frac{1}{2}$ and $\frac{1}{4}$ respectively and so it follows from [11] that its similarity dimension is the solution to $(\frac{1}{2})^x + (\frac{1}{4})^x = 1$; that is, $\log \tau/\log 2$.

We now present another example of partial self-similarity involving the data structure, used in computer science, called a quadtree. A quadtree is a hierarchical data structure based on recursive decomposition of the data. See [12] for a survey of quadtrees. In computer graphics for example, a region of a square can be defined using a quadtree as follows. Subdivide the square into four equal sized quadrants and note in which of the quadrants the region lies. Subdivide each of the quadrants containing parts of the region into four smaller quadrants and continue the process. The region can be described by means of a tree with at most four leaves at each note; hence the term quadtree. A node at the k^{th} level corresponds to a square in the k^{th} subdivision, and the subtrees leaving the given node correspond to the quadrants containing the region. In practice, in computer science, the region is only subdivided a finite number of times so that the resulting quadtree is finite. However, if the region is an arbitrary geometric set then an infinite quadtree would be required.

If the subtree structures leaving two different nodes are the same then form a new graph by equating these nodes; this new graph will not be a tree. If all such pairs of similar nodes are equated and the resulting graph has only a finite number of dissimilar nodes, then this is a directed graph of a partially self-similar set, whose contractions are all similitudes of ratio $\frac{1}{2}$. Conversely, any directed graph with the property that each node has at most four edges, defined by distinct quadrants, leaving it, defines a partially self-similar region in the square by choosing one node as the initial node for a quadtree.

For example, the directed graph in Figure 4 yields the partially self-similar set in Figure 5. The transition matrix of the graph is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

whose eigenvalues are approximately 3.2143, 1, 0.4608 and −0.6751. Hence the similarity dimension is approximately 1.6845.

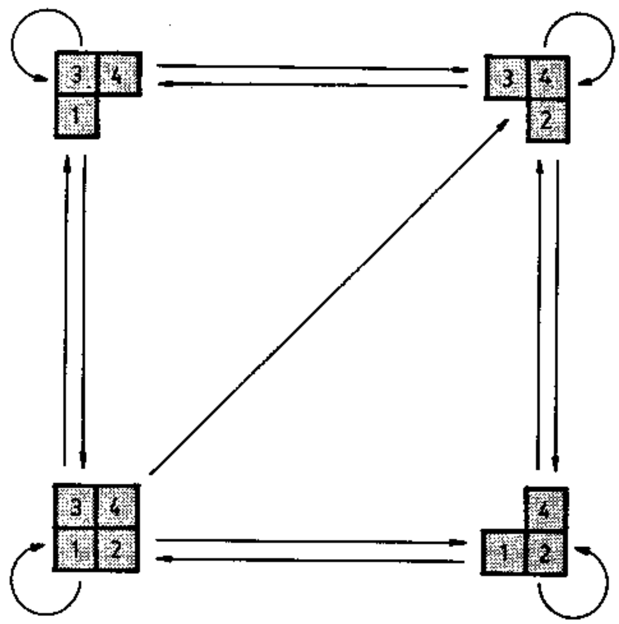
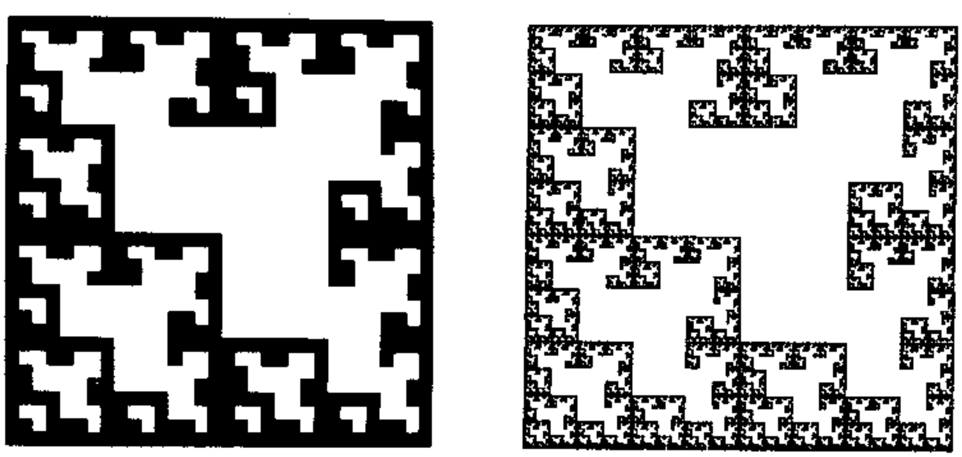


Figure 4. The directed graph of a quadtree.



(a) (b)

Figure 5. A quadtree fractal.

This method can easily be applied to higher dimensions and, in particular, to three dimensional octrees.

3. Fractal sets derived from complex bases

In this section we illustrate a method for finding the fractal boundary of the set of points with zero integer part in a complex base. This procedure is more general than the ad hoc method given in [7]. For each base, a directed graph will be constructed using the points with two radix expansions in that base. This graph determines the partial self-similarity of the boundary and can be used to generate the boundary on a computer, as well as to calculate its similarity dimension.

Let $\mathcal{D}$ be a feasible digit set for a complex base b. Any point on the boundary, ∂S, of the set of complex numbers with zero integer part, has two different expansions in the base b of the form

$$(*) \qquad (0.p_1p_2p_3\ldots p_j\ldots)_b = (Q.q_1q_2q_3\ldots q_j\ldots)_b$$

where $p_j, q_j \in \mathcal{D}$ and $Q \in Z[i]$. The boundary set, ∂S, can be approximated by taking the points in the above expansions to k radix places; that is, by considering points of the form $(0.p_1p_2p_3\ldots p_k)_b$. All the possible expansions for points on the boundary can be obtained by the methods used in [6]. That paper can be modified to find all the numbers with two expansions instead of those with three expansions. For example, Figure 6 is the modification of [6; Figure 5] required to find all the complex numbers with two different expansions in the base $-1+i$ using the digit set $\mathcal{D} = \{0,1\}$.

The labels on the directed edges in Figure 6 refer to the digits p_j and q_j in the j^{th} position of the expansion $(*)$. Each path in the directed graph, starting at the top node, yields two sequences of digits corresponding to two expansions in the base $-1+i$ representing the same complex number. The radix point can be placed at any place in these sequences. Each point on the boundary ∂S corresponds to a pair of sequences in which one integer part is zero and the other is not.

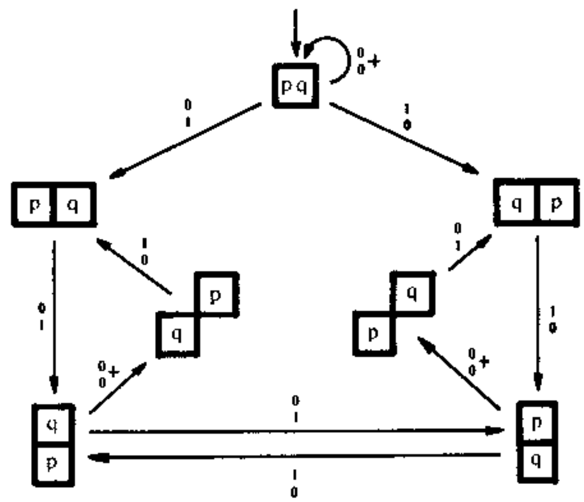


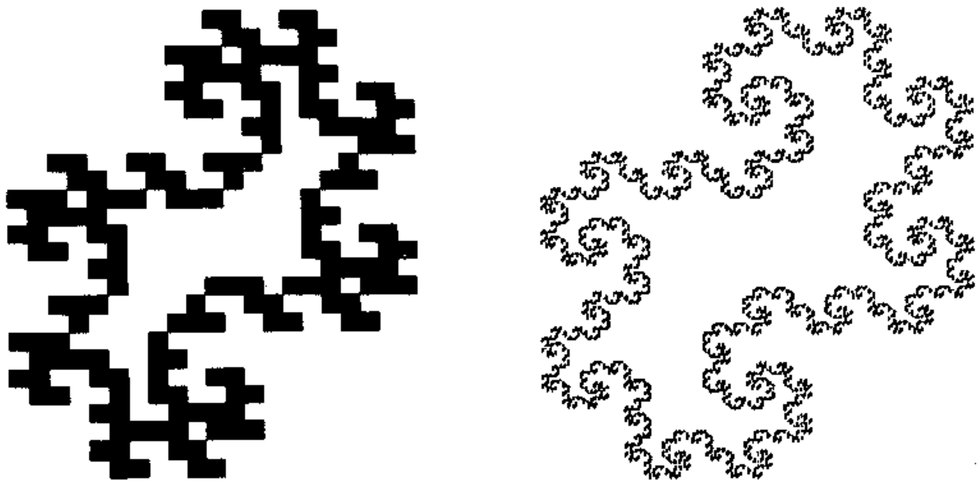
Figure 6. The directed graph for the base $-1+i$.

For example, consider the path that starts at the top and traverses the graph counter-clockwise until it reaches the right-hand triangle; it then loops indefinitely around this triangle. The labels on the edges yield the two sequences

$$\begin{matrix} 0 & 0 & 0 & \overline{0 \quad 0 \quad 1} \\ 1 & 1 & 1 & \overline{0 \quad 1 \quad 0} \end{matrix}$$

where the last three digits are repeated indefinitely. Hence one point on the boundary ∂S is $(0.00\overline{001})_{-1+i} = (1.11\overline{010})_{-1+i}$; another point is $(00000.\overline{100})_{-1+i} = (11101.\overline{001})_{-1+i}$.

The k^{th} approximation to the boundary ∂S can easily be illustrated using a computer. Each expansion, using k radix places in the base b, corresponds to a small square in the complex plane using the grid that is a contraction of the integer grid by the factor b^{-k}. Compute the expansions of all the points on the boundary to k radix places by finding all the paths of length k in the directed graph. For each of these expansions, plot the corresponding square of area $|b|^{-2k}$ in the plane. Figure 7 illustrates the boundary for the base $-1+i$ using approximations to 8 and to 14 radix places.



(a) (b)

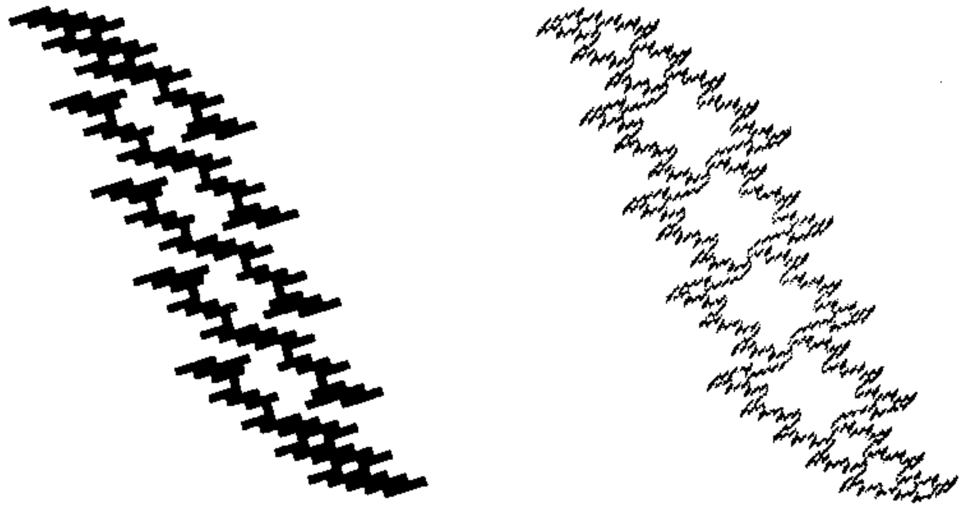
Figure 7
Approximations of the boundary of the set derived from the base $-1+i$.

The directed graph in Figure 6 also yields the partial self-similarity of the boundary. In proceeding from the k^{th} approximation to the $(k+1)^{st}$ approximation, one extra radix place is added to the expansion. Each square in the k^{th} approximation is represented by an expansion $(0.p_1p_2\ldots p_k)_b$ and corresponds to a path of length k in the directed graph, that ends in a certain node K. For each directed edge leaving K there is an expansion $(0.p_1p_2\ldots p_kp_{k+1})_b$ in the $(k+1)^{st}$ approximation. Hence the square in the k^{th} approximation is replaced by a number of smaller squares in the $(k+1)^{st}$ approximation and there is one smaller square corresponding to each directed edge leaving K.

The directed graph for each set derived from the bases $-n+i$ with digits $\{0,1,2,\ldots,n^2\}$ can all be obtained from [6]. Figure 8 shows the approximations of the boundary of the set derived from the base $-2+i$ using 4 and 7 radix places.

The dominant eigenvalue in the transition matrix of the graph obtained from the base $-n+i$ is the largest positive root, λ_n, of

$$\lambda^3 + (2n-1)\lambda^2 - (n-1)^2\lambda - (n^2+1).$$



(a) (b)

Figure 8

Approximations of the boundary of the set derived from the base -2+i.

All the similitudes are obtained from multiplication by b^{-1} and so have a ratio of $\sqrt{n^2+1}$. Hence the similarity dimension of the boundary ∂S is $2 \log \lambda_n / \log(n^2+1)$. In particular, the dimension of Figure 7 is approximately 1.5236 and that of Figure 8 is approximately 1.6087. These results agree with [7].

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