On the Even Permutation Polytope

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Abstract

We consider the convex hull of the even permutations on a set of n elements. We define a class of valid inequalities and prove that they induce a large class of distinct facets of the polytope. Using the inequalities, we characterize the polytope for $n = 4$, and we confirm a conjecture of Brualdi and Liu that, unlike the convex hull of all permutations, this polytope cannot be described as the solution set of polynomially many linear inequalities. We also discuss the difficulty of determining whether a given point is in the polytope.

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Introduction

Let S_n denote the set of permutations of a finite set V of cardinality n. Where $G = (V, E)$ denotes the complete digraph on V (that is, the set E of edges of G is $V \times V$), each element

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σ of S_n can be regarded as a subset K of E. (Namely, $K = \{ij \in E : \sigma(i) = j\}$.) We often regard a permutation to be such a set K. That is, a permutation can be considered to be a set $K \subseteq E$ such that each vertex of the digraph (V, K) has indegree and outdegree equal to 1. Let $x \in \mathbf{R}^E$ be the characteristic vector of an element K of S_n. (We will often use "permutation" as an abbreviation for "characteristic vector of a permutation", and think of a set of permutations as the set of their characteristic vectors.) Then x satisfies

$$\sum_{j \in V} x_{vj} = 1 \quad (v \in V) \tag{1}$$

$$\sum_{i \in V} x_{iv} = 1 \quad (v \in V) \tag{2}$$

$$x_{ij} \geq 0 \quad (ij \in E). \tag{3}$$

A famous theorem of Birkhoff [2] states that the convex hull $P(n)$ of S_n is precisely the set of solutions of the system (1), (2), and (3). ($P(n)$ may be viewed as the set of $n \times n$ doubly stochastic matrices.)

Hoffman (see Mirsky [9]) asked whether there is a similar characterization of the convex hull $Q(n)$ of the even permutations. That is, Hoffman asked whether the polytope $Q(n)$ can, like $P(n)$, be characterized explicitly as the solution set of a system of linear inequalities. Mirsky called the elements of $Q(n)$ *even doubly stochastic matrices*. From the point of view of the digraph G, an even permutation is a permutation K such that the number of components of (V, K) having an even number of edges, is even.

Mirsky gave a family of valid inequalities for $Q(n)$; later, von Below [1] proved that the solution set of this family is equal to $Q(n)$ if and only if $n \leq 3$. Brualdi and Liu [4] proved several results about $Q(n)$. They established its dimension, characterized adjacency of extreme points, and proved that it has diameter 2. They also gave several classes of nonlinear inequalities that must be satisfied by elements of $Q(n)$. Finally, they made the following conjecture, suggesting that $Q(n)$ is much more complicated than $P(n)$.

Conjecture 1. *$Q(n)$ cannot be characterized as the solution set of a system of polynomially many (in n) linear inequalities.*

In this note, we prove Conjecture 1 by explicitly constructing a family of $\frac{1}{2}n(n-1)n!$ linear inequalities, each of which (if $n \geq 5$) induces a distinct facet of $Q(n)$. We show that no inequality in Mirsky's class induces a facet for $n \geq 4$. We also give a complete description of $Q(4)$. Finally, we discuss the difficulty of deciding membership in $Q(n)$.

After this paper was written, we learned of a paper by Hood and Perkinson [6]. It also proves Conjecture 1. Some remarks on the work in that paper can be found after the proof of Theorem 6.

Preliminaries

We recall here a few basic facts from polyhedral theory. More details can be found in Schrijver [11]. The equations (1), (2) are satisfied by every point in $Q(n)$. Since this system of $2n$ equations is easily seen to have rank $2n-1$, the dimension of $Q(n)$ is at most $n^2-(2n-1)=(n-1)^2$. Brualdi and Liu [4] showed that its dimension is exactly $(n-1)^2$, provided that $n \geq 4$. (We will generally assume $n \geq 4$, to avoid some trivial exceptions.) It follows that the solution set of (1), (2) is the affine hull of $Q(n)$.

An inequality $a^Tx \leq a_0$ is *valid* for $Q(n)$ if it is satisfied by every point of $Q(n)$. A *face* of $Q(n)$ is a set of the form $\{\hat{x} \in Q(n) : a^T\hat{x} = a_0\}$ for some valid inequality $a^Tx \leq a_0$ for $Q(n)$. The inequality is said to *induce* the face. An even permutation K is *a-tight* (or just *tight* if a is understood) if it is in the face induced by $a^Tx \leq a_0$. A *facet* of $Q(n)$ is a maximal proper face of $Q(n)$. A face of $Q(n)$ is a facet if and only if it contains an affinely independent set of $(n-1)^2$ even permutations.

Let $Ax = \mathbf{1}$ denote the system (1), (2) of equations. $Q(n)$ is the solution set of a system of the form $Ax = \mathbf{1}$, $A'x \leq b'$ for some A', b'. Any such system must contain an inequality inducing F for every facet F of $Q(n)$ (and need not contain any others). Thus, to show that Conjecture 1 is true, it is enough to exhibit sufficiently many facets of $Q(n)$. Two valid inequalities $a^Tx \leq a_0$ and $b^Tx \leq b_0$ for $Q(n)$ are said to be *equivalent* if they induce the same face. Clearly, $a^Tx \leq a_0$ and $b^Tx \leq b_0$ are equivalent if there exist $\mu \in \mathbf{R}$ with $\mu > 0$ and $y \in \mathbf{R}^{2n}$ such that $(a^T, a_0) = \mu(b^T, b_0) + y^T(A, \mathbf{1})$. It is known that, if $a^Tx \leq a_0$ and $b^Tx \leq b_0$ are facet-inducing, then the converse is true. A *basis* for A is a subset B of E of size $2n-1$ indexing a linearly independent set of columns of A. It is easy to check that, for any $r, s \in V$ (possibly equal) the set $\{rj : j \in V\} \cup \{is : i \in V\}$ is a basis of A. Given a basis B of A and any valid inequality $a^Tx \leq a_0$, there exists an equivalent valid inequality $b^Tx \leq b_0$ such that $b_e = 0$ for all $e \in B$. We say that such an inequality $b^Tx \leq b_0$ is in *B-reduced form*. If $a^Tx \leq a_0$ is facet-inducing, then $b^Tx \leq b_0$ is unique up to multiplication by a positive scalar.

Here is some digraph notation. Let U, W be subsets of V. We write (U, W) to denote $\{ij \in E : i \in U,\ j \in W\}$. For $u \in V$, we may abbreviate $(\{u\}, W)$ to (u, W), and similarly for $(W, \{u\})$. We write $E(U)$ to denote (U, U). For any $J \subseteq E$, let $V(J)$ denote the set $\{v \in V : v \text{ is incident with some } e \in J\}$.

Finally, for a vector $y \in \mathbf{R}^E$ and a subset F of E, we use $y(F)$ to denote $\sum(y_e : e \in F)$. For $y \in \mathbf{R}^E$ and U, W subsets of V, we abbreviate $y((U, W))$ to $y(U, W)$. (To illustrate some of this notation, equation (1) could be written $x(v, V) = 1 \ \ (v \in V)$.)

Mirsky's Class of Inequalities

Mirsky [9] first introduced a class of valid inequalities for $Q(n)$. Until very recently, Mirsky's class was the only known class of valid linear inequalities for $Q(n)$ (other than inequalities that are valid for $P(n)$). This class of inequalities can be described as follows. Let L be an even permutation of V, and let $uv \in L$. Then the Mirsky inequality determined by L and uv is

$$x(L) - 3x_{uv} \leq n - 3. \tag{4}$$

It is easy to see that (4) is valid for $Q(n)$. (If x is the characteristic vector of a permutation K and $uv \in K$, then $x(L) - 3x_{uv} = x(L) - 3 \leq n - 3$. If $uv \notin K$, then $|K \cap L| \leq n - 2$, and equality can hold only if K is odd.) It is also easy to see that, if x is the characteristic vector of an odd permutation of V, then there is an inequality of Mirsky type that it violates. However, as observed in [1], Mirsky's inequalities together with (1), (2), and (3) do not define $Q(n)$ for any $n \geq 4$. We show something stronger here.

Theorem 1 *If $n \geq 4$, no Mirsky inequality is facet-inducing for $Q(n)$.*

Proof. Let $a^T x \leq a_0$ denote the Mirsky inequality determined by the even permutation L and $uv \in L$. Suppose that the even permutation K of V is tight, that is, it satisfies $a^T x \leq a_0$ with equality. If $uv \in K$, then we must have $K = L$. If $uv \notin K$, then we must have $|K \cap L| = n - 3$, so there are three edges $uv, ab, cd \in L \backslash K$. Consider the set $M = L \backslash \{uv, ab, cd\}$. It consists of three directed paths together with a number (possibly zero) of cycles. K is formed by adding to M three edges, none of them uv. It is easy to see that there are exactly two ways to do this. Therefore, since there are $\binom{n-1}{2}$ choices for $\{ab, cd\}$, there are exactly $(n-1)(n-2)+1 = n^2-3n+3$ tight even permutations. The size of a set of affinely independent even permutations in the face induced by $a^T x \leq a_0$ cannot exceed this number, which is smaller (since $n \geq 4$) than $(n-1)^2$. Therefore, $a^T x \leq a_0$ does not induce a facet of $Q(n)$. ∎

A Class of Facet-inducing Inequalities of $Q(n)$

Let t, h be distinct vertices of G, and let R denote $V \backslash \{t, h\}$. Let σ be an even permutation of V. The triple (t, h, σ) determines the following inequality:

$$\sum_{v \in R} x_{v\sigma(v)} + \sum_{v \in R} x_{v\sigma(t)} + \sum_{v \in R} x_{t\sigma(v)} \leq n - 2. \tag{5}$$

This inequality is in the form $x(C) \leq n - 2$ for some $C \subseteq E$ with $|C| = 3n - 6$. It is quite easy to see that for $n \geq 5$, each different choice of (t, h, σ) gives a different set C, and hence a different inequality (5). Since there are $n(n-1)$ choices of t, h and $\frac{1}{2}n!$ even permutations, we get a family of $\frac{1}{2}n(n-1)n!$ different inequalities for $n \geq 5$. For $n = 4$, however, the sets C are not all distinct; in fact, in this case, only 48 different inequalities arise.

Theorem 2 *Inequality (5) is valid for $Q(n)$.*

Proof. Let K be an even permutation of V and let f, g, h denote the values of the three sums in (5) when x is replaced by the characteristic vector of K. Clearly, $f \leq n-2$ and $g, h \leq 1$. It follows that we need only consider the cases in which f is $n-2$ or $n-3$. In the former case, $x_{v\sigma(v)} = 1$ for all $v \in R$, and it follows that $f = g = 0$ and so (5) is satisfied. (We did not need that K is even to make this conclusion.) Now suppose that $f = n-3$. Then there is a unique $u \in R$ such that $x_{u\sigma(u)} = 0$. If the inequality is violated, then $f = g = 1$, from which it follows that $x_{u\sigma(t)} = x_{t\sigma(u)} = 1$. Therefore, K must be the permutation obtained from σ by multiplying it by the transposition (ut), and so K is odd, a contradiction. ∎

Theorem 3 *If $n \geq 4$, inequality (5) induces a facet of $Q(n)$.*

Let σ be a fixed even permutation of V. For any even permutation π of V with characteristic vector $z \in \mathbf{R}^E$, it is easy to see that the vector y defined by $y_{ij} = z_{i\sigma(j)}$ is the characteristic vector of $\sigma \circ \pi$, the permutation mapping i to $\sigma(\pi(i))$. Moreover, the transformation that takes z to y is linear and invertible. (Namely, σ induces a permutation of E, and the corresponding $E \times E$ permutation matrix is invertible.) These observations are very useful for transforming valid inequalities.

Lemma 4 *Let σ be an even permutation of V, let $a^T x \leq a_0$ and $b^T x \leq b_0$ be valid inequalities for $Q(n)$, and define $\hat{a}$ and $\hat{b}$ by $\hat{a}_{ij} = a_{i\sigma(j)}$ and $\hat{b}_{ij} = b_{i\sigma(j)}$. Then*

(a) $\hat{a}^T x \leq a_0$ is a valid inequality for $Q(n)$;

(b) $\hat{a}^T x \leq a_0$ is facet-inducing if and only if $a^T x \leq a_0$ is;

(c) $a^T x \leq a_0$ is equivalent to $b^T x \leq b_0$ if and only if $\hat{a}^T x \leq a_0$ is equivalent to $\hat{b}^T x \leq b_0$.

Proof. Let z be the characteristic vector of an even permutation π of V, and let y be the characteristic vector of the (even) permutation $\sigma \circ \pi$. Then

$$\hat{a}^T z = \sum_{ij \in E} a_{i\sigma(j)} z_{ij} = \sum_{i \in V} a_{i\sigma(\pi(i))} = \sum_{ij \in E} a_{ij} y_{ij} \leq a_0,$$

proving (a). Moreover, if y satisfies $a^T x \leq a_0$ with equality, then z satisfies $\hat{a}^T x \leq a_0$ with equality. Therefore, the set of (characteristic vectors of) even permutations satisfying $a^T x \leq a_0$ with equality is mapped by an invertible linear transformation to a set of even permutations satisfying $\hat{a}^T x \leq a_0$ with equality. It follows that $a^T x \leq a_0$ is facet-inducing if and only if $\hat{a}^T x \leq a_0$ is facet-inducing, proving (b). Now suppose that $F, H, \hat{F}, \hat{H}$ are the faces induced by $a^T x \leq a_0$, $b^T x \leq b_0$, $\hat{a}^T x \leq a_0$, and $\hat{b}^T x \leq b_0$, respectively. Then there is an invertible linear transformation that maps F to $\hat{F}$ and also maps H to $\hat{H}$. Therefore, $F = H$ if and only if $\hat{F} = \hat{H}$, proving (c). ∎

Proof of Theorem 3.
In view of part (b) of Lemma 4, it will be enough to prove the result for the case of inequality (5) in which σ is the identity permutation. So we want to prove that

$$\sum_{v\in R} x_{vv} + \sum_{v\in R} x_{vt} + \sum_{v\in R} x_{tv} \leq n-2 \tag{6}$$

induces a facet of $Q(n)$.

We denote the inequality (6) by $a^T x \leq a_0$. Suppose that the face induced by $a^T x \leq a_0$ is contained in the face induced by the valid inequality $b^T x \leq b_0$, where $b \neq 0$. We will show that this containment cannot be proper, and hence that $a^T x \leq a_0$ induces a facet. Notice that $a^T x \leq a_0$ is B-reduced with respect to the basis $B = (h, V) \cup (V, h)$ of A. We may assume that $b^T x \leq b_0$ is also B-reduced. Since the face induced by $a^T x \leq a_0$ is contained in the face induced b y $b^T x \leq b_0$, every a-tight even permutation is also b-tight. Therefore, if we have two a-tight permutations, then we get an equation involving the components of b. Using this repeatedly we will show that $b^T x \leq b_0$ is a positive multiple of $a^T x \leq a_0$. It is convenient to use J to denote $\{vv : \ v \in R\}$.
Claim 1. $b_{vt} = b_{tv} = b_{vv} + b_{tt}$ for all $v \in R$.
Proof of Claim 1. The permutations $J \cup \{tt, hh\}, (J\backslash\{vv\}) \cup \{vt, th, hv\}$ and $(J\backslash\{vv\}) \cup \{vh, ht, tv\}$ are even and tight. Since $b_e = 0$ for all $e \in B$, the result follows. ∎
Claim 2. $b_{tt} = 0$.
Proof of Claim 2. Let u, v be distinct elements of R. Since $J \cup \{tt, hh\}$ and $(J\backslash\{uu, vv\}) \cup \{tv, vt, hu, uh\}$ are even and tight, we have

$$b_{uu} + b_{vv} + b_{tt} = b_{vt} + b_{tv} = 2(b_{uu} + b_{tt}),$$

where the second equality follows from Claim 1. Therefore, $b_{vv} = b_{uu} + b_{tt}$. Since u and v could be interchanged, the result follows. ∎
Claim 3. There is a number α such that $b_{vt} = b_{ut} = \alpha$ for all $u, v \in R$.
Proof of Claim 3. The permutations $(J\backslash\{vv\}) \cup \{vt, th, hv\}$ and $(J\backslash\{uu\}) \cup \{ut, th, hu\}$ are even and tight. The result now follows from Claims 1 and 2 and the fact that $b_e = 0$ for all $e \in B$. ∎
Claim 4. Let u, v be distinct elements of R. Then $b_{uv} = 0$.
Proof of Claim 4. The permutation $(J \cup \{uv, vt, tu, hh\})\backslash\{uu, vv\}$ is even and tight. This gives $b_{uv} + 2\alpha = 2\alpha$, so $b_{uv} = 0$. ∎

Notice that, since $J \cup \{tt, hh\}$ is even and tight, we now have $b_0 = (n-2)\alpha$. We have shown that $b^T x \leq b_0$ is $\alpha(a^T x \leq a_0)$. Therefore, it induces the same face as $a^T x \leq a_0$, so $a^T x \leq a_0$ is facet-inducing. ∎

To finish the proof of Conjecture 1, we need to show that distinct inequalities (5) induce different facets of $Q(n)$.

Theorem 5 *For $n \geq 5$, inequalities of the form (5) induce $\frac{1}{2}n(n-1)n!$ different facets of $Q(n)$. For $n = 4$, they induce 48 distinct facets.*

Proof. Each inequality (5) can be written in the form $x(C) \leq n-2$, and it will be enough to show that, for all $n \geq 4$, the inequalities for distinct sets C induce distinct facets. Let $x(C) \leq n-2$ and $x(C') \leq n-2$ be two such inequalities, determined by choices t, h, σ and t', h', σ' respectively. We will show that, unless $C = C'$, they are not equivalent.

In the subgraph induced by C', there is a unique vertex $r = h'$ having indegree zero and a unique vertex $s = \sigma'(h')$ having outdegree zero. Let $B = (r, V) \cup (V, s)$. Then B is a basis for A, and $x(C') \leq n-2$ is in B-reduced form. We will use the equations (1), (2) to convert $x(C) \leq n-2$ into an equivalent inequality in B-reduced form.

First, suppose that $rs \in C$. Then we can rewrite $x(C)$ as

$$\begin{array}{lcccccc} x(C \backslash B) & + & x((C \cap B) \backslash \{rs\}) & & & + & x_{rs} \\ = x(C \backslash B) & + & \sum\limits_{rj \in C, j \neq s} x_{rj} & + & \sum\limits_{is \in C, i \neq r} x_{is} & + & 1 - \sum\limits_{j \neq s} x_{rj} \\ = x(C \backslash B) & + & \sum\limits_{rj \in C, j \neq s} (1 - \sum\limits_{i \neq r} x_{ij}) & + & \sum\limits_{is \in C, i \neq r} (1 - \sum\limits_{j \neq s} x_{ij}) & + & 1 - \sum\limits_{j \neq s} (1 - \sum\limits_{i \neq r} x_{ij}). \end{array}$$

Therefore, a B-reduced inequality equivalent to $x(C) \leq n-2$ has left-hand side

$$x(C \backslash B) - \sum_{rj \in C, j \neq s} \sum_{i \neq r} x_{ij} - \sum_{is \in C, i \neq r} \sum_{j \neq s} x_{ij} + \sum_{j \neq s} \sum_{i \neq r} x_{ij}. \tag{7}$$

It will be enough to show that (7) cannot be a positive multiple of $x(C')$. We first prove the following.

Claim. There exist $u, v \in V$ such that $us, rv \in C$ and $uv \notin C$.

Proof of Claim. Since $rs \in C$, we know that one of the following three cases holds: (a) $r \in R$ and $s = \sigma(r)$; (b) $r = t$ and $s = \sigma(w)$ for some $w \in R$; (c) $r \in R$ and $s = \sigma(t)$. Consider case (a). Then we can choose $u = t, v = \sigma(t)$. Now consider case (b). Then we choose any $p \in R \backslash \{w\}$ such that $(t, \sigma(p)) \in C$ and $\sigma(p) \neq \sigma(w)$, and take $u = w$ and $v = \sigma(p)$. Finally, in case (c), we choose some $p \in R \backslash \{r\}$ such that $(p, \sigma(t)) \in C$, and take $v = \sigma(r)$ and $u = p$. In every case it is easy to see that u, v have the desired properties. ∎

Now it follows from the claim that x_{uv} has coefficient -1 in (7), and therefore that the B-reduced form of $x(C) \leq n-2$ cannot be a positive multiple of $x(C') \leq n-2$. Finally, we need to deal with the case in which $rs \notin C$. In this case, the left-hand side of an inequality in B-reduced form equivalent to $x(C) \leq n-2$ is the same as (7) except that the last double sum is missing. But then it will have fewer positive coefficients and/or more negative coefficients than $x(C')$, unless the two double sums in the middle are both empty, which happens only if $C = C'$. Hence if $C \neq C'$, then $x(C) \leq n-2$ cannot be equivalent to $x(C') \leq n-2$. ∎

One may wonder whether it is possible for one of the new inequalities (5) to be equivalent to one of the non-negativity inequalities (3). We show below that it is not. For completeness, we prove also that the non-negativity inequalities induce distinct facets of $Q(n)$.

Theorem 6 *If $n \geq 4$ and $e \in E$, the inequality $x_e \geq 0$ is facet-inducing. Moreover, it is not equivalent to any other inequality (3), nor to any inequality from (5).*

Proof. Note that the transformation of Lemma 4 takes the inequality $x_{ij} \geq 0$ to $x_{i\sigma(j)} \geq 0$. Hence, it suffices to show for any $v \in V$ that $-x_{vv} \leq 0$ induces a facet. We convert this inequality into B-reduced form with $B = (v, V) \cup (V, v)$. So the new inequality $a^T x \leq a_0$ has $a_e = 0$ for all $e \in B$ and $a_e = -1$ for all $e \notin B$. Suppose the face induced by $a^T x \leq a_0$ is contained in the face induced by the valid inequality $b^T x \leq b_0$. Thus, any a-tight even permutation is also b-tight. Choose three distinct vertices i, j, k in $V \setminus \{v\}$. Let $J = \{uu : u \in V \setminus \{v, i, j, k\}\}$. Then the following permutations are easily seen to be even and tight: $J \cup \{jk, kj, vi, iv\}$, $J \cup \{ii, jk, vj, kv\}$, and $J \cup \{ii, kj, vj, kv\}$. It follows that

$$b_{jk} + b_{jk} = b_{ii} + b_{jk} = b_{ii} + b_{kj}.$$

Therefore, $b_{jk} = b_{kj} = b_{ii} = \alpha$ (say), and $2\alpha = b_0 - b(J)$. By symmetry, it follows that $b_e = \alpha$ for all $e \in E(\{i, j, k\})$. Repeating the process for other choices of i, j, k (if necessary), we derive that $b_e = \alpha$ for all $e \notin B$ and $b_0 = \alpha(n-2)$. So $b^T x \leq b_0$ is a multiple of $a^T x \leq a_0$, which implies that $x_e \geq 0$ induces a facet of $Q(n)$.

Now we wish to show that, for any $e \in E$ the inequality $x_e \geq 0$ is not equivalent to any other non-negativity inequality. By Lemma 4, we may assume that $e = \{uv\}$, where $u \neq v$. Each of the following permutations is even and does not contain e:
the identity permutation K;
$(K \setminus \{uu, vv, pp\}) \cup \{vu, up, pv\}$, for any $p \in V \setminus \{u, v\}$;
$(K \setminus \{uu, pp, qq\}) \cup \{up, pq, qu\}$, for any $p, q \in V \setminus \{u, v\}$ with $p \neq q$;
$(K \setminus \{uu, pp, qq\}) \cup \{uq, qp, pu\}$, for any $p, q \in V \setminus \{u, v\}$ with $p \neq q$.
Let f be an edge different from e. It is easy to check that one of the above permutations contains f. Since it does not contain e, it follows that the face induced by $x_e \geq 0$ is not equal to the face induced by $x_f \geq 0$.

Finally, let us show that no inequality (5) is equivalent to a non-negativity inequality. By Lemma 4, it is enough to deal only with the inequality

$$\sum_{v \in R} x_{vv} + \sum_{v \in R} x_{vt} + \sum_{v \in R} x_{tv} \leq n - 2. \tag{8}$$

Each of the following permutations is even and satisfies (8) with equality:
the identity permutation K;
$(K \setminus \{pp, qq, tt\}) \cup \{pq, qt, tp\}$, for any $p, q \in R$ with $p \neq q$;
$(K \setminus \{vv, tt, hh\}) \cup \{vt, th, hv\}$, for any $v \in R$;
$(K \setminus \{vv, tt, hh\}) \cup \{vh, ht, tv\}$, for any $v \in R$.
Let e be any edge. There is a permutation in the above list that contains e. It follows that the face induced by (8) is different from the face induced by $x_e \geq 0$, as required. ∎

We give a brief description of the work of Hood and Perkinson [6] and relate it to our work. They observe that the inequality

$$\sum_{1 \leq i \leq j \leq n} x_{ij} - x_{22} + x_{21} \leq n - 1 \tag{9}$$

is valid for $Q(n)$. They show that it is facet-inducing for $n \geq 6$, and that it provides by symmetry $\frac{1}{2}(n-1)!n!$ distinct facets. The symmetries here are of two types. One is the same as we have used, namely, for any even permutation σ, replacing a_{ij} by $a_{i\sigma(j)}$. The other is, for any permutation ρ such that $\rho(1) = 1$, replacing a_{ij} by $a_{\rho(i)\rho(j)}$. Although the Hood-Perkinson class is larger, it is quite easy to see that it contains none of the facets induced by the inequalities (5). Namely, it is shown in [6] that the facet induced by (9) contains the characteristic vectors of exactly $2^{n-1} - 1$ even permutations, while it is easy to check that for the facet induced by (6), the corresponding number is $2n^2 - 8n + 9$. These two numbers cannot be equal for any integer $n \geq 4$, so the two classes have nothing in common. Another distinction between the two classes, is that our class already provides facets at $n = 4$, which is relevant because it leads to a characterization of the polytope in that case.

A Description of $Q(4)$

In the case $n = 4$ we can prove that the inequalities (5) are all we need to add to the system (1), (2), (3) to get a complete description for $Q(n)$.

Theorem 7 *If $n = 4$, $Q(n)$ is the set of all solutions to the system (1), (2), (3), and (5).*

It should be pointed out that standard computer software for dealing with polyhedra, for example, Avis's *lrs* [7], is perfectly capable of computing the complete list of facets of $Q(n)$ for $n = 4$ and 5. In view of this, it may not be clear why we have included a proof of Theorem 7. One reason is that we believe that our proof has some intrinsic interest. Another is that, because $Q(4)$ is far from full-dimensional, the output of the computer program does not directly provide a proof. It reveals that the dimension is nine, and gives sixty-four facet-inducing inequalities. One is then left with the task of convincing oneself that these inequalities are equivalent to the much more attractive system of Theorem 7.

Our proof follows a method used previously [3]. In particular, we need the following elementary fact, which is proved there.

Lemma 8 *Let P_1, P_2, P_3 be bounded polyhedra in $\mathbf{R}^m$ of equal dimension, and suppose that $P_1 \subset P_2 \subseteq P_3$. Then there exists a point $\bar{x} \in P_2 \backslash P_1$ and an extreme point x^0 of P_3 such that $\bar{x}$ is in the convex hull of $P_1 \cup \{x^0\}$.*

Proof of Theorem 7. Let $P_1 = Q(4)$, let P_2 be the set of solutions of (1), (2), (3), and (5), and let $P_3 = P(4)$. Then $P_1 \subseteq P_2 \subseteq P_3$, and they have equal dimension. If $P_1 = P_2$, we are done, so suppose otherwise. Then all the conditions of Lemma 8 are satisfied. Hence there is a point $\bar{x}$ satisfying (1), (2), (3), and (5) but not in $Q(4)$, permutations $x^0, x^1, \ldots, x^k$, and positive numbers $\lambda_0, \ldots, \lambda_k$ such that x^0 is an odd permutation and $x^1, \ldots, x^k$ are even permutations, $\sum_{j=0}^{n} \lambda_j = 1$ and

$$\bar{x} = \sum_{j=0}^{k} \lambda_j x^j. \tag{10}$$

Given $\bar{x}$, we may choose a collection $\mathcal{X} = \{x^j : \ j = 1, \ldots, k\}$ of even permutations and the expression (10) for $\bar{x}$ such that λ_0 is as small as possible.

By transforming by an even permutation, as in Lemma 4, we can assume that x^0 is the permutation $\{12, 23, 34, 41\}$. Then x^0 violates the instances of (5) indicated below, where for each inequality, written in the form $x(C) \leq 2$, we give the set C. (Note that there are really two kinds of inequalities here. The first four are equivalent under repeated application of the permutation (1234) and the same is true of the other four.)

$C_1 = \{12, 13, 23, 24, 32, 34\}$;
$C_2 = \{41, 42, 12, 13, 21, 23\}$;
$C_3 = \{34, 31, 41, 42, 14, 12\}$;
$C_4 = \{23, 24, 34, 31, 43, 41\}$;
$C_5 = \{12, 14, 22, 23, 33, 34\}$;
$C_6 = \{41, 43, 11, 12, 22, 23\}$;
$C_7 = \{34, 32, 44, 41, 11, 12\}$;
$C_8 = \{23, 21, 33, 34, 44, 41\}$.

We make the following observation: For any C_i, since $x^0(C_i) > 2$ but $\bar{x}(C_i) \leq 2$, it follows that there exists some $x^j \in \mathcal{X}$ such that $x^j(C_i) < 2$. (For otherwise, $2 \geq \bar{x}(C_i) = \lambda_0 x^0(C_i) + 2(1 - \lambda_0) > 2$, a contradiction.) For a given C_i, the list of all possible choices for x^j satisfying $x^j(C_i) < 2$ is given below, and is easily verified. Here, $x \leftarrow K$ means that x is the characteristic vector of K.

C_1: $\hat{x}^1 \leftarrow \{11, 22, 33, 44\}$, $\hat{x}^2 \leftarrow \{14, 42, 21, 33\}$, $\hat{x}^3 \leftarrow \{14, 43, 31, 22\}$;
C_2: $\hat{x}^1 \leftarrow \{11, 22, 33, 44\}$, $\hat{x}^3 \leftarrow \{14, 43, 31, 22\}$, $\hat{x}^4 \leftarrow \{24, 43, 32, 11\}$;
C_3: $\hat{x}^1 \leftarrow \{11, 22, 33, 44\}$, $\hat{x}^4 \leftarrow \{24, 43, 32, 11\}$, $\hat{x}^5 \leftarrow \{13, 32, 21, 44\}$;
C_4: $\hat{x}^1 \leftarrow \{11, 22, 33, 44\}$, $\hat{x}^5 \leftarrow \{13, 32, 21, 44\}$, $\hat{x}^2 \leftarrow \{14, 42, 21, 33\}$;
C_5: $\hat{x}^6 \leftarrow \{13, 31, 24, 42\}$, $\hat{x}^4 \leftarrow \{24, 43, 32, 11\}$, $\hat{x}^5 \leftarrow \{13, 32, 21, 44\}$;
C_6: $\hat{x}^6 \leftarrow \{13, 31, 24, 42\}$, $\hat{x}^5 \leftarrow \{13, 32, 21, 44\}$, $\hat{x}^2 \leftarrow \{14, 42, 21, 33\}$;
C_7: $\hat{x}^6 \leftarrow \{13, 31, 24, 42\}$, $\hat{x}^2 \leftarrow \{14, 42, 21, 33\}$, $\hat{x}^3 \leftarrow \{14, 43, 31, 22\}$;
C_8: $\hat{x}^6 \leftarrow \{13, 31, 24, 42\}$, $\hat{x}^3 \leftarrow \{14, 43, 31, 22\}$, $\hat{x}^4 \leftarrow \{24, 43, 32, 11\}$.

Note that the above observation implies that, for every i, $\mathcal{X}$ contains at least one of the permutations in the list for C_i. Now we consider two cases.

Case 1. Both $\hat{x}^1$ and $\hat{x}^6$ are in $\mathcal{X}$.

Let y^1, y^2, y^3, y^4 be the permutations $\{12, 23, 31, 44\}$, $\{23, 34, 42, 11\}$, $\{13, 34, 41, 22\}$, and $\{12, 24, 41, 33\}$, respectively. Each y^i is an even permutation and

$$2x^0 + \hat{x}^1 + \hat{x}^6 = y^1 + y^2 + y^3 + y^4.$$

Case 2. One of $\hat{x}^1$ or $\hat{x}^6$ is not in $\mathcal{X}$.

Then $\mathcal{X}$ must include both $\hat{x}^3$ and $\hat{x}^5$, or both $\hat{x}^2$ and $\hat{x}^4$. The two cases are symmetrical, so we consider the first. Let y^1, y^2, y^3, y^4 denote the permutations $\{12, 23, 31, 44\}$, $\{23, 32, 14, 41\}$, $\{12, 21, 34, 43\}$, and $\{13, 34, 41, 22\}$, respectively. Notice that each y^i is an even permutation, and that

$$2x^0 + \hat{x}^3 + \hat{x}^5 = y^1 + y^2 + y^3 + y^4.$$

In either case, we can add a (sufficiently) small positive multiple of the derived equation to (10). The resulting expression for $\bar{x}$ will have all of the required properties, but will have a smaller λ_0, a contradiction. This completes the proof. ∎

Theorem 7 can be strengthened, as follows.

Theorem 9 *For $n = 4$, the system consisting of any 7 of the 8 equations (1), (2), the 16 inequalities (3), and the 48 distinct inequalities (5), is a minimal system of linear inequalities describing $Q(n)$.*

Proof. It well known and easy to show that any set of $2n - 1$ of the $2n$ equations (1), (2) implies all of them, but no smaller set does. Moreover, in view of Theorems 3, 5, and 6, each of the inequalities in the system is facet-inducing, and no two of them induce the same facet. It follows that the description is minimal. ∎

The Membership Problem

The description of $Q(n)$ appears to be complicated in general. Therefore, we can expect that it may not be easy in general to test a given point in $\mathbf{R}^E$ for membership in $Q(n)$. In fact, Brualdi and Liu [4] conjectured that there does not exist a polynomial-time algorithm to solve this membership problem. Note that there is a connection between this second conjecture and Conjecture 1. Namely, due to the polynomial-time solvability of linear programming, its truth would imply the truth of Conjecture 1. More precisely, it would imply the truth of a version of Conjecture 1 which requires also that the lengths of the coefficients in the linear inequalities be polynomially-bounded. It would also imply that $Q(n)$ cannot be the projection of a polytope $T(n)$ in dimension $f(n)$, such that $T(n)$ has a polynomial-size description by linear inequalities. Whether such a "compact description" of $Q(n)$ exists, is unknown.

While proving the non-existence of a polynomial-time algorithm for the membership problem seems hopeless, an easier question to answer may be whether the problem is $\mathcal{NP}$-hard. To our knowledge, this remains open. Actually, there is some weak evidence pointing in the direction of solvability of the membership problem, which we now summarize. By a fundamental result of Grötschel, Lovász, and Schrijver (see [5]), the membership problem is solvable in polynomial time if there is a polynomial-time algorithm for the *optimization problem*: "Given $c \in \mathbf{R}^E$ find the maximum of $c^T x$ over $x \in Q(n)$." A special case of the optimization problem is the case in which c is $\{0, 1\}$-valued, and we want to know whether the maximum is n.

The latter problem can be stated more simply as follows: Given a digraph $H = (V, E')$, determine whether E' contains an even permutation. (E' is just $\{ij \in E : c_{ij} = 1\}$.) This problem is equivalent to several other interesting problems, including that of determining whether a given digraph has a directed cycle of even length, and determining whether a given bipartite graph has a Pfaffian orientation. These problems have been solved by Robertson, Seymour, and Thomas [10], based on a characterization due independently to themselves and McCuaig [8].

This problem and the more general optimization problem above, are examples of pairs of problems that occur commonly in combinatorial optimization. Suppose we are given a family of subsets of a set E, such as the family of even permutations of $G = (V, E)$. The optimization problem is, given a weighting of the elements of E, to find the maximum, over all members of the family, of the total weight of that member. The *feasibility problem* is, given a subset of E, to decide whether it contains a member of the family. If the optimization problem is efficiently solvable, then so is the feasibility problem. In fact, families for which the converse is known to fail are rather rare. (This may reflect the current lack of knowledge more than the actual state of affairs.) Since the feasibility problem for the family of even permutations is solvable, there is at least some hope that the optimization problem over $Q(n)$ is solvable, and hence that the membership problem is, too.

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