Restricted 2-Factor Polytopes

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Abstract

The optimal k-restricted 2-factor problem consists of finding, in a complete undirected graph K_n, a minimum cost 2-factor (subgraph having degree 2 at every node) with all components having more than k nodes. The problem is a relaxation of the well-known symmetric travelling salesman problem, and is equivalent to it when $\frac{n}{2} \leq k \leq n-1$. We study the k-restricted 2-factor polytope. We present a large class of valid inequalities, called bipartition inequalities, and describe some of their properties; some of these results are new even for the travelling salesman polytope. For the case $k=3$, the triangle-free 2-factor polytope, we derive a necessary and sufficient condition for such inequalities to be facet inducing.

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1 Introduction

A 2-*factor* of an undirected graph $G = (V, E)$ is a spanning subgraph H of G that has degree 2 at each node. Equivalently, it is a set of node-disjoint circuits that include all of the nodes. Of course, a special case is a Hamiltonian circuit of G. Deciding whether G has a Hamiltonian circuit is a well-known NP-complete problem, whereas deciding whether G has a 2-factor can be done in polynomial time, due to its equivalence to problems in matching. We consider a class of problems intermediate in difficulty to these two problems. A 2-factor H is k-*restricted*, (or just *restricted*), for k an integer, if each component of H has more than k nodes. If $k = n - 1$ (in fact, if $n/2 \leq k \leq n - 1$), then any restricted 2-factor is a Hamiltonian circuit. If $k = 0$, then every 2-factor is a restricted 2-factor.

In fact, much is known about the complexity of determining whether a given G has a restricted 2-factor. For $k = 0, 1, 2$, matching techniques due to Edmonds can be used to answer the question efficiently. (Note that these problems are all equivalent if the graph has no loops or multiple edges, but otherwise there are differences.) For $k = 3$ ("triangle-free 2-factors"), Hartvigsen [9] gave an efficient algorithm. His solution is difficult. On the other hand, for $k \geq 5$ the problem has been proved to be NP-complete by Papadimitriou; see [3]. So from the point of view of computational complexity, only the case $k = 4$ remains open.

The *optimal restricted 2-factor problem* is, given $G = (V, E)$ and edge weights $(c_e : e \in E)$, to find a maximum-weight restricted 2-factor. Of course, this optimization problem is at least as hard as the corresponding decision problem discussed above. So it is NP-hard for $k \geq 5$. In fact, Vornberger [11] has proved that the optimization problem is NP-hard for $k = 4$ also. So from the complexity point of view, the only open case is $k = 3$, the "optimal triangle-free 2-factor problem". If one assumes that the graph is complete and the weight function satisfies the triangle inequality, then the optimization problem remains NP-hard for $k \geq 4$. (This follows from the proof in [11].) However, under this assumption there is a polynomial-time algorithm that guarantees to find a solution of weight at most twice optimal; see Goemans and Williamson [5].

In this paper we study these problems from a polyhedral viewpoint. There is a great deal of evidence that this approach can lead to linear-programming based techniques that provide excellent bounds and even provably optimal solutions. An example, which has received a lot of attention, is the (symmetric) travelling salesman problem (TSP). One motivation for polyhedral results on restricted 2-factors is that they generalize results for the TSP. Another is the open case $k = 3$ for which (unlike the NP-hard cases) we may hope for a complete description of the polyhedron. A third is that there seems to be a sense in which for smaller values of k the polyhedron is better-behaved.

We consider mainly the "bipartition inequalities", a class of inequalities that was introduced for the TSP by Boyd and Cunningham [1]. They include well-known earlier classes like subtour elimination, comb, and clique-tree inequalities. They extend the clique-tree class by dropping the restriction to a tree structure, and by allowing some of the "teeth" to be "degenerate". There is a natural way to choose a subfamily of bipartition inequalities for each k, namely, by requiring teeth to have size at most k. We prove that these "k-bipartition inequalities", are valid for the restricted 2-factor problem for that k. We prove that for a k-bipartition inequality to be facet-inducing for the restricted 2-factor polytope, it must satisfy a certain connectivity condition, namely, it can have no cutset consisting of degenerate teeth. This result is new even for the TSP polytope. Moreover, for $k = 3$, we obtain necessary and sufficient conditions for a k-bipartition inequality to be facet-inducing. The resulting class is large, but it has a nice structure. However, it is unfortunately not complete; we also show that even for $k = 3$, there are facet-inducing inequalities that are not in this class.

The paper is organized as follows. In Section 2, we describe the class of bipartition inequalities and prove their validity for the k-restricted 2-factor polytope. We also establish local necessary conditions for a k-restricted factor to satisfy a k-bipartition inequality with equality. In Section 3, we prove necessary conditions for a bipartition inequality to be facet inducing, and conjecture that they are sufficient. In Section 4 we deal with the triangle-free 2-factor problem, that is, with the case when $k = 3$. We characterize the facet-inducing bipartition inqualities for this case. In fact, we prove the conjecture of the previous section for $k = 3$. Our results also show that, for $k = 3$, the characterization is a good one, as the conditions can be checked by solving a few bipartite matching problems. Finally, we show that the known classes of inequalities are still not sufficient to define the triangle-free 2-factor polytope.

We end this section with some terminology and notation. We shall use the word *factor* to mean k-restricted 2-factor, whenever it is possible to do so without confusion. It is convenient to treat the optimal factor problem as a problem on a simple complete graph. This has the slight disadvantage that we lose the distinction among the cases $k = 0, 1, 2$, but our contributions do not apply to these cases anyway. So we work with the complete graph $K_n = (V, E)$, where $|V| = n \geq 4$, and we write elements of E as (i, j) or ij. Notice that $ij = ji$.

For $S \subseteq V$, we denote $V \backslash S$ by $\overline{S}$. For $S, T \subseteq V$, $E(S : T)$ denotes $\{ij \in E : i \in S, j \in T\}$. For $S \subseteq V$, $E(S)$ denotes $E(S : S)$ and $\delta(S)$ denotes $E(S : \overline{S})$. For $v \in V$, we may abbreviate $\delta(\{v\})$ to $\delta(v)$. For $B \subseteq E$ and $x \in \mathbf{R}^E$, $x(B)$ denotes $\sum(x_{ij} : ij \in B)$. We may write $x(S)$ instead of $x(E(S))$ for $S \subseteq V$ whenever no confusion arises. Generally, we do not distinguish between a subset C of E and its incidence vector $x \in \{0, 1\}^E$ defined by $x_{ij} = 1$ if and only if $ij \in C$.

Given $c \in \mathbf{R}^E$, the following is an integer linear programming formulation of the optimal factor problem.

$$\text{(1)} \quad \text{minimize} \quad \sum (c_{ij} x_{ij} : ij \in E)$$

subject to

$$\text{(1a)} \quad x(\delta(v)) = 2, \quad v \in V;$$

$$\text{(1b)} \quad x(S) \leq |S| - 1, \quad S \subset V \text{ and } 2 \leq |S| \leq k;$$

$$\text{(1c)} \quad x_{ij} \geq 0, \quad ij \in E;$$

$$\text{(1d)} \quad x_{ij} \text{ integer}, \quad ij \in E.$$

The constraints (1a) are called *degree* constraints and (1b) are called *subtour elimination* (SE) constraints. A factor is the union of node-disjoint cycles, called *subtours*, covering all nodes in K_n, and moreover, each subtour contains more than k edges.

The convex hull of feasible solutions to (1) is a bounded polyhedron, which we denote by P^k. We use standard terminology and results from polyhedral theory. See Schrijver [10].

We say that two sets *meet* or *intersect* if they have non-empty intersection. The *intersection graph* of a family of sets is the graph having a node for each member of the family, with two nodes being adjacent if they meet.

2 Bipartition inequalities

In this section, a large class of inequalities, called *bipartition inequalities*, will be shown to be valid for P^k. This class was first introduced by Boyd and Cunningham [1] for the TSP polytope as a generalization of the clique-tree class.

Let $\mathcal{H} = \{H_1, \ldots, H_h\}$ be a collection of mutually disjoint subsets of V called *handles*, and let $T_1, T_2, \ldots, T_{t+m}$ be mutually disjoint proper subsets of V called *teeth.* A tooth is called *degenerate* if it is contained in the union of the handles; otherwise it is called *nondegenerate.* Assume that no T_j is contained in any H_i, and that T_j is nondegenerate if and only if $1 \leq j \leq t$. Assume also that each handle H_i intersects $2k_i + 1$ teeth, where k_i is a positive integer, and d_j denotes the number of handles intersected by tooth T_j for all j, $1 \leq j \leq t + m$. The *bipartition inequality* associated with these handles and teeth is given by

$$
\begin{aligned}
(2)\qquad & \sum_{i=1}^{h} x(H_i) + \sum_{j=1}^{t} x(T_j) + \sum_{j=t+1}^{t+m} \frac{d_j}{d_j - 1} x(T_j) \\
& \leq \sum_{i=1}^{h} |H_i| + \sum_{i=1}^{h} k_i + \sum_{j=1}^{t} (|T_j| - d_j - 1) + \sum_{j=t+1}^{t+m} \frac{d_j}{d_j - 1} (|T_j| - d_j).
\end{aligned}
$$

Notice that in the special case when $h = 0$, $t = 1$, and $m = 0$, the bipartition inequality is just an SE inequality. In the special case when $h = 1$ and $T_j \cap H_1 \neq \emptyset$ for all j, it is the well-known *comb* inequality. Finally, when $m = 0$ and the intersection graph of the H_i and T_j is a tree, we have the *clique-tree* inequalities of Grötschel and Pulleyblank [8]. Recently, Carr [2] has shown that, when the numbers of handles and teeth are fixed, there is a polynomial-time algorithm to solve the separation problem for bipartition inequalities.

Figure 1 represents two bipartition inequalities. The hollow nodes, labelled v_0 and v_0', represent optional nodes that may or may not exist. In the case when the optional nodes do not exist, both inequalities have right-hand side $c_0 = 8$. The coefficient c_e of the left-hand side depends on the total weight of the sets that contain e. Weights of 1 are omitted for simplicity, and thus only the weights of degenerate teeth are given. For instance, $c_{vv'} = c_{ss'} = 2$ for the inequality on the left and $c_{vv'} = 2, c_{ss'} = 0$ and $c_{rw} = 1$ for the one on the right.

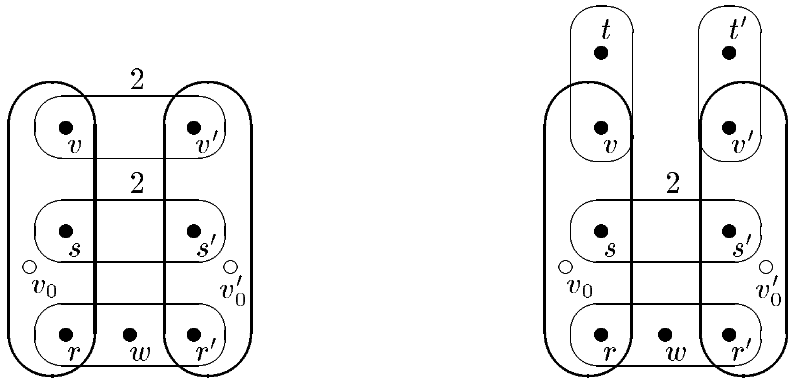

Figure 1: Two bipartition inequalities

A *k-bipartition inequality* is one for which every tooth has size at most k. The main result of this section is the following. It generalizes a result of [1] for the travelling salesman problem.

Theorem 2.1 *Any k-bipartition inequality $cx \leq c_0$ is valid for P^k.*

For the remainder of this section $cx \leq c_0$ denotes a k-bipartition inequality defined by (2). Consider the following maximization problem:

(3) $\quad z^*(c) = \max\{cx : x \in P^k\}.$

A factor x^* is said to be *c-optimal* if $cx^* = z^*(c)$, and to be *c-tight* if $cx^* = c_0$. The inequality $cx \leq c_0$ is valid for P^k if and only if $z^*(c) \leq c_0$ and is face-inducing if and only $z^*(c) = c_0$. We will show the validity of $cx \leq c_0$ by induction on the number of handles defining $cx \leq c_0$. To do so, we first apply a procedure to transform a c-optimal factor x^* to into a c-optimal factor $\hat{x}$ having a special structure. Then, we "decompose" $cx \leq c_0$ with respect to $\hat{x}$ into two bipartition inequalities with smaller number of defining handles, and use induction.

For any factor x, let $\kappa(x)$ denote the number of subtours in x. Given any subset S of V and a factor x such that $x(E(S)) < |S| - 1$, we can apply the following procedure *2OPT* . This will be used repeatedly in our proofs. It is analogous to the well-known local optimization procedure of the same name. In fact, we will need it only for the case where $S = T_j$ or $S = H_i \cap T_j$, where T_j is a tooth and H_i is a handle of the bipartition inequality $cx \le c_0$.

Procedure 2OPT

Given: a vector $c \in \mathbf{R}^E$, a factor x, and a subset S of V such that $x(E(S)) < |S| - 1$.

Step 0. If there is exactly one subtour of x that visits S, then go to Step 1; else go to Step 2.

Step 1. The subtour must enter (and leave) S at least twice, so it contains a path $uv \cdots pq \cdots u'v'$ such that $v, p, v' \in S$ and $u, q, u' \in \overline{S}$. Replace uv, $u'v'$ in x by uu' and vv' to obtain $\bar{x}$. STOP.

Step 2. Choose a pair of edges $uv, u'v'$ that are in different subtours of x, with $v, v' \in S$ and $u, u' \in V \setminus S$, such that $c_{uv} + c_{u'v'}$ is minimum over all such pairs. Replace uv and $u'v'$ by uu' and vv' to obtain $\bar{x}$. STOP.

It is easy to see that 2OPT has the following property.

Proposition 2.2 *2OPT terminates with a factor $\bar{x}$ such that $\bar{x}(E(S)) > x(E(S))$ and $\kappa(\bar{x}) \le \kappa(x)$. Moreover, $c\bar{x} \ge cx$ if and only if $c_{uv} + c_{u'v'} \le c_{uu'} + c_{vv'}$.*

▮

Lemma 2.3 *Let T be a tooth and H be a handle of a bipartition inequality $cx \le c_0$, and let $S = H \cap T$. Suppose that x^* is a c-optimal factor with $x^*(H \cap T) < |H \cap T| - 1$. If 2OPT is applied with $x = x^*$ and $S = H \cap T$, then 2OPT replaces an edge in $E(T \cap H : T \backslash H)$ and an edge in $E(T \cap H : H \backslash T)$ by an edge in $E(T \cap H)$ and an edge in $E(T \backslash H : H \backslash T)$. The resulting factor $\hat{x}$ is also c-optimal.*

Proof: 2OPT replaces a pair of edges uv and $u'v'$ in $\delta(S)$ with uu' and vv', where $vv' \in E(S)$. By definition of c, $c_{uv} + c_{u'v'} \le c_{vv'}$. Since x^* is c-optimal, this inequality must hold with equality, and moreover $c_{uu'} = 0$. As $c_{vv'} \ge 2$ we have $c_{uv} = c_{u'v'} = 1$, which implies that $u \in T \backslash S$ and $u' \in H \backslash S$. It follows that $uu' \in E(H \setminus S : T \setminus S)$, proving the lemma. ▮

Lemma 2.4 *Let x^* be a c-optimal factor. Then there exists a c-optimal factor $\hat{x}$ such that* *(a)* $\hat{x}(H_i \cap T_j) = |H_i \cap T_j| - 1$ *for all* i *and* j *with* $H_i \cap T_j \neq \emptyset$*;*

(b) $\hat{x}(H_i) = x^*(H_i)$ *and* $\hat{x}(T_j) = x^*(T_j)$ *for all* i *and* j*;*

(c) $x^*(\delta(H_i) \cap \{e : c_e = 0\}) \leq \hat{x}(\delta(H_i) \cap \{e : c_e = 0\})$ *for all* $H_i \in \mathcal{H}$*.*

Proof: Apply 2OPT with $S = H_i \cap T_j$ for any handle-tooth pair violating (a). It follows from Lemma 2.3 that (b) and (c) are satisfied by the resulting factor $\hat{x}$. We can repeat this until (a) is also satisfied. ∎

For any $S \subseteq V$, a factor x is said to *saturate* S if $x(S) = |S| - 1$. We say that a factor $\hat{x}$ is *simple* if it satisfies condition (a) of Lemma 2.4.

We describe a decomposition of $cx \leq c_0$ relative to a fixed handle H and a fixed c-optimal simple factor $\hat{x}$. This construction will be used not only in the proof of Theorem 2.1, but in the proofs of subsequent results. Let D be the index set of the degenerate teeth, and let N be the index set of the nondegenerate teeth. For a fixed $H \in \mathcal{H}$, define $S_j \equiv T_j \cap H$ for all j with $T_j \cap H \neq \emptyset$. Let J' denote the index set of the teeth that intersect H, and define J to be $\{j \in J' : d_j \geq 3$, or $d_j = 2$ and $j \in N\}$. With respect to the factor $\hat{x}$ and H, define $J^0(\hat{x})$ to be $\{j \in J : \hat{x}(S_j : T_j \setminus S_j) = 0\}$, and $J^+(\hat{x})$ to be $J \backslash J^0(\hat{x})$. Define the following set of teeth relative to $\hat{x}$ and H: for every $j \in J^0(\hat{x})$, $T'_j = S_j \cup \{v_j\}$, where v_j is a fixed element of $T_j \setminus S_j$; for every $j \in J^+(\hat{x})$, $T'_j = S_j \cup \{v_j\}$, where v_j is the element of $T_j \setminus S_j$ satisfying $\hat{x}_{v_j s} = 1$ for some $s \in S_j$, and $T'_j = T_j$ for all $j \in J' \backslash J$. By definition and the property of $\hat{x}$, we have

(4) $\quad \hat{x}(T'_j) = \hat{x}(S_j) = |S_j| - 1 = |T'_j| - 2, \quad$ for all $j \in J^0(\hat{x})$;

(5) $\quad \hat{x}(T'_j) = |T'_j| - 1 = |S_j|, \quad$ for all $j \in J^+(\hat{x})$.

We now construct two bipartition inequalities from $cx \leq c_0$. Let $ax \leq a_0$ be the bipartition inequality obtained from $cx \leq c_0$ by deleting the handle H and all the teeth that intersect only H, and replacing T_j by $T_j \setminus S_j$ for each $j \in J^0(\hat{x})$. Let $bx \leq b_0$ be a comb inequality defined by the handle H and all teeth T'_j. So $ax \leq a_0$ and $bx \leq b_0$ are k-bipartition inequalities with fewer than h handles.

Using (4) and (5), we can now express $c\hat{x}$ and c_0 in terms of $a\hat{x} + b\hat{x}$ and $a_0 + b_0$, respectively, as follows

$$
\begin{aligned}
c\hat{x} &= a\hat{x} + b\hat{x} + \gamma_1 + \gamma_2 + \gamma_3, \quad \text{where} \\
\gamma_1 &\equiv \sum \left(-\frac{\hat{x}(T_j \setminus S_j)}{(d_j-1)(d_j-2)} + \frac{\hat{x}(T'_j)}{(d_j-1)} : j \in J^0(\hat{x}) \cap D \right), \\
\gamma_2 &\equiv \sum \left(\frac{\hat{x}(T_j)}{d_j - 1} - \hat{x}(T'_j) : j \in J^+(\hat{x}) \cap D \right), \\
\gamma_3 &\equiv \sum \left(-\hat{x}(T'_j) : j \in J^+(\hat{x}) \cap N \right).
\end{aligned}
$$

$$
\begin{aligned}
c_0 &= a_0 + b_0 + \gamma'_1 + \gamma'_2 + \gamma'_3, \quad \text{where} \\
\gamma'_1 &\equiv \sum \left(\frac{d_j}{d_j - 1}(|T_j| - d_j) - \frac{d_j - 1}{d_j - 2}(|T_j \setminus S_j| - d_j + 1) - |S_j| + 1 : j \in J^0(\hat{x}) \cap D \right) \\
\gamma'_2 &\equiv \sum \left(\frac{d_j}{d_j - 1}(|T_j| - d_j) - |T_j| + d_j - |S_j| + 1 : j \in J^+(\hat{x}) \cap D \right), \\
\gamma'_3 &\equiv \sum \left(-|S_j| : j \in J^+(\hat{x}) \cap N \right)
\end{aligned}
$$

Lemma 2.5 *Let $\hat{x}$ be a simple c-optimal factor. Suppose that $a\hat{x} \le a_0$ and $b\hat{x} \le b_0$ hold. Then $c\hat{x} \le c_0$ also holds, and moreover, $c\hat{x} = c_0$ if and only if $a\hat{x} = a_0$, $b\hat{x} = b_0$ and $\gamma_i = \gamma'_i$ for $i = 1, 2, 3$.*

Proof: It suffices to show that $\gamma_i \le \gamma'_i$, $i = 1, 2, 3$. Note first that by (5), $\gamma_3 = \gamma'_3$. Next, observe that for $j \in D \cap J^0(\hat{x})$, we have

(6) $\quad \hat{x}(T_j \setminus S_j) \ge \sum (|T_j \cap H_i| - 1 : H_i \in \mathcal{H} \setminus \{H\}, H_i \cap T_j \ne \emptyset) = |T_j \setminus S_j| - d_j + 1.$

It follows from (4) that

(7) $\quad \gamma_1 \le \sum \left(-\frac{|T_j \setminus S_j| - d_j + 1}{(d_j - 1)(d_j - 2)} + \frac{|S_j| - 1}{(d_j - 1)} : j \in J^0(\hat{x}) \cap D \right) = \gamma'_1,$

and from (5) that

(8) $\quad \gamma_2 \le \sum \left(\frac{|T_j| - 1}{d_j - 1} - |S_j| : j \in J^+(\hat{x}) \cap D \right) = \gamma'_2.$

∎

Lemma 2.5 *seems* to say that there are other inequalities that hold with equality whenever $cx \le c_0$ does. However, what it *really* says is that for each factor satisfying $cx \le c_0$ with equality, one can define other inequalities that the same factor also satisfies with equality. The new inequalities depend on the given factor. We are now ready to prove the validity of the bipartition inequalities.

Proof of Theorem 2.1. We prove the theorem by induction on the number h of handles defining $cx \le c_0$. For $h = 0$, the inequality is the sum of SE constraints $x(T_j) \le |T_j| - 1$, where $|T_j| \le k$. For $h = 1$, the inequality $cx \le c_0$ is a comb with a handle H intersected by all teeth $T_1, \dots, T_{2k_h+1}$ of size at most k. To prove its validity, we use the usual technique known for the TSP. We add the inequalities:

$$\frac{1}{2}x(\delta(v)) = 1, \quad \text{for all } v \in H,$$

$$\frac{1}{2}x(T_j) \le \frac{1}{2}(|T_j| - 1), \quad j = 1, \dots, 2k_h + 1,$$

$$\frac{1}{2}x(T_j \setminus H) \le \frac{1}{2}(|T_j \setminus H| - 1), \quad \text{for all } j \text{ such that } |T_j \setminus H| \ge 2,$$

$$\frac{1}{2}x(T_j \cap H) \le \frac{1}{2}(|T_j \cap H| - 1), \quad \text{for all } j \text{ such that } |T_j \cap H| \ge 2.$$

Taking the integer part of each coefficient and the right-hand side of the resulting inequality yields $cx \le c_0$.

Assume now that the theorem holds for the number of handles less than h. By Lemma 2.4, it suffices to check the validity of $cx \le c_0$ for any simple c-optimal solution $\hat{x}$ to (3). With respect to $\hat{x}$, we can construct as above $ax \le a_0$, $bx \le b_0$, as well as γ_i, γ_i', $i = 1, 2, 3$. By the induction hypothesis, $ax \le a_0$ and $bx \le b_0$ are valid for P^k, and therefore $cx \le c_0$ is valid by Lemma 2.5. ∎

Similar methods allow us to establish some further properties of c-tight factors. These properties will be useful later. They are new even for the TSP. The first one indicates that there are exactly two ways for a tight factor to traverse a degenerate tooth. These are indicated in Figure 2.

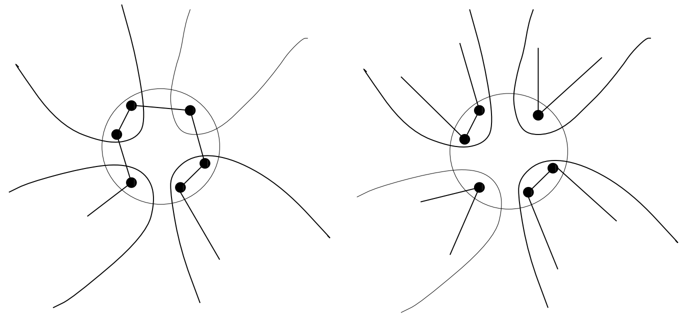

Figure 2: The two ways of traversing a degenerate tooth

Theorem 2.6 *Let T be a degenerate tooth, let $H_1, H_2, \ldots, H_d$ be the handles intersecting T, and let x^* be a c-tight factor. If x^* does not saturate T, then*

$x^*(T) = |T| - d$ *and* $x^*(T \cap H_i) = |T \cap H_i| - 1$ *for* $i = 1, \ldots, d$.

Proof: First, we show that $x^*(T) = |T| - d$. Let $\hat{x}$ be the simple factor produced from x^* via Lemma 2.4. It will be enough to prove that $\hat{x}(T) = |T| - d$. First, suppose that $d = 2$, and let H, H' be the two handles met by T. Then, since T is degenerate, $T = (H \cap T) \cup (H' \cap T)$. Therefore, since $\hat{x}$ is simple,

$$\begin{aligned} \hat{x}(T) &\geq \hat{x}(H \cap T) + \hat{x}(H' \cap T) \\ &= |H \cap T| - 1 + |H' \cap T| - 1 \\ &= |T| - 2. \end{aligned}$$

It follows, since $\hat{x}(T) < |T| - 1$, that $\hat{x}(T) = |T| - 2 = |T| - d$, as required.

Now suppose that $d \geq 3$. Since $\hat{x}$ does not saturate T, there is a proper nonempty subset A of T such that $\hat{x}(A : T \backslash A) = 0$. First suppose that A can be chosen so that $A = T \cap H$ for some handle H. For this H we can apply the decomposition procedure to $cx \leq c_0$, and we will have $T = T_j$, where $j \in J^0(\hat{x})$. By Lemma 2.5, we have $\gamma_1 = \gamma_1'$, and thus by (4), (7), $\hat{x}$ satisfies (6) with equality. Finally, using Lemma 2.4, we have

$$x^*(T) = \hat{x}(T) = \hat{x}(A) + \hat{x}(T \setminus A) = (|A| - 1) + |T \setminus A| - d + 1 = |T| - d,$$

as required.

Now suppose that any such A above meets at least two handles. Then there must be a proper partition $(Q_1, Q_2, \ldots, Q_q)$ of T such that for each i, Q_i is contained in a union of $d_i \geq 2$ handles, and moreover

$$\hat{x}(Q_i) = |Q_i| - 1, \quad \hat{x}(Q_i : T \setminus Q_i) = 0.$$

Form the bipartition inequality $c'x \le c'_0$ from $cx \le c_0$ by replacing the tooth T by the teeth $Q_i, i = 1, \ldots, q$. (Notice that all these new teeth are degenerate.) We compute

$$c'\hat{x} = c\hat{x} + \sum_{i=1}^{q} \frac{d_i \hat{x}(Q_i)}{d_i - 1} - \frac{d\hat{x}(T)}{d-1},$$

and

$$c'_0 = c_0 + \sum_{i=1}^{q} \frac{d_i(|Q_i| - d_i)}{d_i - 1} - \frac{d(|T| - d)}{d-1}.$$

Therefore, since $c\hat{x} = c_0$,

$$\begin{aligned} c'_0 - c'\hat{x} &= \sum_{i=1}^{q} \frac{d_i(|Q_i| - d_i - \hat{x}(Q_i))}{d_i - 1} - \frac{d(|T| - d - \hat{x}(T))}{d-1} \\ &= \sum_{i=1}^{q} \frac{d_i(|Q_i| - d_i - |Q_i| + 1)}{d_i - 1} - \frac{d(|T| - d - \hat{x}(T))}{d-1} \\ &= -d - \frac{d(|T| - d - \hat{x}(T))}{d-1} = -\frac{d(|T| - 1 - \hat{x}(T))}{d-1} < 0, \end{aligned}$$

a contradiction to $c'\hat{x} \le c'_0$. Hence the supposition is false.

It remains to prove that $x^*(T \cap H_i) = |T \cap H_i| - 1$ for all i. Suppose that $x^*(T \cap H_i) < |T \cap H_i| - 1$ for some i. Apply 2OPT relative to x^* and $S = T \cap H_i$. Note that by Lemma 2.3, the resulting $\bar{x}$ must contain an edge $e_0 \in \delta(T)$ with $c_{e_0} = 0$, and moreover satisfy $c\bar{x} = cx^*$ and $\bar{x}(T) = x^*(T) < |T| - 1$. So we can apply 2OPT again relative to $\bar{x}$ and $S = T$. Then 2OPT replaces e_0, $e_1 \in \delta(T)$ with $e \in E(T)$ and $e' \in E(V \setminus T)$. Since

$$c_{e_0} + c_{e_1} \le 1 < \frac{d_j}{d_j - 1} \le c_e + c_{e'},$$

the new factor $\tilde{x} \in P^k$ satisfies $c\tilde{x} > c\bar{x} = c_0$, a contradiction. ∎

The second result states that there are just two ways in which a tight factor can traverse a handle, which we indicate in Figure 3.

Theorem 2.7 *Let x^* be a c-tight factor, let H be a handle, and let $2k_h + 1$ be the number of teeth intersecting H. Then exactly one of the following is true:*

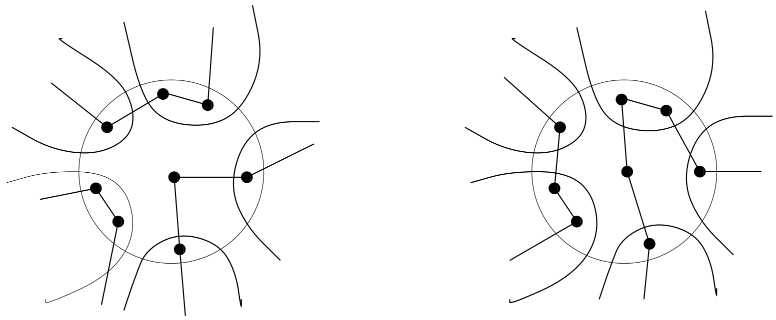

Figure 3: The two ways of traversing a handle

(a) $x^*(H) = |H| - k_h - 1$, $x^*(\delta(H)) = 2k_h + 2$, *and* $x^*(H \cap T_j : T_j \backslash H) \geq 1$ *for all teeth* T_j *meeting* H*;*

(b) $x^*(H) = |H| - k_h$, $x^*(\delta(H)) = 2k_h$, *and* $x^*(H \cap T_j : T_j \backslash H) \geq 1$ *for all but one of the teeth* T_j *meeting* H*.*

(Note that if case (b) holds, we will actually have $x^*(H \cap T_j : T_j \backslash H) = 1$ for all but one of the teeth T_j meeting H.) We need the following observation.

Lemma 2.8 *Suppose that* $\hat{x}$ *is a tight factor for the comb inequality having teeth* $T_1, \ldots, T_{2h+1}$*, each of size at most* k*. Then* $\sum_{j=1}^{2h+1}(|T_j| - 1 - \hat{x}(T_j))$ *is equal to 0 or 1.*

Proof: If the comb inequality holds with equality, then in its derivation in the proof of Theorem 2.1, at most one of the added inequalities is not tight, and that one must be satisfied with a slack of exactly $\frac{1}{2}$. The result follows. ∎

Proof of Theorem 2.7. Consider the simple c-tight factor $\hat{x}$ obtained from x^* via Lemma 2.4. Then $\hat{x}(H) = x^*(H)$ and (by the identity $2x(H) + x(\delta(H)) = 2|H|$) $\hat{x}(\delta(H)) = x^*(\delta(H))$. Finally, since $\hat{x}$ is constructed from x^* by applying 2OPT with respect to sets S of the form $H_i \cap T_j$, by Lemma 2.3, the last condition of (a) and (b) is satisfied by $\hat{x}$ if and only if it is satisfied by x^*. Therefore, it will be enough to prove the theorem for $\hat{x}$ rather than x^*.

If we decompose $cx \leq c_0$ relative to $\hat{x}$ and the handle H, we get the comb inequality $bx \leq b_0$ defined by

$$(9) \quad x(H) + \sum_{j=1}^{2k_h+1} x(T'_j) \leq |H| + \sum_{j=1}^{2k_h+1} (|T'_j| - 1) - k_h - 1.$$

By Lemma 2.5, $b\hat{x} = b_0$, that is, (9) holds with equality for $x = \hat{x}$, so

$$\hat{x}(H) = |H| - k_h - 1 + \sum_{j=1}^{2k_h+1} \left(|T'_j| - 1 - \hat{x}(T'_j)\right).$$

Now by Lemma 2.8, $\hat{x}(H) = |H| - k_h - 1$ or $\hat{x}(H) = |H| - k_h$. By the identity $2x(H) + x(\delta(H)) = 2|H|$, we have that $\hat{x}(\delta(H)) = 2k_h + 2$ in the first case, and $\hat{x}(\delta(H)) = 2k_h$ in the second. In the first case, again by Lemma 2.8, $\hat{x}$ saturates each T'_j and therefore there is an edge of $\hat{x}$ from $H \cap T_j$ to $T_j \backslash H$ for each j, as required. Finally, in the second case, there is an edge of $\hat{x}$ from $H \cap T_j$ to $T_j \backslash H$ for at least $2k_h$ values of j. But since $\hat{x}(\delta(H)) = 2k_h$, it follows that this is true for exactly $2k_h$ values of j, as required. ▮

3 Degenerate cuts

In this section we describe an important necessary condition for a bipartition inequality to be facet-inducing. We begin with some simple examples.

Suppose that the intersection graph of the H_i, T_j is not connected, for example, that there is no handle and more than one tooth. Then it is easy to see that the inequality is the sum of the inequalities corresponding to the connected components, and thus is contained in the faces induced by those inequalities. There is one very special case in which such a bipartition inequality can be facet-inducing, namely, if there are just two teeth which are complements of each other. Then the inequality induces the same face as does the SE inequality determined by one of the teeth.

As a second example, suppose there are just two handles and every tooth intersecting both of them is degenerate. Then it is easy to see that the inequality is the sum of two comb inequalities, each defined by one of the handles and the teeth that intersect it. Again, there is one case in which such an inequality can be facet-inducing, namely, when the two handles are complements of each other, for then the inequality induces the same face as each of the comb inequalities.

The above examples both have the property that there is a set $W \subset V$ such that each of $W, \overline{W}$ contains at least one node in some tooth or handle, and no handle or nondegenerate tooth intersects both W and $\overline{W}$. In this situation we call the set of degenerate teeth intersecting both W and $\overline{W}$ a *degenerate cut*. The main result of this section is the following.

Theorem 3.1 *Let $cx \leq c_0$ be a bipartition inequality that does not have two complementary handles or teeth. If $cx \leq c_0$ has a degenerate cut, then it does not define a facet of P^k.*

Notice that it follows from the theorem that a bipartition inequality having a tree as its intersection graph and having a degenerate tooth (and not having two complementary handles), cannot be facet-inducing. It is possible to prove this fact by showing that such an inequality is a non-negative linear combination of d other bipartition inequalities, where d is the number of handles intersected by the degenerate tooth. For more general bipartition inequalities, it need not be true that if there is a degenerate cut, then the inequality is a non-negative combination of other bipartition inequalities. This may suggest why the proof of Theorem 3.1 is more difficult than one might expect.

For the remainder of this section, $cx \leq c_0$ denotes a bipartition inequality containing a degenerate cut determined by $W \subset V$. Let C be the index set of the teeth intersecting both W and $\overline{W}$. For each $j \in C$, let d'_j denote the number of handles that are intersected by T_j and contained in W, and let d''_j denote the number of handles that are intersected by T_j and contained in $\overline{W}$. Clearly, $d_j = d'_j + d''_j$. Define $ax \leq a_0$ to be the bipartition inequality whose handles are the handles contained in W and whose teeth are the teeth not contained in $\overline{W}$, and $bx \leq b_0$ to be the bipartition inequality whose handles are the handles contained in $\overline{W}$ and whose teeth are the teeth not contained in W. It is easy to verify that $cx \leq c_0$ can be represented as:

$$(10) \quad ax + bx - \sum\left(\frac{d_j - 2}{d_j - 1}x(T_j) :\; j \in C\right) \leq a_0 + b_0 - \sum\left(\frac{d_j-2}{d_j-1}(|T_j| - 1) :\; j \in C\right).$$

Lemma 3.2 *Any c-tight factor x^* satisfies*

$$(11) \qquad ax^* + 2bx^* - \sum\left(\left[\frac{d''_j - 1}{d_j - 1} + 2\frac{d'_j - 1}{d_j - 1}\right] x^*(T_j) :\; j \in C\right)$$
$$= a_0 + 2b_0 - \sum\left(\left[\frac{d''_j - 1}{d_j - 1} + 2\frac{d'_j - 1}{d_j - 1}\right](|T_j| - 1) :\; j \in C\right).$$

Proof: Let $I(x^*)$ denote $\{j \in C : x^*(T_j) = |T_j| - d_j\}$. By Theorem 2.6, we have $x^*(T_j) = |T_j| - 1$ for all $j \in C \backslash I(x^*)$, and $x^*(T_j \cap H_i) = |T_j \cap H_i| - 1$ for all $j \in I(x^*)$ and i such that H_i intersects T_j.

For every $j \in I(x^*)$, choose a handle $H_i \subset \overline{W}$ such that H_i intersects T_j and define T'_j to be $T_j \cap (W \cup H_i)$. Define $\hat{a}x \leq \hat{a}_0$ to be the bipartition inequality obtained from $ax \leq a_0$ by replacing T_j by T'_j for each $j \in I(x^*)$. Notice that $x^*(T'_j) = |T'_j| - d'_j - 1$ for all $j \in I(x^*)$. (This follows from Theorem 2.6 and the definition of T'_j.) Also notice that each T'_j is a nondegenerate tooth of $\hat{a}x \leq \hat{a}_0$. Therefore, $\hat{a}x$ is just $ax + \sum(x(T'_j) - x(T_j) : j \in I(x^*))$ and $\hat{a}_0$ is just $a_0 + \sum(|T'_j| - |T_j| : j \in I(x^*))$. Since $\hat{a}x^* \leq \hat{a}_0$, we have

$$\begin{aligned} ax^* &\leq \hat{a}_0 - \sum(x^*(T'_j) - x^*(T_j) : j \in I(x^*)) \\ &= a_0 + \sum(|T'_j| - |T_j| : j \in I(x^*)) - \sum(|T'_j| - d'_j - 1 - (|T_j| - d_j) : j \in I(x^*)) \\ &= a_0 - \sum(d_j - d'_j - 1 : j \in I(x^*)) \\ &= a_0 - \sum(d''_j - 1 : j \in I(x^*)). \end{aligned}$$

Therefore

$$(12) \quad ax^* \leq a_0 - \sum(d''_j - 1 : j \in I(x^*)).$$

By symmetry we have

$$(13) \quad bx^* \leq b_0 - \sum(d'_j - 1 : j \in I(x^*)).$$

Adding (12) and (13), we get

$$(14) \quad ax^* + bx^* \leq a_0 + b_0 - \sum(d_j - 2 : j \in I(x^*)).$$

Since $cx^* = c_0$, we have from (10) that

$$ax^* + bx^* - \sum\left(\frac{d_j - 2}{d_j - 1}x^*(T_j) : j \in C\right) = a_0 + b_0 - \sum\left(\frac{d_j - 2}{d_j - 1}(|T_j| - 1) : j \in C\right).$$

Since $x^*(T_j) = |T_j| - 1$ for all $j \in C \backslash I(x^*)$, it follows that (14) holds with equality, and therefore that (12) and (13) also hold with equality. Now from this and the facts that $x^*(T_j) = |T_j| - 1$ for $j \in C \backslash I(x^*)$ and $x^*(T_j) = |T_j| - d_j$ for $j \in I(x^*)$, the truth of (11) follows by a straightforward calculation. ∎

We have shown that each point x^* of P^k such that $cx^* = c_0$ satisfies an additional equation (11), which we denote as $gx = g_0$. We can show that the set F of such points is not a facet, by showing that $gx = g_0$ is not a linear combination of $cx = c_0$ and equations that are satisfied by all points of P^k. The latter equations are described as follows.

Lemma 3.3 *The degree constraints (1a) constitute a minimal equality system for P^k.*

Proof: It is well known (and easy to prove) that the degree constraints define a minimal equality system for the TSP polytope. Since P^k contains the TSP polytope and the degree constraints are valid for P^k, the result follows. ∎

Proof of Theorem 3.1: Let A be the node-edge incidence matrix of K_n, so $Ax = \mathbf{2}$ is the system of degree constraints. We must show that g, or equivalently $g' = c - g$, is not in the row space of $\binom{A}{c}$. Notice that

$$g'x = (c-g)x = -bx + \sum_{j \in C} \frac{d'_j - 1}{d_j - 1} x(T_j).$$

Let H_1 be any handle in W. Choose three teeth T_1, T_2 and T_3, intersecting H_1, and choose nodes $v \in H_1 \cap T_1$, $u \in H_1 \cap T_2$, $w \in H_1 \cap T_3$. Let $c'x \leq c'_0$ be obtained from $cx \leq c_0$ by *complementing* H_1, that is,

$$(15) \quad c' = c - \frac{1}{2} \sum (A_i : i \in H_1) + \frac{1}{2} \sum (A_i : i \in V \setminus H_1), \quad c'_0 = c_0 + |V \setminus H_1| - |H_1|,$$

where A_i is the row of A indexed by node i. So we only need to show that g' is not in the row space of $\binom{A}{c'}$.

Consider $B = \{vv' : v' \in V \setminus T_1\} \cup \{uw, uu' : u' \in T_1 \setminus \{v\}\}$. (All of these edges exist, because we are working in a complete graph.) Notice that B consists of a spanning tree plus one additional edge. Therefore, the columns of A indexed by B are linearly independent, and, since $|B| = n$, B forms a basis of E. Further observe that $g'_e = c'_e = 0$ for all $e \in B$, and so what remains to show is that c' and g' are linearly independent. Since $cx \leq c_0$ does not have two complementary handles, by symmetry we may assume without loss of generality that W itself is not a handle. Let $w' \in W \setminus H_1$ and choose $\bar{w} \in \overline{W}$ with $c_{w'\bar{w}} = 0$. The proof is complete since $c'_{w'\bar{w}} \neq 0$ and $g'_{w'\bar{w}} = 0$. ∎

Now we describe a second necessary condition for a bipartition inequality to be facet-inducing. It is simpler than the degenerate cut condition. However, it is not clear that it can be checked efficiently. The valid inequality $ax \leq a_0$ is *dominated by* the valid inequality $bx \leq b_0$ if every $x \in P^k$ satisfying $ax = a_0$ also satisfies $bx = b_0$. (Equivalently, the face induced by $ax \leq a_0$ is contained in the face induced by $bx \leq b_0$.)

Lemma 3.4 *Let $cx \leq c_0$ be a k-bipartition inequality. If $cx \leq c_0$ is dominated by the nonnegativity constraint $-x_e \leq 0$ for some $e \in E$, then it is not facet inducing for P^k.*

Proof. No subtour elimination constraint is dominated by $-x_e \leq 0$. So $cx \leq c_0$ is defined by at least one handle. If $cx \leq c_0$ has at least two handles, we choose a handle, say H_1 such that $e \notin E(H_1)$. If $cx \leq c_0$ is a comb inequality, then we can assume that it is defined to have handle H_1 with $e \notin E(H_1)$. Let T_1, T_2 and T_3 be teeth intersecting H_1. Since $e \notin E(H_1)$, there exist two teeth, say T_1 and T_2, such that $e \notin \delta(T_1 \cap H_1) \cup \delta(T_2 \cap H_1)$. Let $c'x \leq c'_0$ be obtained from $cx \leq c_0$ by complementing H_1; see (15). Let $v \in T_1 \cap H_1, u \in T_2 \cap H_1, w \in T_3 \cap H_1$. So $B = \{vv' : v' \in V \setminus T_1\} \cup \{uw, uu' : u' \in T_1 \setminus \{v\}\}$ forms a basis with $c'_f = 0$ for all $f \in B$. Now $e \notin B$ by construction of B, so it is enough to show that $c'x \leq c'_0$ is not a multiple of $-x_e \leq 0$. This is true because any edge f of $E(T_1) \cup E(T_2)$ different from e has $c'_f \neq 0$. It follows that $cx \leq c_0$ is not facet inducing. ∎

We conjecture that the two necessary conditions are together sufficient for a bipartition inequality to be facet inducing.

Conjecture 3.5 *Let $cx \leq c_0$ be a k-bipartition inequality having no complementary handle or tooth. Then $cx \leq c_0$ is facet-inducing for P^k if and only if it has no degenerate cut and it is not dominated by a non-negativity inequality.*

In the next section we prove this conjecture for $k = 3$. In the process, we show that the second necessary condition can be checked efficiently if $k = 3$.

4 Facet-inducing bipartition inequalities for $k = 3$

In this section we characterize the 3-bipartition inequalities that induce facets of P^3. In fact, we prove Conjecture 3.5 for the case when $k = 3$.

Theorem 4.1 *Let $cx \leq c_0$ be a 3-bipartition inequality having no complementary handle or tooth. Then $cx \leq c_0$ is facet-inducing for P^3 if and only if it has no degenerate cut and it is not dominated by a non-negativity inequality.*

It is not at all obvious that the above characterization is a good one, in that it is not clear how easy it is to see that a given 3-bipartition inequality is not dominated by a non-negativity inequality. We are going to show that this property is equivalent to a matching condition in a certain bipartite graph. This is Theorem 4.4 below. Theorem 4.4 not only shows that Theorem 4.1 is a good characterization; it also is essential in its proof.

For the remainder of this section $cx \leq c_0$ denotes a 3-bipartition inequality. The condition that $cx \leq c_0$ is not dominated by a non-negativity inequality is equivalent to the condition that, for every edge e, there is a c-tight factor using e. As a preliminary to finding a condition for this, let us consider the problem of determining whether there is a c-tight factor at all. (In other words, is the inequality *supporting*, that is, does it induce a non-empty face?) We will obtain a a necessary condition from the results of Section 3. First, we need the following result.

Lemma 4.2 *If there exists a c-tight factor, then there exists a c-tight simple factor that saturates every tooth meeting three handles.*

Proof: Let x^* be a c-tight factor. By Lemma 2.4, we can choose x^* to be simple. Let T be a tooth that meets three handles H_1, H_2, H_3 and is not saturated by x^*. Choose $r_i \in H_i \cap T$, $i = 1, 2, 3$. Note that by Theorem 2.6 $x^*_e = 0$ for $e \in E(T)$.

For $i = 1, 2, 3$, let C_i be the subtour in x^* through r_i, let $v_i r_i$ and $r_i s_i$ be the two edges of C_i incident with r_i, and let $2k_i + 1$ be the number of teeth intersecting H_i. Notice that $x^*(H_i \cap T : T \backslash H_i) = 0$, which implies that x^* satisfies (with respect to $H_i = H$) (b) of Theorem 2.7. Therefore, $x^*(\delta(H_i)) = 2k_i$ and each tooth T' meeting H_i other than T has exactly one edge in $(H_i \cap T' : T' \backslash H_i)$. Therefore, $v_i r_i, r_i s_i \notin \delta(H_i)$, so $v_i, s_i \in H_i$.

We now show that there exists a c-tight simple factor such that each such C_i above contains at least 5 nodes. Suppose that C_i in x^* contains 4 nodes r_i, v_i, s_i, q.

Case 1. C_i is contained in H_i (or equivalently, $q \in H_i$).

In this case, since x^* is simple, only r_i in C_i meets a tooth, for otherwise C_i meets another tooth T', and thus there are two teeth T, T' meeting H_i with $x^*(H_i \cap T : T \setminus H_i) =$

$x^*(H_i \cap T' : T' \setminus H_i) = 0$, contradicting (b) of Theorem 2.7. So $c_f = 1$ for all $f \in C_i$. Next, we claim that x^* must contain some other edge uv in $E(H_i \setminus C_i)$ with $c_{uv} = 1$, for otherwise by (b) of Theorem 2.7 (where $T_1, \ldots, T_{2k_i+1}$ are the teeth meeting H_i),

$$|H_i| - k_i = x^*(H_i) = \sum_{j=1}^{2k_i+1} (|H_i \cap T_j| - 1) + 4 \leq |H_i| - 3 - (2k_i + 1) + 4 = |H_i| - 2k_i,$$

a contradiction. (Note that the inequality depends on the fact that none of v_i, s_i, q is in a tooth.) So replacing uv and $r_i v_i$ by ur_i and vv_i merges C_i with another subtour.

Case 2. $q \notin H_i$.

By (b) of Theorem 2.7, we must have both $c_{v_i q} \geq 1$ and $c_{q s_i} \geq 1$, which implies that q is in some tooth T' and both v_i and s_i are in $T' \cap H_i$, a contradiction to the fact that x^* is simple.

Repeating this argument for each C_i of cardinality four yields the desired x^*. Therefore, we may assume that $|C_i| \geq 5$ for $i = 1, 2, 3$. A new simple c-tight factor saturating T can be constructed by replacing $v_1 r_1, r_1 s_1, v_2 r_2, r_2 s_2, v_3 r_3$ with $v_1 s_1, v_2 s_2, r_1 r_2, r_2 r_3, r_1 v_3$. Repeating this argument for other nonsaturated teeth meeting three handles, we obtain the desired factor. ∎

Now let x be a factor as in the above lemma, and let T be a (degenerate) tooth meeting three handles. If a handle H meeting T satisfies $x(H \cap T : T \setminus H) = 2$, then we say that T *occupies* H. Notice that since T is saturated by x, T must occupy one of the handles that it meets, say H. Now by Theorem 2.7 there can be at most one tooth occupying H. (If we have two such teeth and (a) holds, then we have $x(\delta(H)) \geq 2k_h + 3$, and if (b) holds, we have $x(\delta(H)) \geq 2k_h + 1$, so in both cases we have a contradiction to Theorem 2.7.) Therefore, the number of teeth meeting three handles cannot exceed the number of handles. (As an example of a 3-bipartition inequality that cannot be supporting because it violates this condition, consider the one having three handles of size five, and five teeth of size three, such that each tooth intersects each handle in a single node.) More generally, there must be an injection from the set of such teeth to the set of all handles so that each such tooth is mapped to a handle that it meets. This condition can be described in terms of the existence of a matching in a bipartite graph, where there is a node for each handle and a node for each tooth meeting three handles, and adjacency corresponds to non-empty intersection.

The above necessary condition for a bipartition inequality to be supporting is almost sufficient, but it needs to be amended to handle some exceptions. To give one example of such an exception, consider the bipartition inequality having three handles of size three and three teeth of size three, such that each tooth meets each handle in exactly one node, and there are ten nodes in total. Here we see that the matching condition is satisfied; in fact, if there were nine nodes only, the inequality would be supporting. However, there must be a subtour through the node that is in no handle, and this makes it impossible to obtain a tight factor. Notice that this difficulty persists if there are one, two, or three nodes not in any handle, but disappears if there are four or more (because we can make a subtour on just those additional nodes). We can deal with the exceptions by modifying the definition of the bipartite graph mentioned above.

Given the bipartition inequality $cx \leq c_0$, let n_T denote the number of pendent teeth, and let n_0 denote the number of *isolated* nodes, that is, nodes in no handle or tooth, let V_H denote $\{H_i : 1 \leq i \leq h\}$, let

$$S = \begin{cases} \{v, v'\}, & \text{if } n_T = 0 \text{ and } |V_H| \geq 2 \text{ with } 1 \leq n_0 \leq 3; \\ \{v\}, & \text{if } n_T = 1 \text{ and } |V_H| \geq 2; \\ \emptyset, & \text{otherwise,} \end{cases}$$

and let V_T denote $\{T_j : d_j = 3\} \cup S$. We define the graph $G(c)$ to have nodeset $V_H \cup V_T$, with node H_i adjacent to a node T_j if and only if $H_i \cap T_j \neq \emptyset$ and every node of S adjacent to every handle node H_i. (There are no other adjacencies.) Note that $G(c)$ is bipartite with bipartition $\{V_H, V_T\}$. In particular, no matching of $G(c)$ can have cardinality larger than V_T. Whether this bound is tight or not determines whether the inequality is supporting. (We remark that it is easy to check that every bipartition inequality having fewer than two handles is supporting.)

Theorem 4.3 *Let $cx \leq c_0$ be a 3-bipartition inequality having at least two handles. Then $cx \leq c_0$ is supporting for P^3 if and only if there exists a matching in $G(c)$ of cardinality $|V_T|$.*

A further refinement of the matching approach allows us to characterize the 3-bipartition inequalities not dominated by a non-negativity inequality. We state this result next. (Notice that it does provide the promised good characterization, and hence shows that Theorem 4.1 is also a good characterization.) In fact, we will not actually prove Theorem 4.3, since we do not need it, and its proof is similar to the proof of Theorem 4.4. (Again, it is easy to check that a bipartition inequality having fewer than two handles cannot be dominated by a non-negativity inequality.)

Theorem 4.4 *The 3-bipartition inequality $cx \leq c_0$ is not dominated by a non-negativity inequality if and only if there exists a matching of cardinality $|V_T|$ in $G(c)\backslash\{H_i, H_l\}$ for every pair of nodes H_i, H_l in V_H.*

The proof of this theorem requires some technical ideas that will also be useful later. A node v is a *tip* of $cx \leq c_0$ if it is in a tooth but in no handle. A factor x^* is said to *strongly saturate* a tooth T_j of $cx \leq c_0$ if it saturates T_j and, if $d_j = 2$, $T_j = \{p, q, r\}$, and q is a tip, then $x^*_{pq} = x^*_{qr} = 1$. A factor is *special* if it is c-tight, simple, and strongly saturates every tooth.

Lemma 4.5 *Let $cx \leq c_0$ be a supporting 3-bipartition inequality, and let x^* be a c-tight simple factor. Let $\tilde{E} = \{e \in E : c_e = 0$ and e is not incident to any tip $\}$. Then there exists a special factor $\tilde{x}$ such that for all $e \in \tilde{E}$, $\tilde{x}_e = 1$ whenever $x^*_e = 1$.*

Proof: Let x^* be a c-tight simple factor. If every tooth is strongly saturated by x^*, there is nothing to prove. So suppose that there exists a tooth T not strongly saturated by x^*. Let d be the number of handles intersected by T. We demonstrate below how a new c-tight simple factor $\hat{x}$ can be constructed from x^* such that the new factor satisfies $\hat{x}_e = 1$ for all $e \in \tilde{E}$ with $x^*_e = 1$ and strongly saturates T as well as all teeth that are strongly saturated by x^*. By repeating this process, we can construct $\tilde{x}$, as required.

We distinguish four cases:

Case 1. $d = 1$. Then there exists a tip $u \in T$ such that x^* contains $e = uv$ with $c_e = 0$. Applying 2OPT with respect to T, we obtain $\hat{x}$ from x^* by replacing e and another edge $e' \in E(H)$ (since otherwise $c_{e'} = 0$ implies $c\hat{x} > c_0$), where H intersects T, with an edge in $E(T)$ and some other edge, as required.

Case 2. $d = 2$. Let $r \in H_1 \cap T$ and $r' \in H_2 \cap T$. If T is degenerate, then $c_{rr'} = 2$. Applying 2OPT with respect to T, we obtain $\hat{x}$ from x^* by replacing some edges rv and

$r'v'$ in x^* with rr' and vv'. Since $c\hat{x} \leq c_0$, we must have $c_{rv} = c_{rv'} = 1$ and $c_{vv'} = 0$, and the required $\hat{x}$ is obtained.

Now consider $T = \{r, w, r'\}$ with a tip w, and consider the subcases:

Case 2a. If x^* satisfies $x^*_{rr'} = x^*_{rw} = 1$, then we have some $q \notin T$ with $x^*_{wq} = 1$, and replacing rr' and wq with wr' and qr yields the desired $\hat{x}$.

Case 2b. If $x^*_{rr'} = 1$ and $x^*_{rw} = x^*_{wr'} = 0$, then applying 2OPT with respect to T either results in a factor violating $cx \leq c_0$ (a contradiction), or in a factor as in Case 2a.

Case 2c. If $x^*_{rw} = 1$ and $x^*_{rr'} = x^*_{wr'} = 0$, then applying 2OPT with respect to T gives the desired $\hat{x}$.

Case 2d. If $x^*_{rw} = x^*_{rr'} = x^*_{wr'} = 0$, then applying 2OPT with respect to T yields a new factor. Set the new factor to be x^*, and we are in the Case 2c.

Case 3. $d = 3$. This case is handled by the same construction as was used in the proof of Lemma 4.2.

The proof is complete. ∎

We are now able to prove that the matching condition of Theorem 4.4 is necessary.

Proof of necessity in Theorem 4.4. Let $v_i \in H_i \cap T_j$ and $v_l \in H_l \cap T_p$ be a pair of nodes satisfying $c_{v_i v_l} = 0$. Since $cx \leq c_0$ is not dominated by a non-negativity inequality, there exists a c-tight factor x^* such that $x^*_{v_i v_l} = 1$. Moreover, since $v_i v_l \in \tilde{E}$, by Lemma 4.5, there exists a special factor $\tilde{x}$ with $\tilde{x}_{v_i v_l} = 1$. This implies that no degenerate tooth can occupy H_i or H_l, because if a handle H is occupied, then by Theorem 2.7, we have must be in case (a), have $|\{e \in \delta(H) : \tilde{x}_e > 0 < c_e\}| = 2k_h + 2$, which means that no edge of $\tilde{x}$ having $c_e = 0$ can be in $\delta(H)$. Now we construct a matching M in $G(c) \backslash \{H_i, H_l\}$.

We begin with $M = \emptyset$. For every degenerate tooth T_j that occupies a handle H in $\tilde{x}$, we put the corresponding edge (T_j, H_i) into M. As observed above, M remains a matching. If $S = \emptyset$, we are done. So we just have to handle the two special cases.

Case 1. $n_T = 0$ and $|V_H| \geq 2$ with $1 \leq n_0 \leq 3$. (So $S = \{v, v'\}$.)

Let C_0 be a subtour of $\tilde{x}$ containing at least one isolated node. Since $n_0 \leq 3$, $\tilde{x}$ is special, and $n_T = 0$, C_0 must contain at least one node from some handle H. We have $c_e = 0$ for every edge e of C_0 in $\delta(H)$. It follows that H is not occupied by any tooth meeting three handles. (As above, by Theorem 2.7, such a handle H satisfies $c_e > 0$ for all $e \in \delta(H)$ for which $\tilde{x}_e > 0$.) If H is the only handle having a node in C_0, then

$\tilde{x}$ uses two edges in $\delta(Hi)$ for which $c_e = 0$, but this would contradict Theorem 2.7, which implies that there can be at most one such edge. So C_0 visits at least two handles. Suppose that, beginning from an isolated node u, and proceeding in both directions on C_0, we let H_1, H_2, be the two handles first encountered. We add (v, H_1) and (v', H_2) to M. To show that we still have a matching, we need to show that neither H_1 nor H_2 is occupied by a tooth, or is one of H_i, H_l. Both follow from Theorem 2.7. The first one is a consequence of the fact that occupied handles satisfy $c_e > 0$ for all $e \in \delta(H)$ for which $\tilde{x}_e > 0$. The second follows from the fact that the only edge $e \in \delta(H_i)$ for which $\tilde{x}_e > 0 = c_e$, is v_iv_l, and similarly for H_l. But the edge of C_0 that first enters H_1 (or H_2) is incident to an isolated node, so it cannot be v_iv_l. So M is the desired matching.

Case 2. $n_T = 1$ and $|V_H| \geq 2$. (So $S = \{v\}$.)

Let $T_1 = \{v_1, t_1\}$ be the pendent tooth and let H be the handle it meets. Let C_0 be the subtour of $\tilde{x}$ meeting T_1. Then C_0 must contain edge t_1v_1 where t_1 is a tip and $v_1 \notin T_1$. (We are using the fact that $\tilde{x}$ saturates both T_1 and $H \cap T_1$.) If we proceed along C_0 from t_1 to v_1 and beyond, we find an edge of e of C_0 on which C_0 first enters a handle H'. (It cannot enter a tooth, because $\tilde{x}$ is special, and T_1 is the only pendent tooth.) We add (v, H') to M. (It is possible that $H' = H$.) To show that M remains a matching, we must show that H' is not occupied by a tooth, and that H is different from both H_i and H_l. The former is true because $c_e = 0$ and an occupied handle cannot be entered by such an edge of $\tilde{x}$. The latter follows from the fact that e enters H' from an isolated node or a tip, whereas the only edge e' of $\tilde{x}$ with $c_{e'} = 0$ entering H_i or H_l is v_iv_l, which does not have this property. ∎

The following construction, called the *C-construction*, will be useful in the sequel. Let M be a matching of $G(c)$ of cardinality $|V_T|$. For each tooth T_j, construct a path P_j of length $|T_j| - 1$ in T_j such that

(i) If T_j intersects a handle H_i in two nodes r, s, then rs is an edge of P_j;

(ii) P_j enters and leaves a handle H_i if and only if $H_iT_j \in M$.

Then for each handle H_i, there are $2k_i + 1$ paths P_j visiting H_i, and at least $2k_i$ of them have an end in H_i. Choose $2k_i$ such nodes and k_i paths $Q_{i\ell}$ joining them in pairs, so that every node of H_i not in any P_j is in exactly one of these paths. Define the set C to be the union of the edge sets of all of the P_j and all of the $Q_{i\ell}$. Note that $G(V,C)$ has no vertex of degree more than two, it has a node of degree one in each handle not covered by M, and $|C \cap E(H_i)| = |H_i| - k_i - 1$ for each handle H_i. Also,

$$
\begin{aligned}
c(C) &= \sum_{i=1}^{h}(|H_i| - k_i - 1) + \sum_{j=1}^{t}(|T_j| - 1) + \sum_{j=t+1}^{t+m} \frac{d_j}{d_j - 1}(|T_j| - 1) \\
&= \sum_{i=1}^{h}(|H_i| - k_i - 1) + \sum_{j=1}^{t+m} d_j + \sum_{j=1}^{t}(|T_j| - d_j - 1) + \sum_{j=t+1}^{t+m} \frac{d_j}{d_j - 1}(|T_j| - d_j) \\
&= \sum_{i=1}^{h} |H_i| + \sum_{i=1}^{h} k_i + \sum_{j=1}^{t}(|T_j| - d_j - 1) + \sum_{j=t+1}^{t+m} \frac{d_j}{d_j - 1}(|T_j| - d_j).
\end{aligned}
$$

This construction allows us to construct (many) special factors. In particular, if $G(c)\backslash\{H_p, H_q\}$ has a matching of cardinality $|V_T|$, then $G(V,C)$ will have a node of degree 1 in each of H_p, H_q. This allows us to add edges to C to form a special factor. Besides the flexibility in the choice of p, q, there may be flexibility in the choice of the paths P_j, of the $2k_i$ nodes in H_i for each i, and of the $Q_{i\ell}$. We use this construction repeatedly to prove both Theorem 4.4 and Theorem 4.1.

Proof of Theorem 4.4. We have already proved the necessity of the matching condition. For sufficiency, we need to show that for any edge uv, there exists a tight factor using uv. For many choices of uv, this is easy.

If $uv \in E(H_i)$ for some i, we first choose M so that H_i is not covered. Then it is easy to ensure that uv is an edge of some P_j or some $Q_{i\ell}$.

If $uv \in E(T_j)$ for some j, then it is easy to ensure that $uv \in P_j$, with two exceptions. In the first exceptional case, $d_j = 2$. Say that $T_j = \{u, v, w\}$, and H_i is the handle containing v. We choose M so that H_i is not covered. Then P_j will use uv. In the second exceptional case, $T_j = \{u, v, w\}$, with w a tip, $u \in H_i$ and $v \in H_m$. Then we choose M so that H_i is not covered, and choose the $Q_{i\ell}$ so that u has degree one in $G(V,C)$. Then the special factor resulting contains vw, wu and an edge us such that $c_{us} = 0$. Replacing vw and us by uv and ws gives the required tight factor. (It is not special.)

If $c_{uv} = 0$ it is easy in most cases to construct C so that u, v both have degree one or zero in $G(V,C)$. We treat only the cases where it is not. One case is where u, say, is a node of some handle H_i but of no tooth. We choose M so that H_i is not covered, and construct C. Then u is incident with two edges su, ur of C and there is a $w \in H_i$

incident to just one edge of C. We replace su, ur in C by sr, wu. The other case is where u, say, is a tip of a tooth $T_j = \{u, a, b\}$ with $a \in H_i$ and $b \in H_m$. We choose M so that H_m is not covered, and choose the $Q_{i\ell}$ so that b has degree one in $G(V, C)$. Then we replace ub by ab in C. Notice that we can apply these last techniques independently for either of u or v, since we can choose M to miss any two handles. ▮

We are now ready to prove the main result of this section, Theorem 4.1. Since the necessity of the conditions follows from Theorem 3.1, Lemma 3.4, and Theorem 4.4, we need to prove sufficiency. We consider only the case where the number h of handles defining $cx \le c_0$ is more than 1, for otherwise $cx \le c_0$ is an SE constraint or a comb inequality, and hence the theorem holds by the well known polyhedral results for the travelling salesman polytope P^n. Since $cx \le c_0$ is not dominated by a non-negativity inequality, by Theorem 4.4, there exists a matching of cardinality $|V_T|$ missing any two handles H_p, H_q. We will use these matchings and the C-construction to obtain tight factors containing specific edges.

Since there are at least two handles and no degenerate cut, there exist handles H and H' and a tooth T such that $T = \{r, w, r'\}$, $T \cap H = \{r\}$, and $T \cap H' = \{r'\}$. Let T' be a tooth different from T that intersects H', and let $s' \in H' \cap T'$. Note that T' may intersect H. Let T'' be a tooth different from T that intersects H. Figure 4 gives a picture of the situation, but it is not completely general. Choose a node $s \in H \cap T''$. We define B to be

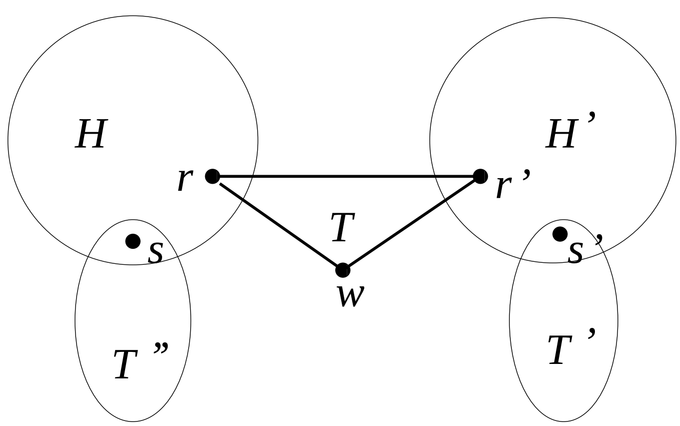

Figure 4: Proving sufficiency.

$\{rs', r's, ss'\} \cup \{wv : v \in V \backslash \{r, r'\}\}$. Thus, B forms a basis and $c_e = 0$ for all $e \in B$. Let

$fx \leq f_0$ be a facet-inducing inequality that dominates $cx \leq c_0$ with $f_e = 0$ for all $e \in B$. Let $\alpha = f_{rr'}$. We successively derive all values of f_e by *comparing* tight factors, that is, given tight factors x, x', using the fact that $0 = cx - cx' = fx - fx'$ to derive values for the f_e. For a factor x, $x \triangle (e_1, \ldots, e_l; e'_1, \ldots, e'_l)$ denotes the factor obtained from x by replacing edges $e_1, \ldots, e_l$ with $e'_1, \ldots, e'_l$. (That is, we take the symmetric difference of the factor x and the edge set $\{e_1, \ldots, e_l, e'_1, \ldots, e'_l\}$, but our notation makes explicit which edges are in and not in x.)

Claim 4.6 *$f_e = \alpha$ for all $e \in E(T)$.*

Proof: It is easy to arrange in the C-construction for C to contain rw and wr' and for r and s' to be incident with exactly one edge of C. Therefore, there is a tight factor x using rw, wr', and rs'. Comparing x with the tight factor $x \triangle (rs', wr'; rr', ws')$ gives $f_{wr'} = \alpha$. Similarly, we have $f_{wr} = \alpha$, and $f_{rr'} = \alpha$ by definition. ▮

Claim 4.7 *$f_e = 0$ for all $e \in \delta(T)$ with $c_e = 0$.*

Proof: For any node u such that $c_{ur} = 0$, it is easy to arrange in the C-construction for C to contain rw and wr', and for u and r to be incident with at most one edge from C. Thus there is a tight factor x using rw, wr', and ur. Comparing x with the tight factor $x \triangle (ru, wr'; rr', wu)$ gives $f_{ru} = 0$. Similarly, for each u such that $c_{r'u} = 0$, we have $f_{r'u} = 0$. Finally, for every u such that $c_{wu} = 0$, we have $f_{wu} = 0$ by definition. ▮

Claim 4.8 *If $e \in E(H) \cup E(H')$ with $c_e = 1$, then $f_e = \alpha$.*

Proof: As before, we can construct a tight factor x using ss', rw, wr'. Notice that $x' = x \triangle (ss', rw; ws', rs)$ is also a factor, since by Theorem 2.7, x contains edges $su \in E(T'')$ and $s'u' \in E(T')$, but neither ur nor $s'r'$. So, comparing x and x' yields $f_{rs} = \alpha$.

Next, consider any pair of nodes $u, v \in H \setminus \{r\}$. We can construct a tight factor x using uv, rs', rw, wr'. (The C-construction will automatically use the last two edges, and we can arrange that uv is in some $Q_{i\ell}$, and that r and s' be incident to exactly one edge of C.) Observe that $x' = x \triangle (uv, wr, rs'; ur, vr, ws')$ and $x'' = x \triangle (uv, rs', wr'; vr, wu, r's')$ are also tight factors, and thus comparing x' and x'' yields $f_{ur} = f_{r's'}$. So we can derive that $f_{r's'} = f_{ur} = f_{rs} = \alpha$ for all $u \in H \setminus \{r\}$. Now comparing x and x' gives $f_{uv} = \alpha$. It follows that $f_e = \alpha$ for all $e \in E(H)$ with $c_e = 1$. By symmetry, $f_e = \alpha$ for all $e \in E(H')$ with $c_e = 1$. ∎

Claim 4.9 *If $e \in \delta(H) \cup \delta(H')$ with $c_e = 0$, then $f_e = 0$.*

Proof: By Claim 4.7 and the symmetry between H and H', we need only consider $e = uv$ with $v \in H$ and $\{u, v\} \cap T = \emptyset$. Suppose that there exists a tight factor x that uses uv, qr, rw, wr' for some $q \in H$, and moreover, uses edges qq' and vv', where q', v' are distinct and different from q, v, r. Then $x \triangle (uv, qr; qv, ur)$ is also a tight factor, and by Claims 4.6, 4.7, and 4.8, comparing it with x shows $f_{uv} = 0$. Now we explain how to construct x.

If v is contained in some other tooth, then by Theorem 2.7, a tight factor x containing uv, rw, wr' satisfies the desired property and such an x is easily constructed using the C-construction.

If v is in no tooth, let x' be a special factor containing uu' for some $u' \in H \cap \hat{T}$ for some tooth $\hat{T}$ and satisfying $c_{uu'} = 0$, where $\hat{T}$ is another tooth meeting H. Then let $v \ldots v_l$ be a path of all nodes in H that are not covered by any tooth (v_l may be v itself), and the required x can be contructed from x' by inserting $v \ldots v_l$ between uu' in x' and removing those nodes from other positions in x'. ∎

Now suppose that T' is intersected by another handle H'', where $\{s''\} = T' \cap H''$ and w' is a tip in T'.

Claim 4.10 *There exists some scalar α' such that $f_e = \alpha'$ for all $e \in E(T')$.*

Proof: Let $\alpha' = f_{s's''}$. By the C-construction, we obtain a special factor x containing $rs', s'w', w's''$. Comparing x with $x \triangle (rs', w's''; rw', s's'')$ implies by Claim 4.7 $f_{w's''} = \alpha'$. Similarly, let x be a special factor containing $r's'', s'w', w's''$; comparing x with $x \triangle (r's'', w's'; r'w', s's'')$ implies $f_{w's'} = \alpha'$. ∎

Claim 4.11 *$f_e = 0$ for all $e \in \delta(T')$ with $c_e = 0$.*

Proof: By Claims 4.6-4.9 we only need to consider edges $e = uv$ with $c_{uv} = 0$, $v \in \{w', s''\}$ and $u \notin T \cup H \cup H'$. Let x be a special factor containing $us', s'w', w's''$. Using Claim 4.10, comparing x with $x \triangle (us', w's''; uw', s's'')$ yields $f_{uw'} = f_{us'} = 0$. Now consider us''. Since $c_{us''} = 0$, $u \notin H''$. Let x' be a special factor containing $us'', s''w', w's'$; comparing x' with $x' \triangle (us'', w's'; uw', s's'')$ shows $f_{us''} = f_{uw'} = 0$. ∎

Now, notice that f_e's for $e \in E(T') \cup \delta(T')$ are proportional to those for $e \in E(T) \cup \delta(T)$, and so we can apply Claim 4.8 with respect to T'. It follows that $\alpha = \alpha'$. Since $cx \le c_0$ has no degenerate cut, by repeated applications of Claims 4.7–4.11, we derive that for any handle $\hat{H}$ and any nondegenerate nonpendent tooth $\hat{T}$,

$f_e = \alpha$, for all $e \in E(\hat{H}) \cup E(\hat{T})$ with $c_e = 1$, and

$f_e = 0$, for all $e \in \delta(\hat{H}) \cup \delta(\hat{T})$ with $c_e = 0$.

The above properties of f are used implicitly in the sequel.

Let $\tilde{T}$ be any pendent tooth intersecting some handle $\tilde{H}$, and $u \in \tilde{T} \setminus H'$.

Claim 4.12 $f_e = \alpha c_e$ *for all* $e \in E(\tilde{T})$.

Proof: We may assume without loss of generality that $H' = \tilde{H}$. Suppose $\{v\} = \tilde{T} \cap H'$. Then let x be the special factor containing vr, rw, wr' and $r'v'$. Clearly, if $|\tilde{T}| = 2$, x contains uv. For $|\tilde{T}| = 3$, let $q \in \tilde{T} \setminus H'$, and we may assume that this x contains qu, uv. Comparing x with $x \triangle (vr, uv, r'v'; ur, r'v, vv')$ implies $f_{uv} = \alpha$. If $|\tilde{T}| = 3$, by symmetry, we derive $f_{qv} = \alpha$. Then comparing x with $x \triangle (vr, uq; ur, vq)$ yields $f_{uq} = \alpha$.

Now suppose that $\{v, q\} = \tilde{T} \cap H'$. First, let x be a special factor containing rv, rw, wr', vq, qu and $r'v' \in E(H')$. So comparing x with $x \triangle (rv, qu, r'v'; ru, r'v, qv')$ implies $f_{uq} = \alpha$. By symmetry $f_{uv} = \alpha$. Next, observe that there exists a special factor x' containing $r'u, rw, wr', vq, qu$, and x' must contain vv' for some $v' \in H'$. Comparing x' with $x' \triangle (ur', vq; uv, qr')$ implies $f_{vq} = 2\alpha$. ∎

Claim 4.13 $f_e = 0$ *for* $e = uu'$ *with* u' *in any pendent tooth different from* $\tilde{T}$.

Proof: If u' is contained in a handle, then $f_{uu'} = 0$ by previous derivations. So suppose that u' is contained in no handle. If $\hat{T}$ is the tooth containing u', then there exists a special factor x' containing $u'v, u'u'', r'w, wr, v'r'$, where $u'u'' \in E(\hat{T})$, $v \in H' \cap \tilde{T}$ and $v' \in H'$. If

x contains uu'', then comparing x with $x \triangle (u'v, uu''; uu', vu'')$ shows $f_{uu'} = 0$. Otherwise we have two cases. First, if $\tilde{T} \cap H' = \{v\}$, then assume that x contains uv, and comparing x with $x \triangle (u'v, uv, r'v'; uu', vr', vv')$ gives $f_{uu'} = 0$. Second, if $\tilde{T} \cap \{v, q\}$, we may assume that x contains vq, uq, and then comparing x with $x \triangle (u'v, uq, r'v'; uu', vr', qv')$ yields $f_{uu'} = 0$. ∎

Claim 4.14 *For any degenerate tooth $\tilde{T}$, $f_e = \alpha c_e$ for all $e \in E(\tilde{T})$.*

Proof: First, suppose that $\tilde{T}$ intersects two handles H_1 and H_2 with $u_i \in H_i \cap \tilde{T}$, $i = 1, 2$. Assume without loss of generality that $H = H_1$ since each handle intersects a nondegenerate tooth that connects two handles. There exists a special factor x containing ru_2, rw, wr', and some edge $u'v' \in E(H_2)$. For $|\tilde{T}| = 2$, comparing x with $x \triangle (ru_2, u'v', u_1u_2; ru_1, u'u_2, v'u_2)$ yields $f_{u_1u_2} = 2\alpha$. For $|\tilde{T}| = 3$, we may assume without loss of generality that $\{u_2, q\} = \tilde{T} \cap H_2$. The factor x then contains u_2q, qu_1. Comparing x and $x \triangle (ru_2, qu_1, u'v'; ru_1, u'u_2, v'q)$ implies $f_{u_1q} = 2\alpha$. By symmetry, $f_{u_1u_2} = 2\alpha$. Finally, let $q' \in H_2 \cap \hat{T}$ with $\hat{T} \neq \tilde{T}$, and x' be a special factor containing u_1q'. So x' contains u_1q, qu_2. Comparing x' with $x' \triangle (u_1q', u_2q; u_1u_2, qq')$ yields $f_{qu_2} = 3\alpha$.

Second, suppose that T intersects three handles, say H_1, H_2, H_3, and let $r_i \in H_i \cap \tilde{T}$, $i = 1, 2, 3$. By the C-construction, there exists a special tight factor x containing $v_1r_2, r_2r_3, r_3r_1, r_1s_1$ where $v_1, s_1 \in H_1$, and v_1 and s_1 are each contained in teeth. Furthermore, there exist nodes $v_2, s_2 \in H_2$ and $v_3, s_3 \in H_3$ such that x contains v_2s_2 and v_3s_3. Since x is special, then $x^* = x \triangle (v_1r_2, r_2r_3, r_3r_1, v_2s_2, v_3s_3; v_1r_1, v_2r_2, r_2s_2, v_3r_3, r_3s_3)$ is a tight factor. Comparing x with x^* implies $f_{r_1r_3} + f_{r_2r_3} = 3\alpha$. By symmetry, we have $f_{r_1r_2} + f_{r_2r_3} = f_{r_2r_1} + f_{r_3r_1} = 3\alpha$. It follows that $f_e = \frac{3}{2}\alpha$ for $e \in E(\tilde{T})$. ∎

Let V_0 be the collection of all nodes not contained in any handle or tooth, and T_0 be the collection of all nodes contained in some pendent tooth but in no handle.

Claim 4.15 *$f_e = 0$ for all $e \in E(V_0) \cup E(V_0 : T_0)$.*

Proof: Let x be a special factor containing rs'. (Recall $r \in H \cap T$, $s' \in H' \cap T'$.) For any $uv \in E(V_0)$, if x contains uv, let $x^* = x$. Otherwise let x^* be obtained from x by applying 2OPT with respect to $S = \{u, v\}$. So x^* is a special factor containing rs', uv.

Now observe that either $x^* \triangle (uv, rs'; ur, vs')$ or $x^* \triangle (uv, rs'; us', vr)$ is a tight factor, and comparing the resulting factor with x^* yields $f_{uv} = 0$.

Next, consider any $u \in V_0$ and $v \in T_0$. Assume that v is contained in pendent tooth $\hat{T}$, and $v' \in \hat{T} \cap H'$. We distinguish the following two cases:

Case 1. If $\hat{T}$ is the only pendent tooth, let x be a special factor containing $vr', r'w, wr$ and vv' with $v' \in H' \cap \hat{T}$. Thus we may assume that x contains some uu' with u' in some handle, and hence $f_{uu'} = 0$. Further, note that $x \triangle (uu', vr', vv'; v'r', uv, vu')$ is a tight factor, and so comparing it with x implies $f_{uv} = 0$.

Case 2. If there exists a pendent tooth T'' different from $\hat{T}$, let $v'' \in T'' \cap H''$. Let x be a special factor containing $v'v''$. Set $x^* = x$ if x contains uv. If not, let x contain vq with $c_{vq} = f_{vq} = 0$, and replace vq with some subpath containing u to obtain x^* containing uv. Now comparing x^* with $x^* \triangle (uv, v'v''; uv', vv'')$ shows $f_{uv} = 0$. ∎

Combining the above lemmas, we have $f_e = \alpha c_e$ for all $e \in E$. It follows that $cx \leq c_0$ induces a facet.

5 Another facet

Since the problem of existence of a restricted factor is solvable in polynomial time when $k = 3$, we may hope that the optimal restricted factor problem is solvable in this case. Hence, we may hope that one could find a complete description by linear inequalities for P^3. A natural first candidate for such a description is the set of all degree, non-negativity, and 3-bipartition constraints. However, this list is not sufficient in general. Consider the inequality $cx \leq c_0$ indicated in Figure 5, where numbers on edges are coefficients, missing edges have coefficient zero and the right-hand side is 16. Let us first explain where this inequality comes from. There is a 3-bipartition inequality $dx \leq d_0$ having the same support. It has three handles and four degenerate teeth, one of size three and the others of size two. This inequality has a degenerate cut, and so is not facet-inducing, by Theorem 3.1. In fact, the proof of that result allows us to identify a comb inequality $px \leq p_0$ inducing a facet that properly contains the face F of $P^3(9)$ induced by $dx \leq d_0$. Of course, there must be other inequalities inducing faces properly containing F. One of them can be obtained as follows. Consider the inequality$qx \leq q_0$

defined to be $dx - \alpha px \leq d_0 - \alpha p_0$, where α is chosen as large as possible so that it is valid for $P^3(9)$. Then $qx \leq q_0$ is equivalent to the inequality of Figure 5.

Proposition 5.1 *The inequality* $cx \leq c_0$ *of Figure 5 is facet-inducing for* $P^3(9)$*, and is not equivalent to any non-negativity or bipartition inequality.*

The proof of this result is elementary, but not particularly short or illuminating, so it is not included here. It is not at all clear to us what class of inequalities this one might belong to, so we have no conjecture as to a complete description for $P^3(n)$.

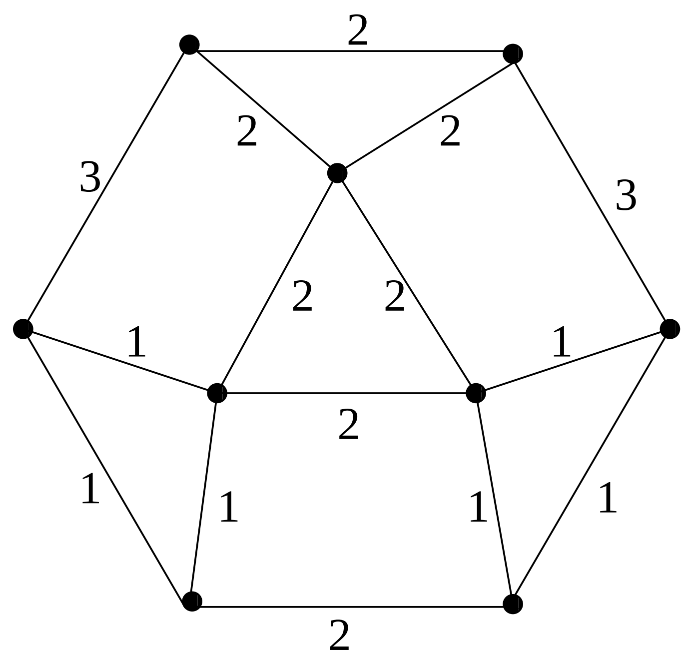

Figure 5: Another facet for P^3.

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