Ladders for travelling salesmen *

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Abstract

We introduce a new class of valid inequalities for the symmetric travelling salesman polytope. The family is not of the common handle-tooth variety. We show that these inequalities are all facet-inducing and have Chvátal rank 2.

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1 Introduction

The *symmetric travelling salesman polytope* $STSP(V)$ is the convex hull of incidence vectors of edge-sets of Hamiltonian cycles of the complete graph on node set V. A description of this polytope by linear inequalities would essentially reduce the travelling salesman problem to a linear program. While there are reasons to believe that we cannot hope to obtain such a complete description, known partial descriptions of the polytope have proved to be remarkably useful in cutting plane approaches to the problem. (See [4, 9], for example.) A good deal of progress has been made in extending these partial descriptions by finding new classes of facet-inducing inequalities, and in incorporating this additional knowledge into the computational approaches.

In this paper we introduce a new class of valid inequalities for $STSP(V)$, called *ladder inequalities*. These inequalities differ from most of the inequalities discovered so far, in that they are not of the usual "handle-tooth" variety. On the other hand, they arise from a strengthening of certain inequalities of this type. A computational study in [4] demonstrates use of ladder inequalities to improve the bounds of LP relaxations. We prove that all ladder inequalities are facet-inducing. We also show that they all have Chvátal rank exactly 2.

2 Preliminaries

Let V be any node set with $n \equiv |V| \geq 3$. We deal with the undirected complete graph $K_n = (V, E)$, and we write elements of E as (i, j) or ij. Note that $ij = ji$. For $S \subseteq V$, let $E(S)$ denote $\{ij \in E : i, j \in S\}$. For $S, T \subseteq V$ with $S \cap T = \emptyset$, let $E(S : T)$ denote $\{ij \in E : i \in S, j \in T\}$. For any $v \in V$, define $\delta(v)$ to be $E(\{v\} : V \setminus \{v\})$. For $B \subseteq E$ and $x \in \boldsymbol{R}^E$, let $x(B)$ denote $\sum(x_{ij} : ij \in B)$. Given $c \in \boldsymbol{R}^E$, the (*symmetric*) *travelling salesman problem* (TSP) can be stated as

(1) minimize $\sum (c_{ij} x_{ij} : ij \in E)$

subject to

(1.a) $\sum (x_{ij} : 1 \leq j \leq n,\ j \neq i) = 2, \quad i \in V;$

(1.b) $x(E(S)) \leq |S| - 1, \quad S \subset V,\ 2 \leq |S| \leq n - 2;$

$$(1.c) \quad x_{ij} \geq 0, \quad ij \in E;$$
$$(1.d) \quad x_{ij} \text{ integer}, \quad ij \in E.$$

Any feasible solution x^0 of (1) is the incidence vector of (the edge-set of) a Hamiltonian circuit or *tour* of K_n. We identify a tour (or more generally a path) of K_n with its edge-set or its node-sequence. The convex hull of feasible solutions to (1) is called an *STS polytope*, and is denoted by by $STSP(V)$. The symmetric TSP is equivalent to the linear program

$$\min \left(\sum (c_{ij}x_{ij} : ij \in E) : x \in STSP(V) \right),$$

and in order to apply the methods of linear programming, we would like to describe it as an optimization subject to linear constraints. It is known ([6], for example) that the affine hull of $STSP(V)$ is just the set of solutions of the *degree constraints* (1.a), and hence its dimension is $\binom{n}{2} - n$. Therefore, an inequality $ax \leq a_0$ that is valid for $STSP(V)$ is *facet-inducing* if and only if $\{x \in STSP(V) : ax = a_0\}$ has dimension $\binom{n}{2} - n - 1$. Moreover, two such inequalities $ax \leq a_0$ and $bx \leq b_0$ are *equivalent* (that is, induce the same face) if and only if there exist $\lambda \in \boldsymbol{R}^V$ and a positive scalar λ_0 such that $(b, b_0) = \lambda(A, \bar{2}) + \lambda_0(a, a_0)$, where A is the node-edge incidence matrix of K_n, and $\bar{2}$ is a vector of 2's. One such class of inequalities consists of the *nonnegativity* constraints (1.c). Another consists of the *subtour elimination* (SE) constraints (1.b).

Many of the known classes of valid inequalities arose from generalizations of the comb inequalities, which we now describe. They were first defined by Chvátal [3] and later generalized by Grötschel and Padberg [5]. Given a *handle* $H \subset V$ and mutually disjoint *teeth* $T_1, T_2, \ldots, T_{2k+1} \subset V$ (k integer, $k \geq 1$) such that

$$T_j \cap H \neq \emptyset \neq T_j \setminus H, \quad 1 \leq j \leq 2k+1,$$

the associated *comb* inequality is

$$x(E(H)) + \sum_{j=1}^{2k+1} x(E(T_j)) \leq |H| + k + \sum_{j=1}^{2k+1} (|T_j| - 2).$$

It is proved in [5] that every comb inequality is facet-inducing for $STSP(V)$.

3 Ladder inequalities

Let H_1 and H_2 be mutually disjoint subsets of V called *handles*. Let $T_1, T_2, \ldots, T_{t+m}$ be pairwise disjoint proper subsets of V called *teeth*, where $t \geq 2$, $m \geq 0$, and $t+m$ is even and at least 4. A tooth T_j is *degenerate* if $T_j \setminus (H_1 \cup H_2) = \emptyset$; otherwise it is *nondegenerate*. Assume that $T_1, T_2, \ldots, T_t$ are nondegenerate teeth and (if $m \geq 1$) that $T_{t+1}, \ldots, T_{t+m}$ are degenerate teeth. Assume also that T_1 intersects only H_1, T_2 intersects only H_2, and T_k, $k = 3, \ldots, t+m$, intersects both H_1 and H_2. T_1 and T_2 are called *pendent teeth*; the others are *nonpendent*. The *ladder inequality* associated with $H_1, H_2, T_1, \ldots, T_{t+m}$ is defined as follows:

$$\begin{aligned} (2) \quad & \sum_{i=1}^{2} x(E(H_i)) + \sum_{j=1}^{t} x(E(T_j)) + \sum_{j=t+1}^{t+m} 2x(E(T_j)) + x(E(T_1 \cap H_1 : T_2 \cap H_2)) \\ & \qquad \leq \sum_{i=1}^{2} |H_i| + t + m - 2 + \sum_{j=1}^{t}(|T_j| - d_j - 1) + \sum_{j=t+1}^{t+m} 2(|T_j| - 2), \end{aligned}$$

where d_j denotes the number of handles intersected by tooth T_j.

Many of the known classes of valid inequalities for $STSP(V)$ are generalizations of the comb inequalities, and are determined by two families of node subsets, called handles and teeth. These include clique tree inequalities [7], bipartition inequalities [1], and binested inequalities [8]. However, in all of these classes the left hand side is of the form

$$\sum \alpha_i x(E(H_i)) + \sum \beta_j x(E(T_j)).$$

The last term of the left hand side of the ladder inequalities does not fit this model. In fact, if that term is dropped, (2) becomes a special kind of bipartition inequality. The smallest ladder inequality (on 8 nodes) was introduced in [1] to illustrate a way in which a bipartition inequality can fail to be facet-inducing.

A general ladder inequality $ax \leq a_0$ is presented in Figure 1(a). Nodes are numbered in such a way that the handles are $H_1 = \{2k : k = 1, 2, \ldots, t+m-1\}$ and $H_2 = \{2k+1 : k = 1, 2, \ldots, t+m-1\}$, and the pendent teeth $T_1 = \{1, 2\}$ and $T_2 = \{h, 3\}$. The hollow nodes w, u, g and g' are optional; any of them may be present or absent. Any node may appear any number of times, at least once for each node 1,…,7 and h. Additional copies of a node are called *clones* and will be discussed in Section 5. In the dashed box, we allow any *even* number (possibly zero) of additional nonpendent teeth to be present.

Every nonpendent tooth may be either nondegenerate (if a node like g or g' is present) or degenerate (if there is no such node). In the latter case, the tooth is contained in the union of the handles. Every coefficient a_{ij} in the corresponding ladder inequality $ax \leq a_0$ is determined by the total weight of all sets containing both nodes i and j. The weights for the degenerate teeth are 2. (For instance, if node g in Figure 1(a) does not exist, then tooth $\{6, 7\}$ is degenerate and thus has weight 2.) All other weights are one. The weights are not shown on the figure, to avoid overcrowding it. The fourth term on the left hand side of inequality (2) is represented by a bipartite graph, reduced to a single edge in Figure 1(a). Finally, the right hand side a_0 is as given in inequality (2). Part (b) of Figure 1 will be explained in Section 4.

We now prove the validity of the ladder inequalities. For $i = 1, 2$, let $\hat{T}_i = T_i \setminus H_i$ and $\hat{H}_i = H_i \setminus (\cup_{j=1}^{t+m} T_j)$.

Theorem 3.1 *The ladder inequality (2) is valid for $STSP(V)$.*

PROOF: Add the following valid inequalities for $STSP(V)$, and divide the resulting inequality by 3:
(i) the comb inequality obtained by deleting $\hat{H}_2$ and T_2,
(ii) the comb inequality obtained by deleting $\hat{H}_2$, T_2 and $H_2 \cap T_j$ for $j = 3, \ldots, t$,
(iii) the sum of the degree constraints for each $v \in H_2$,
(iv) the sum of the degree constraints for each $v \in (T_1 \cap H_1) \cup (T_2 \cap H_2)$,
(v) the SE inequality for $(\cup_{j=3}^{t+m} T_j) \cup \hat{H}_1 \cup \hat{H}_2$,
(vi) the sum of the SE inequalities for $T_j \cap H_1$, $j = 3, \ldots, t$,
(vii) the sum of the SE inequalities for $\hat{T}_1$, T_2 and $T_2 \cap H_2$,
(viii) twice the sum of the SE inequalities for $T_j \cap H_2$, $j = 3, \ldots, t + m$,
(ix) twice the sum of the SE inequalities for T_j, $j = t + 1, \ldots, t + m$,
(x) twice the sum of the SE inequalities for $T_j \cap H_1$, $j = t + 1, \ldots, t + m$,
(xi) twice the SE inequality for $\hat{T}_2$.

It is straightforward to check that for all edges e, the integer part of the coefficient of x_e in the resulting inequality is its coefficient in (2). The right hand side RHS is

$$\text{RHS} \quad = \quad \frac{1}{3}\left(|H_1| + |T_1| - 2 + \sum_{j=3}^{t+m} (|T_j| - 2) + \frac{t + m - 2}{2} \right)$$

$$
\begin{aligned}
&+ \quad \frac{1}{3}\left(|H_1| + |T_1| - 2 + \sum_{j=3}^{t}(|T_j \setminus H_2| - 2) + \sum_{j=t+1}^{t+m}(|T_j| - 2) + \frac{t+m-2}{2}\right) \\
&+ \quad \frac{2}{3}|H_2| + \frac{2}{3}|T_1 \cap H_1| + \frac{2}{3}|T_2 \cap H_2| + \frac{1}{3}\left(\sum_{j=3}^{t+m}|T_j| + |\hat{H}_1| + |\hat{H}_2| - 1\right) \\
&+ \quad \frac{1}{3}\sum_{j=3}^{t}(|T_j \cap H_1| - 1) + \frac{1}{3}(|\hat{T}_1| - 1) + \frac{1}{3}(|T_2| - 1) + \frac{1}{3}(|T_2 \cap H_2| - 1) \\
&+ \quad \frac{2}{3}\sum_{j=3}^{t+m}(|T_j \cap H_2| - 1) + \frac{2}{3}\sum_{j=t+1}^{t+m}(|T_j| - 1) + \frac{2}{3}\sum_{j=t+1}^{t+m}(|T_j \cap H_1| - 1) + \frac{2}{3}(|\hat{T}_2| - 1) \\
&= \quad \sum_{i=1}^{2}|H_i| + t + m - 2 + \sum_{j=1}^{t}(|T_j| - d_j - 1) + \sum_{j=t+1}^{t+m} 2(|T_j| - 2) + \frac{2}{3}.
\end{aligned}
$$

Rounding down each coefficient and the right hand side to the nearest integer, we obtain the desired result. □

4 Primitive ladder inequalities

For any inequality $ax \leq a_0$, we define its *support graph* to be $G_a = (V, E_a)$, where $E_a = \{e \in E : a_e \neq 0\}$. In this section, we consider a subclass of ladder inequalities $ax \leq a_0$ which have a *spanning* support graph (that is, G_a contains no isolated nodes) and satisfy the following properties:

- $|H_i \cap T_j| \leq 1$ for any pair H_i and T_j,
- $|T_j \setminus (H_1 \cup H_2)| = 1$ for $j = 1, \ldots, t$, and
- $|H_i \setminus \left(\cup_{j=1}^{t+m} T_j\right)| \leq 1$ for $i = 1, 2$.

The inequalities in this class are called *primitive* ladder inequalities. Thus, Figure 1(a) shows a general primitive ladder inequality if no node has any clone. (Hollow nodes may be present or absent, and there may be any even number of teeth in the dashed box).

Note that any $ax \leq a_0$ can be written in the following form

$$\sum_{i=1}^{l} \omega_i x(E(L_i)) + bx \leq a_0,$$

where the L_i's are subsets of V. By *complementing* L_i with respect to $ax \leq a_0$, we mean adding to the inequality the multiples of degree constraints $-\frac{\omega_i}{2}x(\delta(v)) = -\omega_i$ for all

$v \in L_i$ and $\frac{\omega_i}{2} x(\delta(v)) = \omega_i$ for all $v \in V \setminus L_i$. The resulting inequality is clearly equivalent to $ax \leq a_0$ but has different coefficients. To facilitate the polyhedral proof, we need a *unique* representation of valid inequalities for $STSP(V)$. This representation is given by the following lemma.

Lemma 4.1 *Let $ax \leq a_0$ be any valid inequality for $STSP(V)$, and let h, u and v be any three distinct nodes in V. Define $B \equiv \delta(h) \cup \{(u,v)\}$. Then there is a unique (up to positive multiples) inequality $cx \leq c_0$ that is equivalent to $ax \leq a_0$ and satisfies $c_e = 0$ for all $e \in B$.*

The lemma follows directly from Remark 4.2 in Grötschel and Padberg [5] by observing that B corresponds to a basis of the column vectors in the node-edge incidence matrix. We call such a representation, $cx \leq c_0$, an (h, uv)-*canonical form*, or an (h, uv)-*canonical inequality*. An example of a ladder inequality in $(h, 13)$-canonical form $cx \leq c_0$ is presented in Figure 1(b). This can be obtained by complementing tooth T_2. Note that $c_{31} = c_{3i} = 0$ for all $i \geq 4$ and even, $c_{21} = c_{24} = c_{26} = 2$, $c_{52} = c_{51} = 1$, etc. Note also that if g is absent, then $c_{67} = 3$.

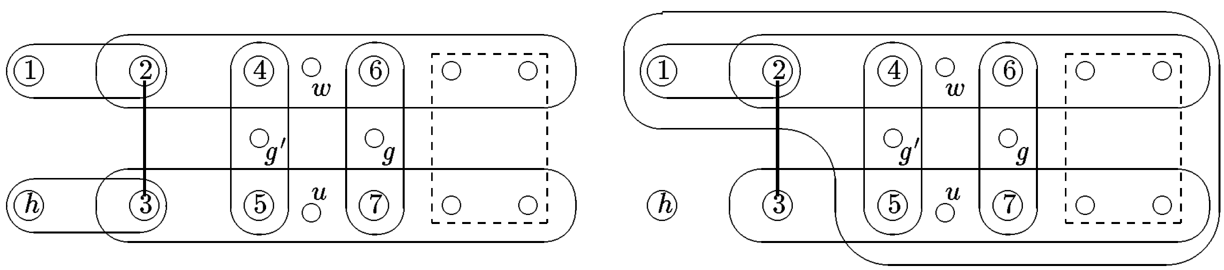
For any valid inequality $bx \leq b_0$ for $STSP(V)$, a Hamiltonian cycle C on V is said to be b-*tight* if $b(C) = b_0$, where $b(C) \equiv \sum_{e \in C} b_e$.

We now outline the polyhedral proof. In this proof, we will make reference to the general primitive ladder inequality shown in Figure 1. In particular, we will use the node labels (numbers 1,...,6, and letters u, w, g, g') as shown in that figure. The hollow nodes g, g' may be assigned to nondegenerate teeth, $\{6, g, 7\}$ and $\{4, g', 5\}$, respectively, as needed in the proof. The other hollow nodes w and u represent the cases that some node in a handle may not be contained in any tooth. Unless otherwise specified, the statements of the proof are true with and without any subset of hollow nodes.

Let $cx \leq c_0$ be the $(h, 13)$-canonical ladder inequality shown in Figure 1(b), and let $fx \leq f_0$ be a facet-inducing $(h, 13)$-canonical inequality that *dominates* $cx \leq c_0$, that is, such that, for all $x \in STSP(V)$, $cx = c_0$ implies $fx = f_0$. Since $f_e = c_e = 0$ for all edges e in $\delta(h)$, the *star* of h, any c-tight Hamiltonian path P, that is, $c(P) = c_0$, on $V \setminus \{h\}$, is also f-tight, that is, $f(P) = f_0$. (Indeed, path P can be converted, in a unique way, into a c-tight cycle C by connecting its endnodes to node h, and thus $f_0 = f(C) = f(P)$.) Therefore, it suffices to *compare* pairs of c-tight paths on $V \setminus \{h\}$: P and P', that is, compute $f(P) - f(P') = 0$ to derive the coefficients of $fx \le f_0$. Each comparison and its implication are denoted by

$$P \sim P' \Longrightarrow \text{“some expression”}.$$

Note that the above implication may involve some obvious node (or tooth) permutations and use earlier results on the f coefficients. Such steps are iterated until $fx \le f_0$ is shown to be some multiple α of $cx \le c_0$. It then follows that $cx \le c_0$, hence $ax \le a_0$, is facet-inducing.



(a) A ladder inequality (b) The ladder in $(h, 13)$-canonical form

Figure 1: Ladder inequalities

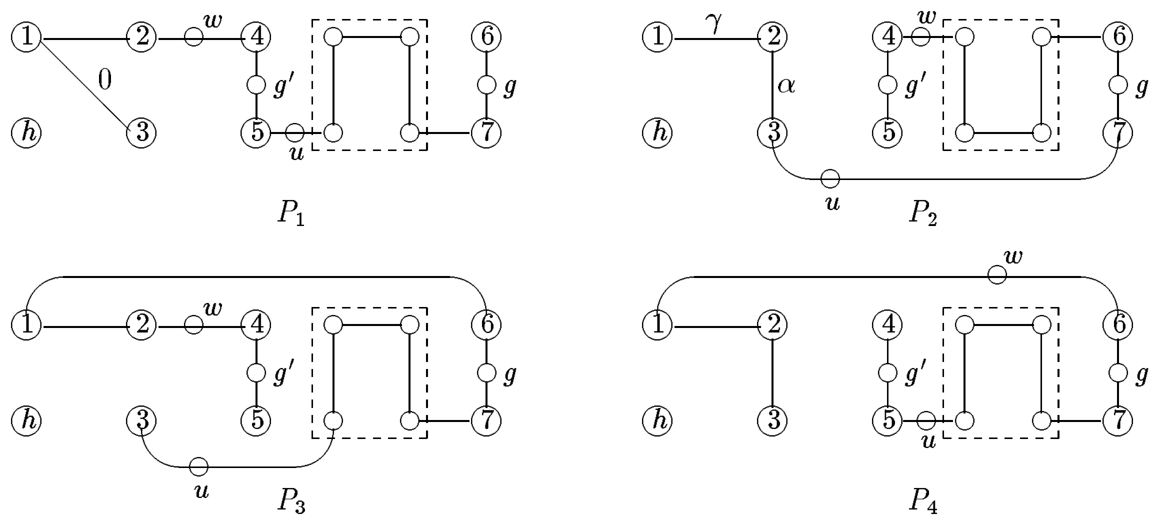


Figure 2: Four c-tight paths

Figure 2 and Figure 3 present 12 types of c-tight paths on $V \setminus \{h\}$ used in the proof. Each path may be represented by either the corresponding edge set or the sequence of nodes.

We are now in a position to prove the following result.

Proposition 4.2 *All primitive ladder inequalities are facet-inducing.*

PROOF: For simplicity, let "+" stand for set union and "−" for set difference. Let $\alpha \equiv f_{23}$ and $\gamma \equiv f_{12}$.

Claim 1. $f_e = 0$ for all e such that $c_e = 0$.

Proof: Since by definition $f_{13} = 0$, $P_1 \sim P_1 - (1,3) + (3,6) \Longrightarrow f_{3i} = f_{13} = 0$ for all $i \geq 4$ and even.

Next, for any nondegenerate tooth, say, $\{6, g, 7\}$, let $P_1' \equiv P_1 - (7,g) + (7,6) = (312w4g'5 \cdots 76g)$.

Then $P_1' \sim P_1' - (1,3) + (3,g) \Longrightarrow f_{3g} = 0$ for all g. If node w does not exist, we are done; else consider edge $(3,w)$. Let $P_1'' \equiv P_1 - (2,w) + (2,6) = (3126g7 \cdots u5g'4w)$. Then $P_1'' \sim P_1'' - (1,3) + (3,w) \Longrightarrow f_{3w} = 0$. $\square$

Claim 2. $f_e = \alpha$ for all e such that $c_e = 1$.

Proof: $P_2 \sim P_2 - (2,3) + (1,5) \Longrightarrow f_{1i} = \alpha$ for all $i \geq 5$ and odd.

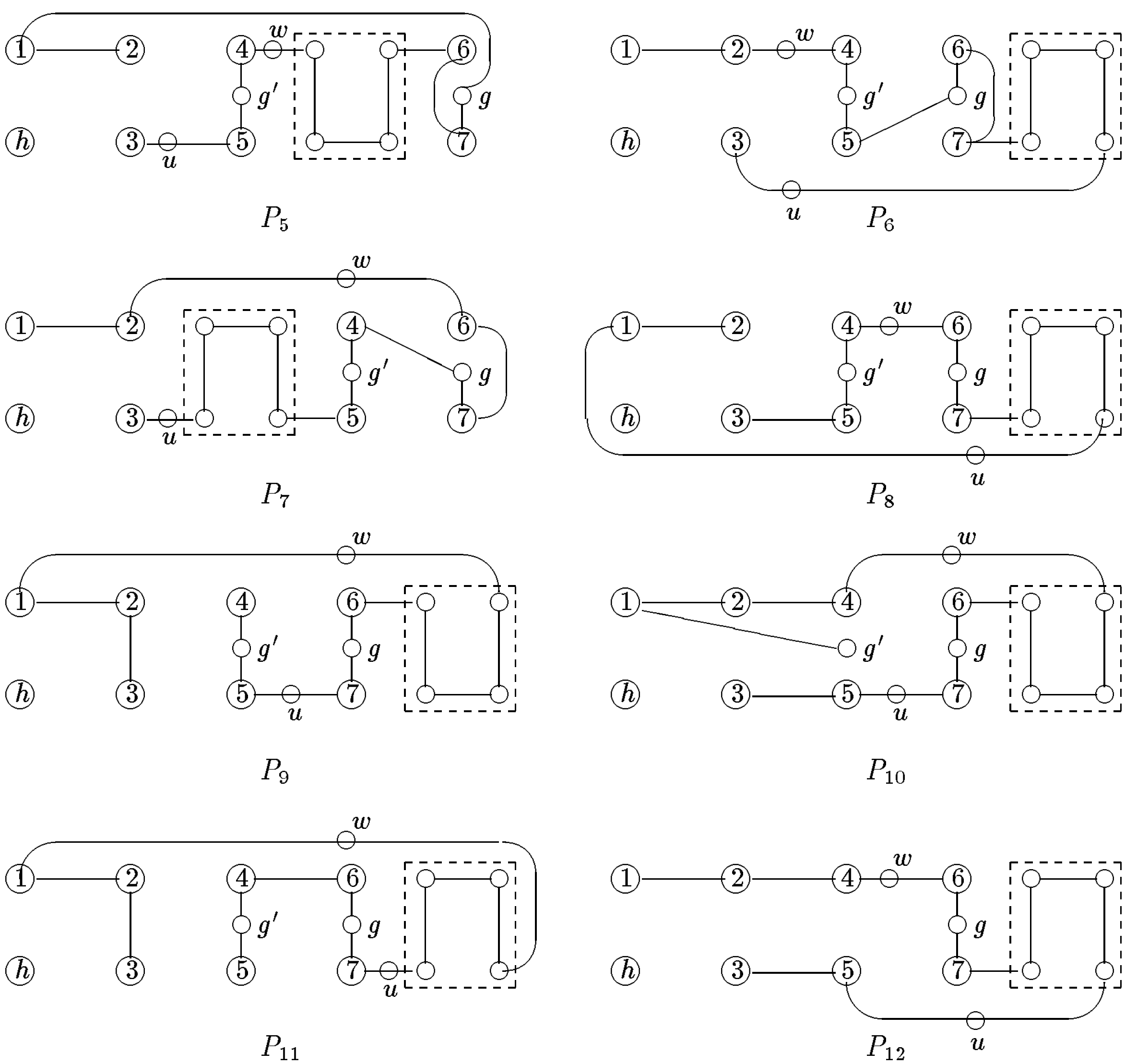


Figure 3: Eight other c-tight paths

$P_{11} \sim P_{11} - (2,3) + (3,5) \Longrightarrow f_{3i} = \alpha$ for all $i \geq 5$ and odd.

$P_2 \sim P_2 - (2,3) + (2,5) \Longrightarrow f_{2i} = \alpha$ for all $i \geq 5$ and odd.

$P_3 \sim P_3 - (1,6) + (3,5) \Longrightarrow f_{1i} = \alpha$ for all $i \geq 4$ and even.

$P_3 \sim P_3 - (1,6) + (5,6) \Longrightarrow f_{ij} = \alpha$ for all $i, j \geq 4$ such that i and j belong to both different teeth and different handles.

If there is a nondegenerate tooth, $\{6, g, 7\}$, use three types of c-tight paths P_5, P_6 and P_7.

$P_5 \sim P_5 - (1,g) + (2,3) \Longrightarrow f_{1g} = \alpha.$

$P_5 \sim P_5 - (1,g) + (2,g) \Longrightarrow f_{2g} = \alpha.$

$P_6 \sim P_6 - (5,g) + (1,g) \Longrightarrow f_{ig} = \alpha$ for all $i \geq 5$, $i \neq 7$ and odd.

$P_7 \sim P_7 - (4,g) + (1,4) \Longrightarrow f_{ig} = \alpha$ for all $i \geq 4$, $i \neq 6$ and even.

If there are at least two nonpendent, nondegenerate teeth, say, $\{6,g,7\}$ and $\{4,g',5\}$, we define $P_6' \equiv P_6 - (4,g') + (4,5) - (5,g) + (g,g') = (12w45g'g67\cdots u3)$. Then we have

$P_6' \sim P_6' - (g,g') + (1,g) \Longrightarrow f_{gg'} = \alpha.$

If all nodes in the handles are contained in the union of teeth, we are done. Otherwise, do the following:

(i) If node w exists, the values of f_e for all edges $e \in \delta(w)$ such that $c_e = 1$ are derived as follows.

$P_4 \sim P_4 - (1,w) + (1,4) \Longrightarrow f_{1w} = \alpha.$

Let $P_3' \equiv P_3 - (2,w) - (4,w) + (2,4) - (1,6) + (1,w) + (6,w) = (5g'421w6g7\cdots u3)$ and, if g' exists, $P_3'' \equiv P_3' - (4,g') + (4,5) = (g'5421w6g7\cdots u3)$.

$P_3' \sim P_3' - (1,w) + (5,w) \Longrightarrow f_{5w} = f_{1w} = \alpha$. So $f_{kw} = \alpha$ for all $k \geq 5$ and odd.

$P_3'' \sim P_3'' - (1,w) + (g',w) \Longrightarrow f_{g'w} = f_{1w} = \alpha.$

When both w and u exist, construct $P_3''' \equiv (u5g'421w6g7\cdots 3)$.

$P_3''' \sim P_3''' - (1,w) + (u,w) \Longrightarrow f_{uw} = f_{1w} = \alpha.$

(ii) If node u exists, the values of f_e for all edges $e \in \delta(u)$ such that $c_e = 1$ are derived as follows.

$P_3 \sim P_3 - (3,u) + (3,5) \Longrightarrow f_{3u} = f_{35} = \alpha.$

$P_2 \sim P_2 - (3,u) + (1,u) \Longrightarrow f_{1u} = f_{3u} = \alpha.$

$P_8 \sim P_8 - (1,u) + (2,u) \Longrightarrow f_{2u} = f_{1u} = \alpha.$

For any nondegenerate tooth, $(354g'u7g6\cdots 21) \sim (g'453u7g6\cdots 21) \Longrightarrow f_{g'u} = f_{3u} = \alpha$.

Let $P \equiv (12w4g'5u3\cdots 7g6)$. $P \sim P - (3,u) + (6,u) \Longrightarrow f_{6u} = f_{3u} = \alpha$. So $f_{ku} = \alpha$ for all $k \geq 4$ and even.

This completes the proof for Claim 2. $\square$

Claim 3. $f_e = \gamma$ for all e such that $c_e = 2$.

Proof: $P_4 \sim P_4 - (1,2) + (2,4) \Longrightarrow f_{2i} = \gamma$ for all $i \geq 4$ and even.

To derive the remaining f_e in the handles with $c_e = 2$, we distinguish, for node w and for node u, the cases with or without that node.

(i) If node w does not exist, then $P_8 \sim P_8 - (4,6) + (2,4) \Longrightarrow f_{ij} = \gamma$ for all distinct $i, j \geq 4$ and even. Otherwise, P_8 includes w and we have

$P_8 \sim P_8 - (4,w) + (2,4) \Longrightarrow f_{kw} = \gamma$ for all $k \geq 4$ and even.

Defining $P_8' \equiv P_8 - (6,w) + (2,w) = (35g'4w21u \cdots 7g6)$, we also have

$P_8' \sim P_8' - (4,w) + (4,6) \Longrightarrow f_{ij} = \gamma$ for all distinct $i, j \geq 4$ and even, and

$P_8 \sim P_8' \Longrightarrow f_{2w} = f_{6w} = \gamma$.

(ii) If node u does not exist, then $P_9 \sim P_9 - (5,7) - (2,3) + (2,4) + (3,5) \Longrightarrow f_{ij} = \gamma$ for all distinct $i, j \geq 5$ and odd. Otherwise, P_9 includes u and we have

$P_9 \sim P_9 - (2,3) - (5,u) + (2,4) + (3,5) \Longrightarrow f_{ku} = \gamma$ for all $k \geq 5$ and odd, and

$P_3 \sim P_3 - (u,v) - (1,6) + (5,v) + (1,u) \Longrightarrow f_{5v} = f_{uv} = \gamma$, where $(u,v) \in P_3$, $v \geq 7$ and odd. This shows that $f_{ij} = \gamma$ for all distinct $i, j \geq 5$ and odd.

For any nondegenerate tooth $\{4, g', 5\}$, we have

$P_{10} \sim P_{10} - (v,5) + (5,g') \Longrightarrow f_{5g'} = \gamma$, where $v = u$ if u exists and $v = 7$ otherwise.

$P_{10} \sim P_{10} - (2,4) + (4,g') \Longrightarrow f_{4g'} = \gamma$.

$P_{11} \sim P_{11} - (4,g') + (4,5) \Longrightarrow f_{45} = f_{4g'} = \gamma$.

This completes the proof for Claim 3. $\square$

Claim 4. $\gamma = 2\alpha$.

Proof: By Claims 1, 2 and 3, $P_1 \sim P_4 \Longrightarrow \gamma = 2\alpha$. $\square$

Claim 5. For every degenerate tooth T, say $T = \{4,5\}$ (without g'), we have $f_{45} = 3\alpha$.

Proof: $P_8 \sim P_{12} \Longrightarrow f_{45} = 2\gamma - \alpha = 4\alpha - \alpha = 3\alpha$. $\square$

From Claims 1–5, it follows that $f_e = \alpha c_e$ for all $e \in E(V)$. The proof of Proposition 4.2 is complete. $\square$

5 Lifting ladder inequalities

We have shown that all *primitive* ladder inequalities are facet-inducing for STS polytopes. In this section, we show by node lifting and cloning that *all* ladder inequalities are facet-inducing. We begin with the following simple lemma on (h, uv)-canonical forms, which is used in our proofs.

Lemma 5.1 *Let $cx \leq c_0$ be an (h, uv)-canonical facet-inducing inequality for $STSP(V)$. If an (h, uv)-canonical inequality $fx \leq f_0$ satisfies $f(P) = f_0$ for all c-tight paths P on*

$V \setminus \{h\}$, then $f = c$ and $f_0 = c_0$, up to a positive multiple.

PROOF: Assume that $cx \leq c_0$ and $fx \leq f_0$ satisfy the assumptions of the lemma. Consider any c-tight cycle C and let $P \equiv C \setminus \delta(h)$. Since P is a Hamiltonian path on $V \setminus \{h\}$ and $c(P) = c(C) = c_0$, we have $f(P) = f_0$, implying $f(C) = f_0$. Since $cx \leq c_0$ is facet-inducing and both $cx \leq c_0$ and $fx \leq f_0$ are in (h, uv)-canonical form, this implies $f = c$ and $f_0 = c_0$, up to a positive multiple. □

We say that a valid inequality induces a *nontrivial* facet if it is not equivalent to either a nonnegativity constraint $x_e \geq 0$ or a bound constraint $x \leq 1$. The following two results show how large classes of nontrivial facets can be obtained by node lifting.

The first theorem allows us to add *isolated nodes*, that is, nodes that are not in the union of all handles and teeth, and therefore whose incident edges have zero coefficients in the ladder inequality (1). Actually, this node lifting theorem applies to a broad class of STSP facet-inducing inequalities, such as the well-known clique tree class. An inequality $ax \leq a_0$ for $STSP(V)$ is a *2-tooth inequality* if it satisfies

(i) it is a nontrivial valid inequality for $STSP(V)$;

(ii) $a \geq 0$;

(iii) there exist (at least) two disjoint *teeth* $T_1 = \{t_1, h_1\}$ and $T_2 = \{t_2, h_2\}$ such that for each $i = 1, 2$, we have $a_{t_i h_i} > 0$, and $a_{t_i v} = 0$ for all $v \neq h_i$;

(iv) either $a_{h_1 v} \geq a_{h_1 t_1}$ or $a_{h_1 v} = 0$ for all $v \in V$.

Many of the known valid inequalities have this property, including all primitive clique tree, ladder and chain inequalities as well as many bipartition inequalities.

Theorem 5.2 (Adding an isolated node) *Suppose that the 2-tooth inequality $ax \leq a_0$ defines a nontrivial facet of $STSP(V)$, and $q \notin V$. Let $a^* x^* \leq a_0^*$ be a lifted inequality for $STSP(V^*)$, where $V^* = V \cup \{q\}$, obtained by letting $a_0^* = a_0$, $a_e^* = a_e$ for all $e \in E(V)$ and zero otherwise. Then $a^* x \leq a_0^*$ is facet-inducing for $STSP(V^*)$.*

PROOF: Consider a facet-inducing 2-tooth inequality $ax \leq a_0$. Without loss of generality, we may assume that $a_{t_1 h_1} = 1$. Define $Y \equiv \{v \in V \setminus \{t_1\} : v = h_1 \text{ or } a_{h_1 v} > 0\}$ and $Z \equiv V \setminus T_1$. Note that (i) implies that both Y and Z are nonempty. Since $h_1 \in Y$ and $t_2 \in Z \setminus Y$, both Y and $Z \setminus Y$ are nonempty subsets of $V \setminus \{t_1\}$. Let $cx \leq c_0$ be the

(t_1, t_2h_1)-canonical inequality obtained from $ax \leq a_0$ by complementing T_1. It is easily verified that this inequality satisfies the following properties:

(P1) $c \geq 0$ and the support graph $G_c = (V, E_c)$ of $cx \leq c_0$ consists of the isolated node t_1 and a *bi-clique structure* induced by subsets Z and Y of V; that is, $E_c = E(Z) \cup E(Y)$, where $Z \cup Y = V \setminus \{t_1\}$, and $Y \setminus Z = \{h_1\}$;

(P2) $c_e \geq 1$ for all $e \in E(Z)$;

(P3) $c_{h_1v} \geq 1$ for all $v \in Y$ and $c_{h_1v} = 0$ for all $v \in Z \setminus Y$; and

(P4) $c_{t_2h_1} = 0$; $c_{t_2h_2} > 1$ and $c_{t_2v} = 1$ for all $v \in Z \setminus \{h_2\}$.

Let $a^*x \leq a_0^*$ be as defined in the theorem. Conditions (i) and(ii) imply that $a^*x \leq a_0^*$ is valid for $STSP(V^*)$. Let $c^*x \leq c_0^*$ be the (t_1, t_2h_1)-canonical inequality obtained from $a^*x \leq a_0^*$ by complementing the tooth $\{t_1, h_1\}$. Comparing this inequality with the (t_1, t_2h_1)-canonical inequality $cx \leq c_0$, we observe that $c_e^* = c_e$ for all $e \in E(V)$, that $c_{qh_1}^* = 0$ and $c_{qv}^* = 1$ for all $v \in Z$, and that $c_0^* = c_0 + 1$.

Let $fx \leq f_0$ be any (t_1, t_2h_1)-canonical facet-inducing inequality for $STSP(V^*)$ that dominates $c^*x \leq c_0^*$. Let $\alpha \equiv f_{qt_2}$.

Claim 1. $f_{qh_1} = 0$ and $f_{qz} = \alpha$ for all $z \in Z \setminus Y$.

Proof: We have assumed that $ax \leq a_0$, and thus $cx \leq c_0$ as well, is not equivalent to a trivial inequality $x_e \geq 0$. Therefore, for every $z \in Z \setminus Y$, there exists a c-tight path P on $V \setminus \{t_1\}$ containing edge (z, h_1). By (P3), $c_{zh_1} = 0$, and thus the edge e connecting the endnodes of P satisfies $c_e = 0$, for otherwise $c(P \cup \{e\} \setminus \{(z, h_1)\}) > c_0$. This implies by (P1) that path P has the form $P = (u \cdots zh_1)$ with $c_{uh_1} = 0$ and $u \in Z \setminus Y$. Let $P' \equiv P \cup (q, u)$, $P'' \equiv (h_1qu \cdots z)$ and note that both P' and P'' are c^*-tight paths on $V^* \setminus \{t_1\}$.

(i) First, let $z = t_2$. Comparing P' with the c^*-tight path $(h_1qu \ldots t_2)$ implies $f_{qh_1} = f_{h_1t_2} = 0$.

(ii) Next, comparing P'' with $(h_1qz \cdots u)$ yields $f_{qz} = f_{qu}$.

(iii) Now, consider any other $z \in Z \setminus Y$, $z \neq t_2$. If $u = t_2$, then comparing P' and the c^*-tight path $(u \ldots zqh_1)$ yields $f_{qz} = \alpha$ and Claim 1 is proved for node z. Else, $u \neq t_2$ and we may write $P = (u \cdots vt_2s \cdots zh_1)$. By (P4), we have $c_{vt_2} = 1$ or $c_{t_2s} = 1$ (or both). If $c_{vt_2} = 1$, then comparing c^*-tight paths $(h_1u \cdots vqt_2s \cdots z)$ and $(h_1u \cdots vqz \cdots st_2)$ yields $f_{qz} = f_{qt_2} = \alpha$. If $c_{t_2s} = 1$, then comparing $(u \cdots vt_2qs \cdots zh_1)$ and $(t_2v \cdots uqs \cdots zh_1)$

yields $f_{qu} = \alpha$, and therefore by (ii), $f_{qz} = f_{qu} = \alpha$. We have shown that $f_{qz} = \alpha$ for all $z \in Z \setminus Y$ and the proof of Claim 1 is complete. $\square$

Claim 2. $f_{qw} = \alpha$ for all $w \in Z \cap Y$.

Proof: Since $cx \leq c_0$ is a nontrivial inequality, for any $w \in Z \cap Y$, there exists a c-tight cycle C on V containing edge (t_1, w). Thus, there exists a c-tight path $P \equiv (w \cdots s)$ on $V \setminus \{t_1\}$, obtained by deleting from C the edges incident with t_1. Note that, by (P2) and (P3), $c_{ws} \geq 1$. By property (P4), path P must contain an edge (u, v) incident with t_2 and with $c_{uv} = 1$. (Otherwise, P would contain (h_2, t_2) and (t_2, h_1) with $c_{t_2 h_1} = 0$, implying that $c(P \cup \{(w, s)\} \setminus \{(t_2, h_1)\}) \geq c_0 + 1$, a contradiction.) Let $P \equiv (w \ldots uv \ldots s)$ and $P' \equiv P \cup \{(w, s)\} \setminus \{(u, v)\}$. Comparing P' and P yields $c_{ws} \leq 1$ and therefore $c_{ws} = 1$. Thus P' is also a c-tight path on $V \setminus \{t_1\}$. Now comparing $P' \cup \{(q, u)\}$ and $P' \cup \{(q, v)\}$ yields $f_{qu} = f_{qv} = \alpha$, since $t_2 \in \{u, v\}$. Finally, comparing the two c^*-tight paths $(w \cdots uqv \cdots s)$ and $(u \cdots wqv \cdots s)$, we obtain $f_{qw} = f_{qu} = \alpha$. The proof of Claim 2 is complete. $\square$

Consider the following inequality for $STSP(V)$,

$$(3) \qquad \sum_{e \in E(V \setminus \{t_1\})} f_e x_e \leq f_0 - \alpha.$$

Denote this inequality by $\hat{f}x \leq \hat{f}_0$ and observe that it is in $(t_1, t_2 h_1)$-canonical form. Consider any Hamiltonian path P on $V \setminus \{t_1\}$, say $P = (u \ldots v)$. By property (P1), P must have at least one endnode v in Z. Letting $P^* \equiv (u \ldots vq)$, we have $f_0 \geq f(P^*) = \hat{f}(P) + \alpha$. This shows that inequality (3) is satisfied by any Hamiltonian path on $V \setminus \{t_1\}$. Furthermore, if P is c-tight on $V \setminus \{t_1\}$, then P^* is c^*-tight, and therefore also f-tight, on $V^* \setminus \{t_1\}$. That is, $f_0 = f(P^*) = \hat{f}(P) + \alpha$. Thus, every c-tight path on $V \setminus \{t_1\}$ satisfies (3) with equality. Since $cx \leq c_0$ is facet-inducing for $STSP(V)$, Lemma 5.1 implies that, with the appropriate positive multiple, $c_0 = \hat{f}_0 = f_0 - \alpha$ and $c_e^* = c_e = \hat{f}_e = f_e$ for all $e \in E(V \setminus \{t_1\})$.

Finally, from the c-tight path $P = (w \cdots uv \cdots s)$ in the proof of Claim 2, we obtain two c^*-tight paths $(w \cdots uqv \cdots s)$ and $(qw \cdots uv \cdots s)$. Since $c_{uv} = 1$, we have $f_{uv} = 1$. Therefore comparing these paths yields $\alpha + 1 = 2\alpha$. So $\alpha = 1$.

This shows that $f_0 = c_0 + 1 = c_0^*$ and $f_e = c_e^*$ for all $e \in E(V^*)$, implying that $c^*x \leq c_0^*$, or equivalently $a^*x \leq a_0^*$, is facet-inducing for $STSP(V^*)$. The proof of Theorem 5.2 is complete. □

We remark that the above theorem is not only of theoretical interest but also of practical importance in polyhedral computations for the TSP. Since all facet-inducing 2-tooth inequalities for small STS polytopes also induce facets for large STS polytopes by adding isolated nodes, they can be effectively used as cutting planes for solving the large TSP's. Moreover, they have small support graphs, and thus require far less computer memory to store. As a consequence, we may expect facet-inducing 2-tooth inequalities derived from the study of small STS polytopes to play a role in the efficient solution of large STS problems. Denis Naddef pointed out to us that an example arose in computation for which ladder inequalities improved the LP bound. This example is discussed in detail in [4].

To show that any ladder inequality is facet-inducing, we use the following *node-cloning* result, which is an extension of Theorem 4.1 in Queyranne and Wang [10].

Theorem 5.3 (A sufficient condition for node cloning) *Let u and q be any two nodes such that $u \in V$ and $q \notin V$. Let $V^* \equiv V \cup \{q\}$. Assume that $cx \leq c_0$ is a nontrivial facet-inducing (u, pw)-canonical inequality for $STSP(V)$ satisfying $c_e \geq 1$ for all e with $c_e \neq 0$, and moreover the following condition:*

Condition $\mathcal{B}(u, D; \omega)$: *There exists a scalar $\omega \geq 1$ and a partition $(\{u\}, D, U, U')$ of V such that:*

B1. *$c_e = 0$ for all $e \in E(D : U')$;*

B2. *$1 \leq c_e \leq \omega$ for all $e \in E(D : U)$; and*

B3. *$c_e \geq \omega$ for all $e \in E(U)$.*

Then the inequality $c^u x \leq c_0^u$, defined by $c_0^u = c_0$, $c_e^u = c_e$ for all $e \in E(V)$ and $c_e^u = 0$ for all $e \in \delta(q)$, is facet-inducing for $STSP(V^*)$.

PROOF: Let $d \in D$, and let $fx \leq f_0$ be a (u, qd)-canonical inequality that dominates $c^u x \leq c_0^u$ and defines a facet of $STSP(V^*)$.

Claim 1. $f_{qv} = 0$ for all $v \in U' \cup D$.

Proof: Consider any nodes $v \in D$ and $v' \in U'$. Note $c_{vv'} = 0$ by (B1). Since $cx \leq c_0$ is not equivalent to any $x_e \geq 0$, there is a c-tight cycle C on V, thus a c-tight path $P \equiv C \setminus \delta(u) = (s \cdots vv' \cdots t)$ on $V \setminus \{u\}$ containing (v, v'). Let $P' \equiv P \cup \{(s,t)\} \setminus \{(v, v')\}$. Since $0 \leq c(P) - c(P') = c_{vv'} - c_{st}$, P' is also c-tight. Comparing $P' \cup \{(q, v)\}$ and $P' \cup \{(q, v')\}$ yields $f_{qv} = f_{qv'}$. Since $f_{qd} = 0$, the claim follows.

Claim 2. $f_{qv} = 0$ for all $v \in U$.

Proof: Consider any node $v \in U$. Let C' be a c-tight cycle on V containing uv. Then $P' \equiv C' \setminus \delta(u) = (v \cdots v')$ is the c-tight path on $V \setminus \delta(u)$. If $v' \in U' \cup D$ then construct two c-tight paths as in (i) to show that $f_{qv} = 0$. Otherwise $v' \in U$. In this case P' has the form $(v \cdots rs \cdots v')$ where $r \in U$ and $s \in D$. (Note that P' contains no edge $e_0 \in E(D : U')$, since otherwise $c(P' \cup \{(vv')\} \setminus \{e_0\}) > c_0$, a contradiction.) By (B2) and (B3), $P'' \equiv P' \cup \{(r, v')\} \setminus \{(r, s)\}$ is a c-tight path on $V \setminus \delta(u)$. Comparing $P'' \cup \{(q, v)\}$ and $P'' \cup \{(q, s)\}$ yields $f_{qv} = f_{qs} = 0$. So Claim 2 also holds.

Finally, consider any c-tight cycle C on V. Clearly $C^* = C \cup \{(q, u), (q, v)\} \setminus \{(u, v)\}$, where $(u, v) \in C \cap \delta(u)$, is a Hamiltonian cycle on V^* satisfying $c^u(C^*) = c_0^u$, and hence is f-tight. Further using the above claim, we have $f(C) = f(C^*) = f_0$. Since $cx \leq c_0$ defines a facet, by Lemma 5.1, we have $f_e = c_e$ for all $e \in E(V)$ and $f_0 = c_0$. □

Theorem 5.4 *All ladder inequalities are facet-inducing.*

PROOF: Let $bx \leq b_0$ be any ladder inequality. Clearly, there exists a corresponding facet-inducing primitive ladder inequality $a'x \leq a'_0$ obtained by discarding all isolated nodes in G_b and shrinking each nonempty set $H_i \cap T_j$, $T_j \setminus (H_1 \cup H_2)$ and $H_i \setminus \left(\cup_{j=1}^{t+m} T_j\right)$ into a singleton set. If G_b contains s isolated nodes, we apply Theorem 5.2 s times to $a'x \leq a'_0$ to obtain a facet-inducing ladder inequality $ax \leq a_0$ with G_a containing s isolated nodes. To clone any other node u, we consider $ax \leq a_0$ as being a general facet-inducing ladder inequality for $STSP(V)$. Recall that $V^* = V \cup \{q\}$. We need to show that the inequality $a^u x \leq a_0^u$, obtained by replacing $\{u\}$ with $\{u, q\}$, is also facet-inducing for $STSP(V^*)$. Let $cx \leq c_0$ and $c^u x \leq c_0^u$ be their respective (u, vw)-canonical inequalities. Then $c^u x \leq c_0^u$ is exactly the inequality obtained in Theorem 5.3 from $cx \leq c_0$. Thus, to show that $a^u x \leq a_0^u$ is facet-inducing, it is enough to check that $cx \leq c_0$ satisfies the conditions of Theorem 5.3 for each of the following cases. (Note that by symmetry, the following also applies to the cases with respect to H_1.)

Case 1: $u \in T_2 \setminus H_2$. Construct $cx \leq c_0$ by complementing T_2, as in Figure 1(b). Then, $cx \leq c_0$ satisfies the required conditions and $\mathcal{B}(u, T_2 \cap H_2; 1)$.

Case 2: $u \in T_j \setminus (H_1 \cup H_2)$, $3 \leq j \leq t$. Construct $cx \leq c_0$ by complementing T_j. Then, $cx \leq c_0$ satisfies the required conditions and $\mathcal{B}(u, T_j \cap H_2; 1)$.

Case 3: $u \in H_2 \setminus \left(\cup_{j=1}^{t+m} T_j\right)$. Construct $cx \leq c_0$ by complementing H_2. Then, $cx \leq c_0$ satisfies the required conditions and $\mathcal{B}(u, T_2 \cap H_2; 1)$.

Case 4: $u \in H_2 \cap T_j$. $j \geq 3$. Construct $cx \leq c_0$ by complementing H_2 and T_j. Then, $cx \leq c_0$ satisfies the required conditions and $\mathcal{B}(u, T_j \setminus (H_1 \cup H_2); 1)$ if $3 \leq j \leq t$; or $\mathcal{B}(u, T_j \cap H_1; 2)$ if $t + 1 \leq j \leq t + m$.

Case 5: $u \in H_2 \cap T_2$. Construct $cx \leq c_0$ by complementing H_2, T_2 and then adding the degree constraints $-x(\delta(s)) = -2$ for all $s \in S \equiv T_1 \cap H_1$. Then, $cx \leq c_0$ satisfies the required conditions and $\mathcal{B}(u, T_2 \setminus H_2; 1)$. (Note that $U = V \setminus (T_2 \cup H_2 \cup (T_1 \cap H_1))$ in the partition $(\{u\}, D, U, U')$.)

The proof is complete. □

6 The Chvátal rank of ladder inequalities

Let P be a rational polyhedron in $\boldsymbol{R}^E$, that is, $P = \{x : Ax \leq b\}$, where A and b are rational, and let P_I denote the convex hull of the integral points in P. Define P^0 to be P and for $i \geq 1$, P^i to be the set of points satisfying all *integral* inequalities $ax \leq a_0$ derived from P^{i-1} by the following rounding procedure: For any finite set of m (say) inequalities $Cx \leq d$ valid for P^{i-1} and $\lambda \in \boldsymbol{R}_+^m$ such that λC is integral, take $a = \lambda C$ and $a_0 = \lfloor \lambda d \rfloor$. So each P^i contains P_I and $P^0 \supseteq P^1 \supseteq \cdots \supseteq P^i$. These definitions were introduced by Chvátal [3], and the rounding procedure is closely related to the cutting plane methods of Gomory. It can be proved that each P^i is itself a polyhedron, and that there is an integer k, depending on P, such that $P^k = P_I$. (See Chvátal [3] for details.)

The (Chvátal) *rank* of an inequality $ax \leq a_0$ valid for P_I is the least i such that $ax \leq a_0$ is valid for P^i. It is a measure of the complexity of the derivation of the inequality by the above procedure. Suppose that we take P to be a *subtour polytope*, that is, the solution set of (1.a), (1.b) and (1.c). Then P_I is $STSP(V)$, and it is of interest to classify facet-inducing inequalities by their rank. Of course, the non-negativity and SE inequalities have rank 0. It is well known that comb inequalities have rank 1 [2].

From this and our proof for the validity of the ladder inequalities, it follows that each ladder inequality has rank at most 2.

In the remainder of this section, we prove that each ladder inequality has rank at least 2, hence exactly 2. There is an apparently "obvious" technique for proving that an integral inequality $ax \leq a_0$, which is valid for P_I, cannot be obtained from inequalities of rank 0 by the rounding procedure. Namely, we show that there is no solution λ to

$$\lambda A = a, \quad \lambda \geq 0, \quad \lambda b < a_0 + 1.$$

By the duality theorem of linear programming, this is equivalent to showing that there is $\bar{x} \in P$ with $a\bar{x} \geq a_0 + 1$. However, there is a difficulty with this argument. It may be that there are inequalities of which $ax \leq a_0$ is a non-negative combination, that are obtainable by rounding, although $ax \leq a_0$ itself is not. This difficulty does not disappear even if we know that $ax \leq a_0$ is facet-inducing for P_I, since it still may have an equivalent form that is obtainable by rounding.

An instructive example that arises from the 6-node TSP is the following inequality:

$$ax = x_{12} + x_{13} + x_{23} + 2x_{14} + 2x_{25} + 2x_{36} + x_{45} + x_{46} + x_{56} \leq 8 = a_0.$$

This inequality is facet-inducing for $STSP(V)$ with $|V| = 6$. In fact, it is equivalent to a comb inequality with handle $\{1,2,3\}$ and teeth $\{1,4\}, \{2,5\}, \{3,6\}$. Hence it has rank 1. However, the point $\bar{x} = \frac{1}{2}a$ satisfies (1.a), (1.b) and (1.c) with $a\bar{x} = 9 = a_0 + 1$.

Actually, this difficulty was overlooked in some previous papers [1, 2], where it was claimed using the above argument that certain inequalities have rank at least two. These results are correct, but their proofs contain gaps that can be filled by the following result from [11]. Let $Gx = g$ be the *equality system* for P_I, that is, the linearly independent equations whose solution set is the affine hull of P_I.

If G is written (G_B, G_N) such that G_B is a nonsingular square matrix, we say that a valid inequality $ax \leq a_0$ for P_I is an *integral B-canonical form* if $a = (a_B, a_N)$ with $a_B = 0$ and all components of a_N being relatively prime integers. Notice that for every rational valid inequality, there is a unique integral B-canonical form to which it is equivalent.

Proposition 6.1 *Let $ax \leq a_0$ be an integral B-canonical form that is facet-inducing for P_I, and suppose that $G_B^{-1}G$ is integral. Then $ax \leq a_0$ has Chvátal rank at most 1 if and only if $z(a) \equiv \max\{ax : x \in P\} < a_0 + 1$.*

For the STSP case, the equality system $Gx = g$ consists of the degree constraints (1.a). Consider the integral B-canonical form of Lemma 4.1. It is easy to see that, for any column g_{pq} of G_N, the vector $G_B^{-1}g_{pq}$, that is, the vector d that satisfies $G_B d = g_{pq}$ has components $0, -1, +1$. Namely, the $+1$ and -1 components alternate on the edges of the unique odd-length edge-simple path in B joining p to q. Hence Proposition 6.1 can be applied.

We are now in a position to prove the main result of this section.

Theorem 6.2 *The ladder inequality (2) has Chvátal rank two.*

PROOF: From the proof of Theorem 3.1, it follows that every ladder inequality $cx \le c_0$ has Chvátal rank at most two. We now show that it has Chvátal rank at least two. To do so, we first construct its $(h, 13)$-canonical form $ax \le a_0$ where, as in Section 4 and Figure 1(a), nodes $h \in T_2 \setminus H_2$, $1 \in T_1 \setminus H_1$ and $3 \in H_2 \cap T_2$. Hence by Proposition 6.1, we just need to construct a feasible solution $\bar{x}$ to the subtour polytope satisfying $a\bar{x} \ge a_0 + 1$.

For $j = 3, \ldots, t+m$, let P_j be a Hamiltonian path on T_j that saturates both $T_j \cap H_1$ and $T_j \cap H_2$ with the endpoints $v_j^1 \in T_j \cap H_1$ and $v_j^2 \in T_j \cap H_2$. Let P_2 be a Hamiltonian path on $V \setminus (\cup_{j=3}^{t+m} T_j)$ that saturates $T_i \setminus H_i$, $H_i \cap T_i$, $\hat{H}_i$ for $i = 1, 2$ with endpoints $v_1 \in H_1$ and $v_2 \in H_2$. Define the edge set

$$P_1 \equiv (\cup_{j=2}^{t+m} P_j) \cup \{(v_{j+i-1}^i, v_{j+i}^i) \in E(H_i) : i = 1, 2;\ j \text{ is even and } 4 \le j \le t+m-2\},$$

and node sets $S_1 \equiv \{v_1, v_3^1, v_{t+m}^1\}$, $S_2 \equiv \{v_2, v_3^2, v_4^2\}$. Then P_1 is a path system with all nodes in $S_1 \cup S_2$ of degree 1 and all other nodes of degree 2. Now define $\bar{x} \in \boldsymbol{R}^E$ by $\bar{x}_e = 1$ for all $e \in P_1$, $\bar{x}_e = \frac{1}{2}$ for all $e \in E(S_1) \cup E(S_2)$ and $\bar{x}_e = 0$ otherwise. It is easily verified, using the $(h, 13)$-canonical form $ax \le a_0$ of the ladder inequality, that we have $a\bar{x} = a_0 + 1$. $\square$

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