Integral Solutions of Linear Complementarity Problems

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Abstract

We characterize the class of integral square matrices M having the property that for every integral vector q the linear complementarity problem with data M, q has only integral basic solutions. These matrices, called *principally unimodular matrices*, are those for which every principal nonsingular submatrix is unimodular. As a consequence, we show that if M is rank-symmetric and principally unimodular, and q is integral, then the problem has an integral solution if it has a solution. Principal unimodularity can be regarded as an extension of total unimodularity, and our results can be regarded as extensions of well-known results on integral solutions to linear programs. We summarize what is known about principally unimodular symmetric and skew-symmetric matrices.

Introduction

Let M be a V by V matrix, where V is a finite set. We call M *principally unimodular (PU)* if every nonsingular principal submatrix of M is unimodular (that is, has determinant ± 1). Principal unimodularity arises as a generalization of total unimodularity as follows: a matrix A is totally unimodular if and only if $\begin{pmatrix} 0 & A \\ \pm A^T & 0 \end{pmatrix}$ is PU. Principally unimodular matrices were introduced by Bouchet [1].

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Totally unimodular matrices are of fundamental importance in combinatorial optimization, due to the connection with integrality in linear programming. We will see that principal unimodularity plays an analogous role with respect to the linear complementarity problem, particularly in the case of "rank-symmetric" matrices.

A V by V matrix M is called *rank-symmetric* if rank $(M[X,Y])$ = rank $(M[Y,X])$ for all $X, Y \subseteq V$. Here $M[X,Y]$ denotes the X by Y submatrix of M, that is, the submatrix of M having rows indexed by elements of X and columns indexed by elements of Y. We shall denote by $M[X]$ the principal submatrix $M[X,X]$. Obviously symmetric and skew-symmetric matrices are rank-symmetric. The first-order optimality conditions for quadratic programming give rise to linear complementarity problems involving rank-symmetric matrices that are neither symmetric nor skew-symmetric.

We give a terse treatment to the linear complementarity problem; for a detailed survey of the problem see Cottle, Pang and Stone [5]. Let M be a V by V matrix, and let q be a column vector indexed by V. The *linear complementarity problem*, with respect to q, M, is to find column vectors w, z indexed by V satisfying:

$$\begin{aligned} w &= Mz + q, && (1)\\ w_v z_v &= 0, \qquad (v \in V) && (2)\\ w, z &\geq 0. && (3)\end{aligned}$$

We denote the above problem by (q, M). Let w, z be column vectors indexed by V. We say that (w, z) is *complementary* if (2) is satisfied, and that (w, z) is *feasible for* (q, M) if (1) and (3) are satisfied. A complementary feasible pair (w, z) for (q, M) is called a *solution* of (q, M). For a solution (w, z) of (q, M), w is uniquely determined by z, so we occasionally represent the pair (z, w) by z alone.

Suppose that $M[X]$ is nonsingular, for some subset X of V. There is a unique pair of vectors w', z' satisfying (1) such that $w'_X = 0$ and $z'_{\overline{X}} = 0$. Here v_X denotes the restriction of the vector v to the set X, and $\overline{X}$ denotes $V \setminus X$. The pair z', w' is defined as follows:

$$\begin{aligned} z'_X &= -(M[X])^{-1} q_X, & w'_X &= 0,\\ z'_{\overline{X}} &= 0, & w'_{\overline{X}} &= q_{\overline{X}} - M[\overline{X}, X](M[X])^{-1} q_X.\end{aligned}$$

Then (w', z') is called a *basic pair* of (q, M) with respect to X. Note that w', z' are not necessarily nonnegative. A *basic solution* is a basic pair that *is* nonnegative. Our main theorem is the following.

Theorem 1 *Let M be a V by V integral matrix. Then the following are equivalent:*

(a) M is principally unimodular.

(b) For every integral vector q, all basic solutions of (q, M) are integral.

Unfortunately it is not the case, for an integral PU-matrix M, that (q, M) has an integral solution for every integral q for which (q, M) has a solution. Indeed, consider (q, M) where

$$M = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \; q = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}.$$

Note that M is PU. Let $z^* = (0, \frac{3}{2}, \frac{1}{2})^T$, and $w^* = (0,0,0)^T$; then z^*, w^* is a solution to (q, M). However, for any solution (z, w) to (q, M), we have $z_2 - z_3 - 1 \geq w_1$. Then, since $w, z \geq 0$, we must have $z_2 > 0$. So, by complementarity, $w_2 = 0$, and $2z_3 - 1 = 0$. Thus z is not integral. However, when M is required to be rank-symmetric, the situation is better.

Theorem 2 *Let M be a V by V rank-symmetric matrix, and let q be a column vector indexed by V. If (q, M) has a solution, then (q, M) has a basic solution.*

As an immediate consequence of Theorems 1 and 2 we have the following result.

Corollary 3 *Let M be a V by V, integral, rank-symmetric, principally unimodular matrix, and let q be an integral column vector indexed by V. If (q, M) has a solution, then (q, M) has an integral solution.* □

It is easy to prove that (a) implies (b) in Theorem 1, using elementary linear algebra.

Proposition 4 *Let $M[X]$ be a unimodular submatrix of $M \in \mathbf{Z}^{V \times V}$, where $X \subseteq V$, and let $q \in \mathbf{Z}^V$. Then the basic pair of (q, M) corresponding to X is integral.*

Proof It suffices to prove that $M[X]^{-1}$ is integral, and this follows from the adjoint formula for the inverse of a matrix. □

Linear programming

Theorem 1 generalizes the following well-known theorem in integer programming. A polyhedron $P \subseteq \mathbf{R}^V$ is *integral* if $\max(c^T x : x \in P)$ has an integral optimal solution whenever it has an optimal solution.

Theorem 5 (Hoffman and Kruskal [8]) *For $A \in \mathbf{Z}^{X \times Y}$, the following are equivalent*

(a) A is totally unimodular.

(b) For every $b \in \mathbf{Z}^X$, the polyhedron $\{x \in \mathbf{R}^Y : Ax \geq b, x \geq 0\}$ is integral. □

Again, that (a) implies (b) is straightforward. To see the converse we need briefly to outline the well-known connection with the LCP.

Let $A \in \mathbf{Z}^{X \times Y}$, $c \in \mathbf{Z}^Y$ and $b \in \mathbf{Z}^X$. We are interested in the following linear programming problem (P) and its dual (D).

$$(P) - \begin{cases} \min & c^T z_1 \\ \text{s.t.} & Az_1 \geq b \\ & z_1 \geq 0. \end{cases}, \qquad (D) - \begin{cases} \max & b^T z_2 \\ \text{s.t.} & A^T z_2 \leq c \\ & z_2 \geq 0. \end{cases}$$

We define

$$M = \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix}, \text{ and } q = \begin{pmatrix} c \\ -b \end{pmatrix}.$$

Then $z \in \mathbf{R}^V$ is a solution of (q, M) if and only if z_Y is optimal to (P) and z_X is optimal to (D).

If A is not totally unimodular, then M is not principally unimodular. Therefore, by Theorem 1, there exist $b \in \mathbf{Z}^X$ and $c \in \mathbf{Z}^Y$ such that at least one of $\{x \in \mathbf{R}^Y : Ax \geq b, x \geq 0\}$ and $\{y \in \mathbf{R}^X : A^T y \leq c, y \geq 0\}$ is not integral. If $\{x \in \mathbf{R}^Y : Ax \geq b, x \geq 0\}$ is not integral, then we are done. So we assume that $\{y \in \mathbf{R}^X : A^T y \leq c, y \geq 0\}$ is not integral. Thus, by an elementary result, there exists $b' \in \mathbf{Z}^X$ such that the optimal value of the linear programming problem (D) is not integral. Hence, by the duality theorem, $\{x \in \mathbf{R}^Y : Ax \geq b, x \geq 0\}$ is not integral. This proves Theorem 5.

Principal pivoting

Let $M \in \mathbf{R}^{V \times V}$; suppose $X \subseteq V$ and $M[X]$ is nonsingular. Define matrices α, β, γ, δ, and $M * X$ by

$$M = \begin{array}{c} \\ X \\ \overline{X} \end{array} \!\!\begin{array}{c} \begin{array}{cc} X & \overline{X} \end{array} \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{array}, \text{ and } M * X = \begin{array}{c} \\ X \\ \overline{X} \end{array} \!\!\begin{array}{c} \begin{array}{cc} X & \overline{X} \end{array} \\ \begin{pmatrix} \alpha^{-1} & -\alpha^{-1}\beta \\ \gamma\alpha^{-1} & \delta - \gamma\alpha^{-1}\beta \end{pmatrix} \end{array}.$$

The operation that converts M to $M * X$ is called a *principal pivot*, and is well known in the context of the linear complementarity problem.

Given the linear complementarity problem (q, M), we denote by $(q, M) * X$ the problem $(q', M * X)$, where q' is defined by

$$q'_X = -\alpha^{-1} q_X, \text{ and } q'_{\overline{X}} = q_{\overline{X}} - \gamma\alpha^{-1} q_X.$$

The following lemma shows that the problems (q, M) and $(q, M) * X$ are essentially the same; the proof follows directly from the definitions.

Lemma 6 (Cottle,Pang, and Stone [5]) *Let $M \in \mathbf{R}^{V \times V}$, $q \in \mathbf{R}^V$, $M[X]$ be a nonsingular principal submatrix of M, and (w, z) be a solution of (q, M). Define w', z' such that $w'_X = z_X$, $z'_X = w_X$, $w'_{\overline{X}} = w_{\overline{X}}$, and $z'_{\overline{X}} = z_{\overline{X}}$. Then (w', z') is a solution of $(q, M) * X$.* □

Let (w, z) be a solution to (q, M), and let (w', z') be the corresponding solution to $(q, M) * X$. It can be easily verified that (w, z) is a basic solution to (q, M) if and only if (w', z') is a basic solution for $(q, M) * X$. Furthermore, nonnegativity, complementarity and integrality are also preserved under such transformations.

We now consider the effect that principal pivoting has on subdeterminants.

Theorem 7 *Let $M[X]$ be a nonsingular principal submatrix of $M \in \mathbf{R}^{V \times V}$. Then, for equicardinal subsets S, T of V,*

$$\det((M * X)[S, T]) = \pm \det(M[(X \setminus T) \cup (S \setminus X), (X \setminus S) \cup (T \setminus X)]) / \det(M[X]).$$

Before proving the theorem we discuss its consequences. It is clear from the definition that principal pivoting preserves skew-symmetry. (It does not preserve symmetry.) Theorem 7 shows that it also preserves rank-symmetry. Theorem 7 also implies, except for the sign, the following theorem of Tucker. For sets A, B we define $A \Delta B$ to be the symmetric difference of A and B; that is, $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Theorem 8 (Tucker [10]) *Let $M[X]$ be a nonsingular principal submatrix of $M \in \mathbf{R}^{V \times V}$. Then, for $S \subseteq V$,*

$$\det((M * X)[S]) = \det(M[X \Delta S]) / \det(M[X]). \qquad \square$$

Tucker's theorem implies that the properties of being principally unimodular, being positive (semi-) definite, and having positive principal minors are all preserved by principal pivoting.

Proof of Theorem 7. Let $Y = V \setminus X$, and let M be partitioned as follows:

$$M = \begin{array}{c} \\ X \\ Y \end{array} \!\! \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{array}.$$

Construct a copy $\tilde{V}$ of V, and for $Z \subseteq V$, denote by $\tilde{Z}$ the corresponding copy of Z. Now define M' to be

$$\begin{array}{c} \\ X \\ Y \end{array} \!\! \begin{array}{c} \begin{array}{cccc} X & Y & \tilde{X} & \tilde{Y} \end{array} \\ \begin{pmatrix} I & 0 & \alpha & \beta \\ 0 & I & \gamma & \delta \end{pmatrix} \end{array}.$$

For equicardinal subsets R, C of V, we have $\det M[R, C] = \pm \det M'[V, \tilde{C} \cup (V \setminus R)]$. Now define matrices D and B by

$$D = \begin{array}{c} \\ X \\ Y \end{array} \!\! \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{pmatrix} \alpha^{-1} & 0 \\ -\gamma\alpha^{-1} & I \end{pmatrix} \end{array}, \text{ and } B = DM' = \begin{array}{c} \\ X \\ Y \end{array} \!\! \begin{array}{c} \begin{array}{cccc} X & Y & \tilde{X} & \tilde{Y} \end{array} \\ \begin{pmatrix} \alpha^{-1} & 0 & I & \alpha^{-1}\beta \\ -\gamma\alpha^{-1} & I & 0 & \delta - \gamma\alpha^{-1}\beta \end{pmatrix} \end{array}.$$

Therefore

$$\begin{aligned} \det(B[V, \tilde{C} \cup (V \setminus R)]) &= \det(M'[V, \tilde{C} \cup (V \setminus R)]) \det(D) \\ &= \pm \det(M[R, C]) / \det(M[X]). \end{aligned} \qquad (4)$$

Now swapping the columns Y and $\tilde{Y}$ pairwise in B we get the matrix B', where

$$B' = \begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & \tilde{X} & \tilde{Y} \end{array} \\ \left(\begin{array}{cccc} I & 0 & \alpha^{-1} & \alpha^{-1}\beta \\ 0 & I & -\gamma\alpha^{-1} & \delta - \gamma\alpha^{-1}\beta \end{array} \right) \end{array}.$$

Note that $B'[X \cup Y, \tilde{X} \cup \tilde{Y}]$ can be obtained from $M * X$ by multiplying the rows and the columns indexed by elements of X by -1. Hence corresponding subdeterminants of the two matrices are equal, up to sign. Therefore,

$$\begin{aligned} \det(B[V, \tilde{C} \cup (V \setminus R)]) &= \pm \det(B'[V, (C \cap X) \cup (V \setminus R \setminus X) \cup (\tilde{C} \setminus \tilde{X}) \cup (\tilde{X} \setminus \tilde{R})]) \\ &= \pm \det(M * X[(R \setminus X) \cup (X \setminus C), (C \setminus X) \cup (X \setminus R)]). \quad (5) \end{aligned}$$

The result is obtained by combining equations (4) and (5), and observing that principal pivoting is an involution. □

Elementary pivots

The following result about pivoting is implied by the quotient formula for the Schur complement (Cottle *et al.* [5, page 76]).

Proposition 9 *Let $M[X]$ be a nonsingular principal submatrix of $M \in \mathbf{R}^{V \times V}$, and let $(M * X)[Y]$ be a nonsingular principal submatrix of $M * X$. Then $(M * X) * Y = M * (X \Delta Y)$.* □

Suppose that $M[X]$ is a nonsingular principal submatrix of M, and there exists $X' \subseteq X$ such that $M[X']$ is nonsingular. Then, by Proposition 9, $M * X = (M * X') * (X \setminus X')$. We call a nonempty set X an *elementary set of* M if $M[X]$ is nonsingular but there exists no proper nonempty subset X' of X such that $M[X']$ is nonsingular. We call the pivot transforming M to $M * X$ *elementary* if X is an elementary set of M. Thus any pivot is equivalent to a sequence of elementary pivots. Note that the elementary sets of a rank-symmetric matrix have cardinality one or two.

Proposition 10 *If X is an elementary set of $M \in \mathbf{R}^{V \times V}$, then every row and column of $M[X]$ contains exactly one nonzero entry.*

Proof Since $M[X]$ is nonsingular, there is a permutation σ of X such that $m_{i\sigma(i)} \neq 0$ for all $i \in X$. Therefore, there exists a cyclic permutation σ' of a subset X' of X such that $m_{i\sigma'(i)} \neq 0$ for all $i \in X'$. Choose σ' so that $|X'|$ is as small as possible. If some row or column of $M[X']$ has more than one nonzero entry, then there exist $i, j \in X'$ such that $m_{ij} \neq 0$ and $j \neq \sigma'(i)$. From this we easily obtain a cyclic permutation on a proper subset of X' that contradicts the choice of X'. Therefore, $M[X']$ is nonsingular, and, since X is elementary, $X' = X$. □

Proofs of Theorems 1 and 2

Proof of Theorem 1. It follows from Proposition 4 that (a) implies (b), so it remains to prove that (b) implies (a). We will need the following result.

Claim *Let X be a subset of V, such that* $\det(M[X]) = \pm 1$. *Then,* $M * X$ *is integral. Furthermore, if q, q' is a pair of vectors such that* $(q', M * X) = (q, M) * X$*, then q is integral if and only if q' is integral.*

Since $M[X]$ is unimodular and integral, $M[X]^{-1}$ is unimodular and integral. Therefore, $M * X$ is also integral. Thus, if q is integral, then q' is integral. The converse follows since pivoting is an involution. This proves the claim.

Suppose that M is not PU, and let Y be a minimum cardinality subset of V such that $M[Y]$ is not unimodular. Suppose that Y is not an elementary set of M. Since $M[Y]$ is nonsingular, there exists a subset Y' of Y, such that Y' is an elementary set of M. By our choice of Y, $M[Y']$ is unimodular. By the claim, it suffices to prove the theorem for $M * Y'$. Now $|Y' \Delta Y| < |Y|$, and so by Theorem 8 with $X = Y'$ and $S = Y \Delta Y'$, $(M * Y')[Y \Delta Y']$ is not unimodular. Thus, inductively, we may assume that Y is an elementary set.

We will create an integral vector q so that the basic solution (w, z) of (q, M), with respect to the set Y, is feasible but not integral. To be basic, (w, z) must satisfy the following equations

$$M[Y]z_Y + q_Y = 0 \tag{6}$$

$$M[\overline{Y}, Y]z_Y + q_{\overline{Y}} = w_{\overline{Y}}. \tag{7}$$

By Proposition 10, every row and column of $M[Y]$ contains exactly one nonzero element. Therefore, every row and column of $M[Y]^{-1}$ contains exactly one nonzero element. Furthermore, since $M[Y]$ is integral but not unimodular, $M[X]^{-1}$ contains some non-integral entries. Thus, it is easy to find an integral q_Y such that the unique solution z_Y to equation (6) is both nonnegative and not integral. Given this z_Y, we can choose an integral $q_{\overline{Y}}$ sufficiently large so that the solution $w_{\overline{Y}}$ to equation (7) is nonnegative. Hence we have an integral q, and a nonintegral basic solution (w, z) to (q, M), as required. □

Proof of Theorem 2. Let $M = (m_{ij})$ be a rank-symmetric matrix, let (w, z) be a solution to (q, M), and denote by X the support of z (that is, the set $\{v \in V : z_v \neq 0\}$). We prove the result by induction on $|X|$; if $|X| = 0$, then (w, z) is basic. Let $Y = \{v \in V : w_v = 0\}$. Note that, by complementarity, X is a subset of Y.

Suppose that $M[Y, X] = 0$. In particular, we have $M[X] = 0$. Choose some $x \in X$. Now define a new vector z' by fixing $z'_v = z_v$ for all $v \in V - x$, and decreasing z'_x as far as possible, while maintaining z' feasible to (q, M). Let $w' = Mz' + q$. Since $M[X] = 0$ and $z'_{\overline{X}} = 0$, we have

$$w'_X = q_X, \; w'_{\overline{X}} = M[\overline{X}, X]z'_X + q_{\overline{X}}.$$

However, since $w_X = 0$, we have $q_X = 0$. Therefore, (w', z') is complementary, and hence

(w', z') is a solution to (q, M). If $z'_x = 0$, then z' has a smaller support than z, so the result follows inductively. Therefore, we may assume that $z'_x > 0$. Since we cannot reduce z'_x further while maintaining feasibility to (q, M), there exists $y \in V$ such that $w'_y = 0$, and $m_{xy} > 0$. Hence, by replacing (w, z) by (w', z'), and redefining Y accordingly, we get $M[X, Y] \neq 0$.

Choose $x \in X$, and $y \in Y$ such that $m_{xy} \neq 0$. If $m_{xx} \neq 0$, then we take $y = x$. Now define S to be $\{x, y\}$. Since M is rank-symmetric, $M[S]$ is nonsingular. Recall that (q, M) has a basic solution if and only if $(q, M) * S$ has a basic solution. Now define z', w' such that

$$z'_S = w_S,\ w'_S = z_S,\ z'_{\overline{S}} = z_{\overline{S}},\ w'_{\overline{S}} = w_{\overline{S}}.$$

Then, by Proposition 6, (z', w') is a solution to $(q, M) * S$. However, since $S \subseteq Y$ and $S \cap X \neq \emptyset$, z' has a smaller support than z. Therefore, the result follows by induction. □

Remarks

We hope that the results of this paper give some mathematical-programming motivation for the study of principally unimodular matrices, in particular, of PU-matrices that are rank-symmetric. The construction that shows that principal unimodularity extends total unimodularity involves a symmetric or skew-symmetric matrix. Therefore, results on either the class of symmetric or skew-symmetric PU-matrices would provide results on the class of totally unimodular matrices. For example, one might ask whether important known facts on totally unimodular matrices can be generalized to either of these two classes of PU-matrices. We summarize here what is known and not known at present in this direction.

First, we mention that there is an interesting subclass of the skew-symmetric PU-matrices, other than those arising from the totally unimodular matrices. It arises from orientations of circle graphs; see Bouchet [1]. Up to now, no such subclass of symmetric PU-matrices is known.

A key observation concerning the recognition of totally unimodular matrices is that the essence of the problem is to recognize whether there is *any* totally unimodular matrix having the same support. The reason is that, given the support of a totally unimodular matrix, it is easy to construct a totally unimodular matrix that has that support, and it is almost unique. Namely, a result of Camion [3] states that two totally unimodular matrices having the same support can be obtained one from the other by a sequence of row and column negations. Similar results hold for the two classes of PU-matrices. It is proved by Geelen [6,7] that two "connected" symmetric PU-matrices having the same support can be obtained one from the other by a sequence of operations of the following form: negating a row and the corresponding column, and negating the whole matrix. (These operations obviously preserve principal unimodularity and symmetry.) Connectedness refers to the absence of a block decomposition, and it is easy to extend this result to a statement about all symmetric PU-matrices. For skew-symmetric matrices the situation is trickier.

It is proved in [7,2] that two "3-connected" skew-symmetric PU-matrices having the same support can be obtained one from the other by a sequence of operations as above. We do not define the notion of the 3-connectivity here, but it is true that if a matrix lacks 3-connectivity then there exists a decomposition that reduces questions about principal unimodularity to such questions about smaller matrices.

There are two famous characterizations of totally unimodular matrices, or equivalently, of regular matroids. They are the excluded minor theorem of Tutte [11] and the decomposition theorem of Seymour [9]. Tutte's theorem has been generalized to the class of symmetric PU matrices (Geelen [6,7]), but not to the class of skew-symmetric ones. Seymour's theorem, which underlies the polynomial-time recognition algorithms for totally unimodular matrices, has not been generalized to either class of PU matrices, and there is no known polynomial-time recognition algorithm for either class. We are optimistic that many of the open problems implicit in these remarks can be solved.

The first-order optimality conditions for a quadratic program give a linear complementarity problem (q, M), where M has the form $\begin{pmatrix} D & A \\ -A^T & 0 \end{pmatrix}$ for some symmetric matrix D. While M is not symmetric, it can be made symmetric by negation of certain rows. Therefore testing principal unimodularity of such matrices is equivalent to testing principal unimodularity of symmetric matrices, which remains unsolved. However, for a *convex* quadratic program, M has the additional property of being positive semidefinite. Chandrasekaran, Kabadi, and Sridhar [4] give a polynomial-time algorithm for testing principal unimodularity of the matrices arising from convex quadratic programming.

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