The Optimal Path-Matching Problem

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March, 1996
Revised June, 1997

Abstract

We describe a common generalization of the weighted matching problem and the weighted matroid intersection problem. In this context we establish common generalizations of the main results on those two problems—polynomial-time solvability, min-max theorems, and totally dual integral polyhedral descriptions. New applications of these results include a strongly polynomial separation algorithm for the convex hull of matchable sets of a graph, and a polynomial-time algorithm to compute the rank of a certain matrix of indeterminates.

1 Introduction

Given a graph $G = (V, E)$, a *perfect matching* of G is a subset of edges such that each vertex of G is incident to exactly one edge of the subset. Tutte [19] gave a necessary and sufficient condition for the existence of a perfect matching. Later Edmonds [6], [7] gave polynomial-time algorithms to decide whether a given graph has a perfect matching, and (given a weighting of the edges) to find a perfect matching of maximum weight. He also gave a polyhedral description of the perfect matchings of G, by characterizing their convex hull as the solution set of a certain system of linear inequalities. Finally, Cunningham and Marsh [5] proved the total dual integrality of the system of inequalities.

Given matroids M_1, M_2 on the same set T, a *common basis* of M_1, M_2 is a subset of T that is a basis in both matroids. Edmonds [9] gave a necessary and sufficient condition for the existence of a common basis, and polynomial-time algorithms to determine whether there exists a common basis and to find a common basis of maximum weight. (In analyzing such "matroid algorithms", we regard each independence test as a single step of the algorithm.) He also found a totally dual integral polyhedral description of the common bases.

Here we propose a common generalization of matching and matroid intersection, and establish common generalizations for the results mentioned above. Let $G = (V, E)$ be a graph and T_1, T_2 disjoint stable sets of G, that is, sets of mutually nonadjacent vertices. We denote $V \backslash (T_1 \cup T_2)$ by R. Let M_i be a matroid on T_i, for $i = 1, 2$, and suppose that M_1 and M_2 have rank r. A *basic path-matching* is a collection of r vertex-disjoint paths, all of whose internal vertices are in R, linking a basis of M_1 to a basis of M_2, together with a perfect matching of the vertices of R not in any of the paths. (Figure 1 shows an example. Here we assume that the only basis of M_1 is T_1 and the only basis of M_2 is T_2. The thick edges form a basic path-matching.) In the

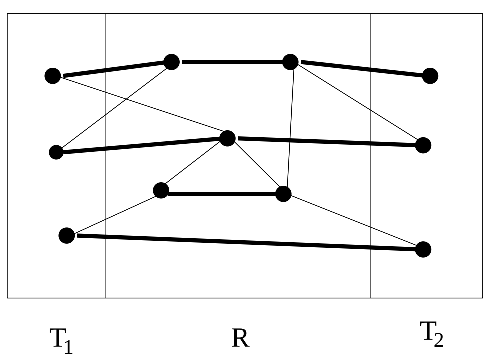

Figure 1: A basic path-matching

special case when $R = V$, a basic path-matching is nothing but a perfect matching of G. In the special case when $R = \emptyset$, and G consists of a perfect matching joining copies T_1, T_2 of a set T, a basic path-matching corresponds to a common basis of M_1 and M_2. Another important special case occurs when there are no special restrictions on G, T_1, T_2, but M_1 and M_2 are *free,* that is, T_i is a basis of M_i for $i = 1$ and 2. In this case we refer to a basic path-matching as a *perfect path-matching with respect to* G, T_1, T_2. Perfect path-matching is itself a nontrivial generalization of matching. With the exception of matroid intersection, all of the applications of basic path-matching that we cite are actually special cases of perfect path-matching.

The existence theorem

Given a graph G, we denote by odd(G) the number of components of G having an odd number of vertices. For a subset S of vertices of G, $G[S]$ denotes the subgraph of G induced by S. A pair of subsets $D_1 \subseteq T_1 \cup R$, $D_2 \subseteq T_2 \cup R$ is called *stable* if no edge of G joins a vertex in $D_1 \backslash D_2$ to a vertex in D_2 or a vertex in $D_2 \backslash D_1$ to a vertex in D_1. (To see where the name comes from, consider the special case in which $R = \emptyset$.) The sets of vertices contained in the ellipses of Figure 2 form a stable pair. We use r_1, r_2 to denote the rank functions of M_1, M_2. We now state the main result on the existence of basic path-matchings.

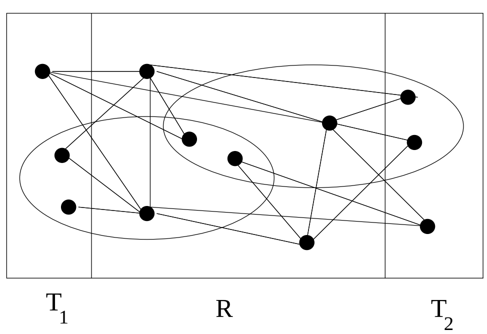

Figure 2: A stable pair

Theorem 1.1 *There exists a basic path-matching with respect to G, M_1, M_2 if and only if there does not exist a stable pair (D_1, D_2) for which*

$$r_1(T_1 \setminus D_1) + r_2(T_2 \setminus D_2) + |R \backslash (D_1 \cup D_2)| < r + odd(G[D_1 \cap D_2]).$$

Proof of necessity in Theorem 1.1: Suppose that there exists a basic path-matching K, and let (D_1, D_2) be a stable pair. We think of the paths of K as being *from* T_1 *to* T_2. There are at least $r - r_1(T_1 \backslash D_1)$ paths of K beginning in $D_1 \cap T_1$. Each of them has a first vertex not in D_1. Since (D_1, D_2) is stable, that vertex must be in $(R \cup T_2) \backslash (D_1 \cup D_2)$. Also, for each odd component H of $G[D_1 \cap D_2]$, either an edge of a path of K leaves H or a matching edge leaves H. In either case the other end of this edge is again in $(R \cup T_2) \backslash (D_1 \cup D_2)$. Now we have identified at least $r - r_1(T_1 \backslash D_1) + \mathrm{odd}(G[D_1 \cap D_2])$ vertices of $(R \cup T_2) \backslash (D_1 \cup D_2)$, and all of them must be distinct. Moreover, at most $r_2(T_2 \backslash D_2)$ of them can be from T_2. Therefore,

$$r - r_1(T_1 \backslash D_1) + \mathrm{odd}(G[D_1 \cap D_2]) \leq |R \backslash (D_1 \cup D_2)| + r_2(T_2 \backslash D_2).$$

The result follows. ∎

The stable pair indicated in Figure 2 shows that there is no perfect path-matching in that example. Now we apply the existence theorem to derive the existence theorems for matching and matroid intersection mentioned above.

Matching. Tutte's Theorem [19] states that $G = (V, E)$ has a perfect matching if and only if there does not exist a subset S of V such that $\mathrm{odd}(G - S) > |S|$. It is easy to see that the condition is necessary. Now suppose that G has no perfect matching, and take $R = V$. Then by Theorem 1.1, there exists a stable pair (D_1, D_2) such that

$$|V \backslash (D_1 \cup D_2)| < \mathrm{odd}(G[D_1 \cap D_2]).$$

Now observe that, because (D_1, D_2) is stable, every odd component of $G[D_1 \cap D_2]$ is also an odd component of $G[D_1 \cup D_2]$. Therefore, $\mathrm{odd}(G[D_1 \cap D_2]) \leq \mathrm{odd}(G[D_1 \cup D_2])$. If we take $S = V \backslash (D_1 \cup D_2)$, it follows that $\mathrm{odd}(G - S) > |S|$, as required.

Matroid Intersection. Suppose that M_1, M_2 are matroids on T, each of rank r. Edmonds' Matroid Intersection Theorem [9] states that there exists a common basis if and only if there does not exist a subset A of T such that $r_1(A) + r_2(T \backslash A) < r$. It is easy to see that this condition is necessary. Now suppose that there does not exist a common basis. If we take G to be a perfect matching joining copies T_1, T_2 of T, and R to be $\emptyset$, then by Theorem 1.1 there exists a stable pair (D_1, D_2) such that

$$r_1(T_1 \setminus D_1) + r_2(T_2 \setminus D_2) < r.$$

We see from the stability of (D_1, D_2) that at least one end of any edge of G is in $T_1 \setminus D_1$ or in $T_2 \setminus D_2$. Thus, if A denotes the subset of T corresponding to $T_1 \setminus D_1$, then $r_1(A) + r_2(T \setminus A) < r$, as required.

Algorithms

The main algorithmic result of this paper is the following.

Theorem 1.2 *There is a polynomial-time matroid algorithm to decide whether there exists a basic path-matching with respect to G, M_1, M_2.*

We have already mentioned that Edmonds also gave polynomial-time algorithms for weighted versions of the matching and matroid intersection problems; we want to generalize these results, too. We need to be careful, however, to find an appropriate weighted generalization of the basic path-matching problem. The simplest choice, to consider the weight of a basic path-matching to be the sum of the weights of its edges, leads to an $\mathcal{NP}$-hard problem. For suppose that all edge-weights are 1, $|T_1| = |T_2| = 1$, and M_1, M_2 have rank 1. Then there exists a basic path-matching of weight $|V|-1$ if and only if G has a hamiltonian path joining T_1 to T_2. Instead, we define the weight of a basic path-matching to be the sum of the weights of the edges of the paths plus *twice* the weights of its other edges. Notice that this choice has the nice property that it does not favour putting edges into paths over putting them into the matching, and the resulting maximum-weight problem still contains the weighted versions of the matching and matroid intersection problems.

Theorem 1.3 *There is a polynomial-time matroid algorithm to find (if there is one) a maximum-weight basic path-matching with respect to G, M_1, M_2.*

Polyhedra

The algorithmic results will be derived as consequences of a polyhedral theorem, which we now describe. First, we introduce some terminology and notation. We use $\mathbf{R}$ to denote the set of real numbers. If B is a finite set, a *polyhedron* in $\mathbf{R}^B$ is a set of the form $\{x \in \mathbf{R}^B : Ax \leq b\}$ for some real matrix A and vector b of appropriate dimensions. A *polytope* is a bounded polyhedron. We say that a polyhedron is *integral* if it is the convex hull of its integral points. It is well-known that a polytope is the convex hull of a finite set of points, and the minimal such set consists of its *extreme points.* For $x \in \mathbf{R}^B$ and $C \subseteq B$, we denote by $x(C)$ the sum $\sum(x_j : j \in C)$. For a graph $G = (V, E)$ and a subset S of V, we denote by $\delta(S)$ (or $\delta_G(S)$) the set of edges of G that have exactly one end in S, and by $\gamma(S)$ (or $\gamma_G(S)$) the set of edges of G having both ends in S. If G is a digraph and S is a subset of its vertices, we denote by $\delta^-(S)$ the set of arcs of G having tail in S and head not in S, and by $\delta^+(S)$ the set of arcs of G having head in S and tail not in S.

Given a basic path-matching K, let K_1 be the set of edges in the paths and let K_2 consist of the remaining edges of K. We define the *basic path-matching vector* corresponding to K to be the vector $\psi^K \in \mathbf{R}^E$ such that, for $e \in E$,

$$\psi_e^K = \begin{cases} 1, & \text{if } e \in K_1 \\ 2, & \text{if } e \in K_2 \\ 0, & \text{if } e \notin K. \end{cases}$$

We denote the set of all basic path-matchings by $\mathcal{K} = \mathcal{K}(G, M_1, M_2)$. Notice that, given a weight vector c, the problem of finding a maximum-weight basic path-matching can be written as $\max(c\psi^K : K \in \mathcal{K})$. The convex hull of all basic path-matching vectors, $\text{conv}(\{\psi^K : K \in \mathcal{K}\})$, is called the *basic path-matching polyhedron.* By a slight abuse of notation, we will sometimes denote this polyhedron by $\text{conv}(\mathcal{K})$. The maximum weight basic path-matching problem is equivalent to a linear programming problem over $\text{conv}(\mathcal{K})$.

Theorem 1.4 *$conv(\mathcal{K}(G, M_1, M_2))$ is the set of all $x \in \mathbf{R}^E$ satisfying:*

$$\begin{aligned}
x(\delta(v)) &= 2 && (v \in R) && (1)\\
x(\delta(S)) &\geq r && (T_1 \subseteq S \subseteq T_1 \cup R) && (2)\\
x(\delta(S)) &\geq 2 && (S \subseteq R,\ |S|\ \textit{odd}) && (3)\\
x(\delta(A)) &\leq r_1(A) && (A \subseteq T_1) && (4)\\
x(\delta(A)) &\leq r_2(A) && (A \subseteq T_2) && (5)\\
x(\delta(T_1)) &= r && && (6)\\
x(\delta(T_2)) &= r && && (7)\\
x &\geq 0. && && (8)
\end{aligned}$$

Theorem 1.4 is proved in Section 3. Here we apply it to matching and matroid intersection. Edmonds [7] proved the following result on the polytope of perfect matchings.

Theorem 1.5 (Matching Polytope Theorem) *The convex hull of incidence vectors of perfect matchings of a graph $G = (V, E)$ is the set of all $x \in \mathbf{R}^E$ satisfying:*

$$\begin{array}{rcll} x(\delta(v)) & = & 1 & \quad (v \in V) \\ x(\delta(S)) & \geq & 1 & \quad (S \subseteq R,\ |S| \text{ odd}) \\ x & \geq & 0. & \end{array}$$

Now consider the special case of Theorem 1.4 in which $T_1 = T_2 = \emptyset$. Then we get a description by linear inequalities of the convex hull of *twice* the incidence vectors of perfect matchings of G. In the resulting description, the inequalities (2), (4), (5), (6), and (7) each collapse to a single redundant inequality. Dividing the right-hand sides of the remaining inequalities by 2, we obtain a description of the perfect matching polyhedron, and it is precisely that of Theorem 1.5, as required.

Edmonds [9] also proved the following polyhedral theorem on common bases.

Theorem 1.6 (Matroid Intersection Polyhedron Theorem) *The convex hull of incidence vectors of common bases of two matroids M_1, M_2 on T with rank functions r_1, r_2 is the set of all $x \in \mathbf{R}^T$ satisfying:*

$$\begin{array}{rcll} x(A) & \leq & r_1(A) & \quad (A \subseteq T) \\ x(A) & \leq & r_2(A) & \quad (A \subseteq T) \\ x(T) & = & r & \\ x & \geq & 0. & \end{array}$$

Applying Theorem 1.4 in the case in which $R = \emptyset$ and G consists of a perfect matching joining T_1 to T_2, we get a description of the convex hull of incidence vectors of common bases of M_1 and M_2. In this description, inequalities (1) and (3) disappear, inequalities (2) are redundant, and inequality (6) is the same as (7). Therefore, Theorem 1.6 also follows from Theorem 1.4.

It is also quite easy to prove the algorithmic results stated above from Theorem 1.4.

Proof of Theorems 1.2 and 1.3 from Theorem 1.4: By the equivalence of optimization and separation—see Grötschel, Lovász, and Schrijver [13]—it is possible to optimize an arbitrary linear function over $\mathrm{conv}(\mathcal{K}(G, M_1, M_2))$ in polynomial time if and only if it is possible to solve the separation problem for the same polytope in polynomial time. (The *separation problem* for a polytope $P \subseteq \mathbf{R}^n$ is, given a point $\hat{x} \in \mathbf{R}^n$, either to determine that $\hat{x} \in P$ or to find a linear inequality $ax \leq \alpha$ that is violated by $\hat{x}$ but satisfied by every point in P.) However, it is straightforward to show that the latter is true for the polyhedron of Theorem 1.4. First, it is easy to check that a given point $\hat{x} \in \mathbf{R}^E$ satisfies inequalities (1) and (8), since there are only a few of them. Henceforth, we may assume that $\hat{x} \geq 0$. Now inequalities (2) can be checked by solving a minimum-cut problem. Inequalities (3) require a more sophisticated use of minimum-cut methods, but these can also be checked in polynomial time; see Padberg and Rao [16]. Next, $\hat{x}$ satisfies inequalities (4) and (6) if and only if the vector $y \in \mathbf{R}^{T_1}$ defined by $y_v = \hat{x}(\delta(v))$ is in the convex hull of incidence vectors of bases of M_1. Polynomial-time algorithms for the separation problem for this polytope are given in [13] and [2]. The inequalities involving M_2 can be handled similarly. This completes the proof. ∎

Note that the algorithms that result from these proofs use the ellipsoid method, and are not practical.

Independent path-matchings

Many important results are formulated in terms of matchings of graphs (rather than perfect matchings), and in terms of common independent sets of two matroids (rather than common bases). There is an analogous theory for path-matchings, which we describe here. In particular, we show how these results lead to a proof of Theorem 1.1.

We begin as before with G, M_1, M_2, except that we no longer require that M_1 and M_2 have the same rank. An *independent path-matching* with respect to G, M_1, M_2 is a set K of edges such that every component of $G(V, K)$ having at least one edge is a simple path from $T_1 \cup R$ to $T_2 \cup R$, all of whose internal vertices are in R, and such that the set of vertices of T_i in any of these paths is independent in M_i, for $i = 1$ and 2. (Of course, any basic path-matching is an independent path-matching.) The thick edges in Figure 3 form an independent path-matching with respect to the free matroids M_1, M_2. It is

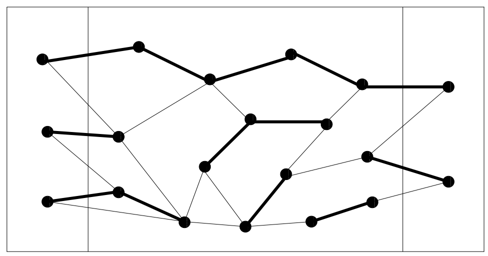

Figure 3: An independent path-matching

easy to see, in the case where $R = \emptyset$ and G consists of a perfect matching of T_1 to T_2, that an independent path-matching corresponds to a common independent set of M_1 and M_2. In the case where $R = V$, we do not get such a simple correspondence to matchings of G, since there may be paths of length more than 1 in $G(V, K)$. However, let us define the *independent path-matching vector* ψ^K corresponding to K in the same way as before, namely, an edge of a one-edge component of $G(V, K)$ having both ends in R gets an entry of 2, the other edges of K get entries of 1, and edges not in K get entries of 0. Also, we define the *weight* of K with respect to a given weight vector c to be $c\psi^K$. Then in the case where $R = V$, it is easy to see that the maximum weight of an independent path-matching is twice the maximum weight of a matching, although there may be maximum-weight independent path-matchings that do not arise directly from a single maximum-weight matching. Finally, there is the important special case in which M_1, M_2 are free; then we may refer to an independent path-matching with respect to G, M_1, M_2 as a *path-matching* with respect to G, T_1, T_2.

The problem of finding a maximum-weight independent path-matching can be reduced by a trick to the problem of finding a maximum-weight basic path-matching (in a different graph with different matroids). However, as mentioned above, there is something to be gained by attacking this problem more directly. A main result is the following polyhedral description of the independent path-matchings. We denote by $\mathcal{K}^* = \mathcal{K}^*(G, M_1, M_2)$, the set of all independent path-matchings with respect to G, M_1, M_2. (As before, we may abbreviate conv($\{\psi^K : K \in \mathcal{K}^*\}$) to conv($\mathcal{K}^*$).)

Theorem 1.7 *conv($\mathcal{K}^*(G, M_1, M_2)$) is the set of all $x \in \mathbf{R}^E$ satisfying:*

$$
\begin{aligned}
x(\delta(v)) &\leq 2 && (v \in R) && (9)\\
x(\gamma(S)) &\leq |S \cap R| && (T_1 \subseteq S \subseteq T_1 \cup R) && (10)\\
x(\gamma(S)) &\leq |S \cap R| && (T_2 \subseteq S \subseteq T_2 \cup R) && (11)\\
x(\gamma(S)) &\leq |S| - 1 && (S \subseteq R, |S| \text{ odd}) && (12)\\
x(\delta(A)) &\leq r_1(A) && (A \subseteq T_1) && (13)\\
x(\delta(A)) &\leq r_2(A) && (A \subseteq T_2) && (14)\\
x &\geq 0. && && (15)
\end{aligned}
$$

Theorem 1.7 is proved from Theorem 1.4 in Section 3. It is easy to derive from it the polyhedral theorems of Edmonds on matchings and common independent sets. We also call attention to the path-matching polyhedron, that is, the special case in which M_1, M_2 are free.

Corollary 1.8 *The convex hull of path-matchings determined by G, T_1, T_2 is the set of all $x \in \mathbf{R}^E$ satisfying $x \in \mathbf{R}^E$ satisfying:*

$$
\begin{aligned}
x(\delta(v)) &\leq 1 && (v \in T_1 \cup T_2) && (16)\\
x(\delta(v)) &\leq 2 && (v \in R) && (17)\\
x(\gamma(S)) &\leq |S \cap R| && (T_1 \subseteq S \subseteq T_1 \cup R) && (18)\\
x(\gamma(S)) &\leq |S \cap R| && (T_2 \subseteq S \subseteq T_2 \cup R) && (19)\\
x(\gamma(S)) &\leq |S| - 1 && (S \subseteq R, |S| \text{ odd}) && (20)\\
x &\geq 0. && && (21)
\end{aligned}
$$

∎

A system $Ax \leq b$ of linear inequalities is *totally dual integral* if for every integral vector c for which the linear programming problem minimize $(yb : yA = c,\ y \geq 0)$ has an optimal solution, it has an optimal solution that is integral. (A fundamental theorem states that, if $Ax \leq b$ is totally dual integral and b is integral, then $P = \{x : Ax \leq b\}$ is an integral polyhedron.) Cunningham and Marsh [5] proved that the system of inequalities describing the convex hull of matchings is totally dual integral, and Edmonds [9] proved the same thing for the convex hull of common independent sets of two matroids. (However, the system of inequalities appearing in Theorem 1.5 is *not* totally dual integral.) These results generalize well-known min-max theorems characterizing the maximum cardinality of a matching and of a common independent set. We prove in Section 4 that the system of Theorem 1.7 is totally dual integral. This theorem generalizes the similar results for matching and matroid intersection. It also can be used to prove a generalization of the corresponding min-max formulas, which we now state.

Theorem 1.9 *The maximum over $K \in \mathcal{K}^*(G, M_1, M_2)$ of $\psi^K(E)$ is the minimum over stable pairs (D_1, D_2) of*

$$r_1(T_1 \backslash D_1) + r_2(T_2 \backslash D_2) + |R \backslash (D_1 \cup D_2)| + |R| - odd(G[D_1 \cap D_2]).$$

We leave to the reader the exercise of obtaining from this theorem the Tutte-Berge matching formula, and Edmonds' matroid intersection min-max theorem. We can now prove the existence result for basic path-matchings, Theorem 1.1.

Proof of Theorem 1.1 from Theorem 1.9: We already showed that the condition is necessary. Now suppose that there is no basic path-matching. Notice that this implies that the maximum of $\sum(x_e : e \in E)$ over independent path-matching vectors x is less than $|R| + r$. It follows from Theorem 1.9 that there exists a stable pair (D_1, D_2) such that

$$r_1(T_1 \backslash D_1) + r_2(T_2 \backslash D_2) + |R \backslash (D_1 \cup D_2)| + |R| - \mathrm{odd}(G[D_1 \cap D_2]) < |R| + r.$$

So (D_1, D_2) is the required stable pair. ∎

2 Applications

In this section we treat some further applications of path-matchings.

Disjoint paths

Suppose that we are given a graph $G' = (V', E')$ whose vertex-set is partitioned into sets T_1', T_2', R with $|T_1'| = |T_2'| = k$. We wish to find, if possible,

k vertex-disjoint paths of G' from T_1' to T_2'. This is, of course, a standard problem, for which Menger's Theorem gives a characterization, and network flow methods give efficient algorithms. Our purpose here is just to show that it can be transformed into a perfect path-matching problem.

Here is the construction. Form a new graph G by adding, for every $r \in R$, vertices r_1, r_2 and edges rr_1, rr_2, r_1r_2 and put $T_1 = R_1 \cup T_1'$, $T_2 = R_2 \cup T_2'$, where R_i denotes $\{r_i : r \in R\}$. Then there is a perfect path-matching of G with respect to T_1, T_2 if and only if the desired paths exist in G'. Thus the existence of a polynomial-time algorithm for the disjoint paths problem is a consequence of Theorem 1.2.

We can also use the construction to derive Menger's Theorem from Theorem 1.1. Menger's Theorem states that the disjoint paths exist if there exists no set S that separates T_1' from T_2' in G' and has size less than k. (A set of vertices *separates* T_1' from T_2' if it meets every path from T_1' to T_2'.) It is easy to show that the condition is necessary. Now suppose that G' does not contain the desired paths. Then there exists a stable pair (D_1, D_2) of G such that

$$|D_1 \cap T_1| + \mathrm{odd}(G[D_1 \cap D_2]) > |T_2 \backslash D_2| + |R \backslash (D_1 \cup D_2)|. \qquad (22)$$

For every $r \in D_1 \cap D_2$, $r_1 \notin D_1$ and $r_2 \notin D_2$. For every such r, delete r from D_1 and add r_1 to D_1. Then we get a new stable pair with $D_1 \cap D_2 = \emptyset$ and still satisfying (22). Now

$$S = (T_1' \backslash D_1) \cup (T_2' \backslash D_2) \cup (R \backslash (D_1 \cup D_2))$$

separates T_1' from T_2' in G', and a calculation shows that $|S| < k$, proving Menger's Theorem.

Rank in the matching matroid

Another application of the maximum path-matching formula occurs when $T_1 = \emptyset$. Then the maximum value of a path-matching is the maximum intersection of a matchable set with R. This is the rank of R in the "matching matroid" determined by G. This matroid was introduced by Edmonds and Fulkerson [10]. The following formula for the rank of R is well known, although it was not explicitly stated in [10]. We use the notation $\mathrm{odd}_T(H)$ for the number of odd components of H, each having all of its vertices contained in T. This result can be derived from the min-max theorem by a method similar to ones used above, and we leave this to the reader.

Theorem 2.1 *Let $G = (V, E)$ be a graph and R a subset of V. The maximum size of a subset of R covered by a matching of G is the minimum over $S \subseteq V$ of*

$$|R| - |S| + odd_R(G[V \backslash S])$$

The Tutte matrix

Let $G' = (V', E')$ be a graph, and let x_e, $e \in E'$ be distinct variables. Let $A = (a_{ij})$ be a V' by V' skew-symmetric matrix such that $a_{ij} = \pm x_e$ if $ij = e \in E'$, and $a_{ij} = 0$ otherwise. We call A the *Tutte matrix* of G', even though it is not quite unique. Given subsets I, J of V', both of size k, we want to determine whether the submatrix $A[I, J]$ is nonsingular, that is, whether its determinant is nonzero (as a polynomial), or more generally, to determine its rank. (Edmonds [8] seems to have been the first to emphasize such algorithmic questions. For example, he proposed the problem of finding a polynomial-time algorithm to compute the rank of a matrix whose entries are multivariate polynomials with integral coefficients.) This problem is in $\mathcal{NP}$, because it is not difficult to show that A is nonsingular if and only if there exist (small) rational values for the variables so that the resulting rational matrix is nonsingular. (This observation is the basis for a well-known approach that provides a randomized polynomial-time algorithm.) However, it is not obvious that it is in *co-*$\mathcal{NP}$, let alone that there is a polynomial-time deterministic algorithm.

There are two important special cases where satisfactory results have been available. If $I \cap J = \emptyset$, then each x_e occurs at most once in $A[I, J]$, so $A[I, J]$ is nonsingular if and only if there is at least one nonzero term in the expansion of its determinant. This property can be tested by solving a bipartite matching problem. Also, if $I = J$, then $A[I, J]$ is nonsingular if and only if $G[I]$ has a perfect matching. (The latter fact, which is not obvious but can be proved by elementary methods, played an important role in Tutte's original paper [19].) We generalize this fact, as follows. Define $G = (V, E)$ to be the graph obtained from G' by deleting the vertices not in $I \cup J$ and the edges having both ends in $I \backslash J$ and those having both ends in $J \backslash I$. The following result can be proved by elementary methods; see [11] or [4].

Theorem 2.2 *The rank of $A[I, J]$ is the maximum of $\psi^K(E)$ over path-matchings K with respect to $G, I \backslash J, J \backslash I$.* ▮

From Theorems 2.2 and 1.2 we get immediately the following consequence.

Corollary 2.3 *There is a polynomial-time algorithm to determine the rank of a given submatrix of the Tutte matrix.* ∎

We can combine Theorem 2.2 with the min-max theorem Theorem 1.9 to obtain a formula for the rank of $A[I, J]$. However, this formula is not really new. It can be proved directly using a linear-algebra method of Lovász [14]. His proof, which can be found in [11], predates our polyhedral proof, also in [11], and our generalization, which first appeared in [4].

Theorem 2.4 *The rank of $A[I, J]$ is equal to the minimum over all stable pairs (I', J') of G with respect to $I \backslash J, J \backslash I$ of*

$$|I \backslash I'| + |J \backslash J'| + |I' \cap J'| - odd(G[I' \cap J']). \tag{23}$$

∎

The matchable set polyhedron

A *matchable set* of a graph is a set of vertices forming the ends of the edges of some matching. The *matchable set polyhedron* $Q(G)$ of a graph G is the convex hull of incidence vectors of matchable sets of G. This polyhedron was introduced by Balas and Pulleyblank [1], who gave a nice description by linear inequalities. (We will not need that description here.)

There are several ways to obtain a polynomial-time algorithm for the separation problem for $Q(G)$. The easiest method to describe briefly goes as follows. We use the equivalence of separation and optimization, so it is enough to show that there is a polynomial-time algorithm to optimize any linear function cx over $Q(G)$. One way to do this, is to reduce the problem to a weighted matching problem, by defining edge weights $c'_{uv} = c_u + c_v$ for any edge uv. The resulting algorithm for the separation problem is based on the ellipsoid method, and so is not combinatorial, and is not strongly polynomial. There does exist a polynomial-time combinatorial algorithm for the separation problem; see [3]. However, that algorithm uses scaling, and is not strongly polynomial. Here we describe a strongly polynomial algorithm, based on the results of this paper. In fact, it was this problem that originally led to the formulation of problems on path-matchings.

Let A be a V by V matrix, and let $\mathcal{F}$ denote $\{I \subseteq V : A[I, I]$ is nonsingular$\}$. (If A is the Tutte matrix of G, then $\mathcal{F}$ is the family of matchable sets of G.) Let $A' = (A, \mathcal{I})$, where $\mathcal{I}$ is a $|V|$ by $|V|$ identity matrix. We suppose that the columns of $\mathcal{I}$ are indexed by $V' = \{v' : v \in V\}$, and for any subset J of V we define J' to be $\{v' : v \in J\}$. It is easy to see that $A[J, J]$ is nonsingular if and only if $B = J \cup (V' \backslash J')$ indexes a column basis of A'. (Of course, there are column bases of A' that are not of this form.) Let $\mathcal{B}$ consist of the sets $B \subseteq V \cup V'$ such that B indexes a column basis of A'. Then $\mathcal{B}$ is the family of bases of a matroid N on $V \cup V'$. Theorem 2.5 below is the key observation. Its proof uses Edmonds' Matroid Intersection Polyhedron Theorem 1.6. We use the following notation: Given a vector $x \in \mathbf{R}^V$, define $x' \in \mathbf{R}^{V'}$ by $x'_{j'} = x_j$ for all $j \in V$.

Theorem 2.5 *Given $x \in \mathbf{R}^V$, define $y \in \mathbf{R}^{V \cup V'}$ by $y = (x, 1 - x')$. Then x is a convex combination of incidence vectors of elements of $\mathcal{F}$ if and only if y is a convex combination of incidence vectors of elements of $\mathcal{B}$.*

Proof: First, suppose that x is a convex combination $\sum \lambda_i x^i$ of incidence vectors x^i of members C_i of $\mathcal{F}$. Then for each i, $C_i \cup (V' \backslash C'_i)$ is a basis of N_1. Let y^i be its incidence vector. Then $y = \sum \lambda_i y^i$, as required.

Now suppose that y is a convex combination of incidence vectors of elements of $\mathcal{B}$. Define a matroid $N_2 = (V \cup V', \mathcal{B}_2)$ by $\mathcal{B}_2 = \{B \subseteq V \cup V' : B = C \cup (V' \backslash C')$ for some $C \subseteq V\}$. Then y is a convex combination of incidence vectors of bases of N_2. (There are many ways to show this. One is to observe that the coefficient matrix of the system $y_i + y_{i'} = 1,\ y \geq 0$ is totally unimodular.) Hence by Theorem 1.6, y is a convex combination $\sum \lambda_i y^i$ of incidence vectors y^i of common bases B_i of N and N_2. But a common basis of N and N_2 is of the form $C \cup (V' \backslash C')$ where $C \in \mathcal{F}$, so for each i, the vector x^i, defined to be y^i restricted to V, is the incidence vector of a member of $\mathcal{F}$. Therefore, since $x = \sum \lambda_i x^i$, we are done. ∎

It follows from Theorem 2.5 that we can use a separation algorithm for the convex hull of incidence vectors of elements of $\mathcal{B}$ to determine whether a given vector x is in the convex hull of incidence vectors of elements of $\mathcal{F}$. (In the alternative case when x is not in the polytope, we also need to be able to find a violated valid inequality, but clearly an inequality $ax + b(1 - x') \leq \alpha$ translates into an inequality $(a - b)x \leq \alpha + \beta$, where β is the sum of the components of b.) There is a strongly polynomial matroid algorithm for the former problem [2]. If we want to apply it to the special case where A is the Tutte matrix of a graph G, then we need an algorithm to decide whether a given subset of $V \cup V'$ is independent in the matroid N. But it is easy to see that a subset $P \cup Q'$ is independent if and only if the submatrix $A[V \backslash Q, P]$ has rank $|P|$, so we can use the algorithm of Corollary 2.3. The latter algorithm is strongly polynomial, so we have a strongly polynomial separation algorithm for the matchable set polytope.

Remark Theorem 2.5 is the basis for a separation algorithm for "linear delta-matroid polyhedra".

3 Proofs of polyhedral theorems

In this section we prove Theorem 1.4, and then we use Theorem 1.4 to prove Theorem 1.7. An important step in the proof of Theorem 1.4 is the proof of the following key fact: If the inequalities (3) are omitted from the list of inequalities, the resulting polyhedron P' is integral. This fact is interesting in its own right; it generalizes both the matroid intersection polyhedron theorem and the "fractional matching polyhedron theorem". Moreover, its proof is the only part of the proof of Theorem 1.4 that uses a new idea.

Theorem 3.1 *The set of solutions of (1), (2), and (4)–(8) is an integral polytope.*

Proof: Let P' denote the polytope that is claimed to be integral. Let $\vec{G} = (V, \vec{E})$ denote the digraph obtained from G by replacing each edge by a pair of oppositely directed arcs. We define a matroid N_1 on $\vec{E}$, as follows. A set A is a basis of N_1 if and only if

- No arc in A has its tail in T_2;
- Each vertex in R is the tail of exactly one arc in A;
- Each vertex in T_1 is the tail of at most one arc in A;
- The set of elements of T_1 that are tails of arcs in A is a basis of M_1.

(N_1 is a matroid because it is the direct sum of matroids of rank at most one and a matroid obtained from M_1 by making parallel copies of its elements.)

We define M_2 similarly, interchanging "head" with "tail" and T_1 with T_2. It follows from Theorem 1.6, the common basis polytope theorem, that the polytope $Q \subseteq \mathbf{R}^{\vec{E}}$ defined to be the set of all y satisfying

$$(Q) \quad \left\{ \begin{array}{rcll} y(\delta^-(v)) & = & 0, & (v \in T_2) \\ y(\delta^+(v)) & = & 0, & (v \in T_1) \\ y(\delta^-(v)) & = & 1, & (v \in R) \\ y(\delta^+(v)) & = & 1, & (v \in R) \\ y(\delta^-(A)) & \leq & r_1(A), & (A \subseteq T_1) \\ y(\delta^+(A)) & \leq & r_2(A), & (A \subseteq T_2) \\ y(\delta^-(T_1)) & = & r & \\ y(\delta^+(T_2)) & = & r & \\ y & \geq & 0 & \end{array} \right.$$

has only integral extreme points.

We define a function $\rho : \mathbf{R}^{\vec{E}} \to \mathbf{R}^E$, by: $\rho(y)_{vw} = y_{vw} + y_{wv}$, for $vw \in E$. Let $\rho(Q)$ denote $\{\rho(y) : y \in Q\}$. It is easy to see that $\rho(Q)$ is an integral polyhedron. (Namely, if $x \in \rho(Q)$, then $x = \rho(y)$ for some $y \in Q$. Now y can be expressed as a convex combination of integral points in Q, and so by linearity of ρ, x is a convex combination of their images, which are integral points of $\rho(Q)$.) So we can complete the proof by proving the following.

Claim $P' = \rho(Q)$.

Given $x \in \rho(Q)$, choose $y \in Q$ with $x = \rho(y)$. For any S such that $T_1 \subseteq S \subseteq T_1 \cup R$, we have

$$\begin{aligned} x(\delta(S)) &= y(\delta^-(S)) + y(\delta^+(S)) \\ &\geq y(\delta^-(S)) - y(\delta^+(S)) \\ &= \sum_{v \in S} (y(\delta^-(v)) - y(\delta^+(v))) = r, \end{aligned}$$

so x satisfies (2). It is straightforward to check that x also satisfies the other inequalities defining P', so $\rho(Q) \subseteq P'$.

Now, suppose that $x \in P'$. Let $\mathcal{L}$ denote the set of all paths in G from a vertex in T_1 to a vertex in T_2 and having internal vertices only from R. For $vw \in E$, we denote by $\mathcal{L}_{vw}$ the set of paths in $\mathcal{L}$ that use the edge vw. By the Max-flow Min-cut Theorem, there exists a nonnegative vector $\lambda \in \mathbf{R}^{\mathcal{L}}$, such that $\lambda(\mathcal{L}) = r$, and, for $vw \in E$, $\lambda(\mathcal{L}_{vw}) \leq x_{vw}$. Let $f \in \mathbf{R}^{\vec{E}}$ be the

(T_1, T_2)-flow in $\vec{G}$, corresponding to the path-flow λ. That is, for $vw \in \vec{E}$, $f_{vw} = \sum \lambda_L$ where the sum is over $L \in \mathcal{L}_{vw}$ such that v precedes w on L. Now, define a vector $y \in \mathbf{R}^{\vec{E}}$, such that, for $vw \in \vec{E}$,

$$y_{vw} = f_{vw} + \frac{1}{2}(x_{vw} - (f_{vw} + f_{wv})).$$

It is easily verified that $y \in Q$, and $\rho(y) = x$. Thus, $x \in \rho(Q)$, so $P' \subseteq \rho(Q)$, and we are done. ∎

Remark One might expect that adding to (Q) the inequalities $y(\delta^-(S)) \geq 1$ and $y(\delta^+(S)) \geq 1$ for all $S \subseteq R$ such that $|S|$ is odd, also results in an integral polyhedron. This is false.

Our proof of Theorem 1.4 follows a technique that was used previously in proofs of Edmonds' description of the perfect matching polyhedron (Theorem 1.5); see Schrijver [17] or Green-Krótki [12]. The proof uses Theorem 1.5, but could easily be modified to avoid doing so.

Proof of Theorem 1.4: Let $P(G, M_1, M_2) \subseteq \mathbf{R}^E$ (or simply P) denote the polyhedron defined by the inequalities (1)–(8). Clearly, $\text{conv}(\mathcal{K}) \subseteq P$. To prove the opposite inclusion, it suffices to prove that P is integral. (Namely, an integral vector x satisfying (1), (4), (5), (6), and (7) must determine a set of r disjoint paths joining pairs of vertices in $T_1 \cup T_2$, together with disjoint circuits and edges in $G[R]$. Moreover, the vertices in T_i that are ends of paths must form a basis of M_i for $i = 1$ and 2. Because of inequalities (2), the paths must go from T_1 to T_2, and because of inequalities (3), the circuits must all be even. If there are no circuits, then x is a basic path-matching vector, and if there are some even circuits, then x is the average of two such vectors.)

We prove that P is integral by induction on the number of vertices of G; the result is obviously true when G has just one vertex. Let $x' \in \mathbf{R}^E$ be an extreme point of P. If x' does not satisfy with equality any of the inequalities (3) for which $|S| \geq 3$, then by Theorem 3.1, x' is integral. Otherwise, there exists $S \subseteq R$ such that $|S| \geq 3$, $|S|$ is odd, and $x'(\delta(S)) = 2$.

Denote by G_1 the graph $G \circ S$, that is, the graph obtained from G by deleting the edges in $\gamma(S)$ and shrinking the vertices in S to a single vertex which we call S. Let x^1 denote the restriction of x' to G_1. Let P_1 denote $P(G_1, M_1, M_2)$, and let $\mathcal{K}_1$ denote $\mathcal{K}(G_1, M_1, M_2)$. It is easily verified that

$x^1 \in P_1$. Then, by induction, $\mathrm{conv}(\mathcal{K}_1) = P_1$. Thus there exists a nonnegative vector $\lambda^1 \in \mathbf{R}^{\mathcal{K}_1}$ such that $\lambda^1(\mathcal{K}_1) = 1$ and

$$x^1 = \sum_{K \in \mathcal{K}_1} \lambda^1_K \psi^K.$$

Let $U = V \backslash S$. Let G_2 denote $G \circ U$, let $\mathcal{K}_2$ denote the set of perfect matchings of G_2, and let x^2 denote the restriction of x' to G_2. It is easily verified, using the Perfect Matching Polyhedron Theorem 1.5, since $x^2(\delta(v)) = 2$ for all vertices v of G_2, that that there exists a nonnegative vector $\lambda^2 \in \mathbf{R}^{\mathcal{K}_2}$ such that $\lambda^2(\mathcal{K}_2) = 1$ and

$$x^2 = \sum_{K \in \mathcal{K}_2} \lambda^2_K \psi^K.$$

In what follows, when we speak of a basic path-matching in G or in G_1, we mean with respect to the matroids M_1, M_2. We will show explicitly that x' can be expressed as a convex combination $\sum(\mu_K \psi^K : K \in \mathcal{K})$. Begin with all $\mu_K = 0$, and do the following until $\lambda^1 = 0$. Consider $L \in \mathcal{K}_1$ such that $\lambda^1_L > 0$. Either there exist two edges $e', e'' \in L$ that are incident with the vertex S of G_1, or there exists a matching edge e' of L that is incident to S. In the latter case, we take $e' = e''$. Now, $x^2_{e'} = x^1_{e'} > 0$, and so there exists $J' \in \mathcal{K}_2$ containing e' such that $\lambda^2_{J'} > 0$. Similarly, there exists $J'' \in \mathcal{K}_2$ containing e'' such that $\lambda^2_{J''} > 0$. (An example is shown in Figure 4, where L is drawn with thin lines, $J' \backslash \{e'\}$ is drawn with thick solid lines, and $J'' \backslash \{e''\}$ is drawn with thick dashed lines.) Let $K = L \cup J' \cup J''$. Note that $J' \cup J''$ may

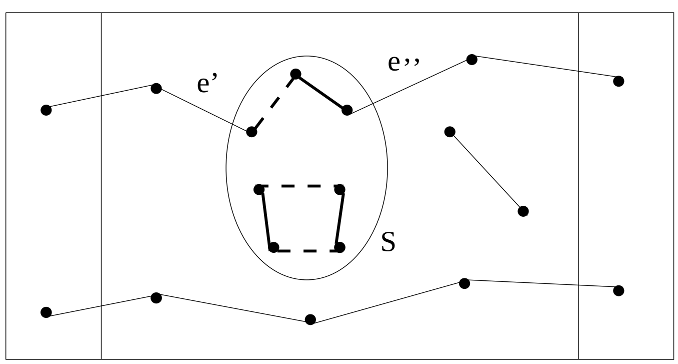

Figure 4: Combining solutions

contain circuits of even length, so K is not necessarily a basic path-matching in G; however, K is the union of two basic path-matchings K', K'' (possibly equal) of G. If $K' = K''$, then take ε to be the minimum of $\lambda^1_L, \lambda^2_{J'}$; increase $\mu_{K'}$ by ε and decrease λ^1_L and $\lambda^2_{J'}$ by ε. If $K' \neq K''$, then take ε to be the minimum of $\frac{1}{2}\lambda^1_L$, $\lambda^2_{J'}$, and $\lambda^2_{J''}$; increase $\mu_{K'}$ and $\mu_{K''}$ by ε, decrease λ^1_L by 2ε and decrease $\lambda^2_{J'}$ and $\lambda^2_{J''}$ by ε.

Thus we can obtain x' as a convex combination of path-matching vectors of basic path-matchings in G. However, x' is an extreme point of P, and so cannot be expressed as a convex combination of other elements of P. Therefore, x' is a basic path-matching vector, and hence is integral. ∎

Remark It can be deduced from the proof that any basic path-matching problem on a bipartite graph can be reduced to a matroid intersection problem.

As a consequence of Theorem 1.4, we get a second description of $\mathrm{conv}(\mathcal{K})$.

Corollary 3.2 *$conv(\mathcal{K}(G, M_1, M_2))$ is the set of solutions of:*

$$\begin{aligned}
x(\delta(v)) &= 2 && (v \in R) && (24)\\
x(\gamma(S)) &\leq |S \cap R| && (T_1 \subseteq S \subseteq T_1 \cup R) && (25)\\
x(\gamma(S)) &\leq |S| - 1 && (S \subseteq R, |S| \ \ odd) && (26)\\
x(\delta(A)) &\leq r_1(A) && (A \subseteq T_1) && (27)\\
x(\delta(A)) &\leq r_2(A) && (A \subseteq T_2) && (28)\\
x(\delta(T_1)) &= r && && (29)\\
x(\delta(T_2)) &= r && && (30)\\
x &\geq 0. && && (31)
\end{aligned}$$

Proof: It is clear that the above inequalities are valid for $\mathrm{conv}(\mathcal{K}(G, M_1, M_2))$. Now suppose that $x \in \mathbf{R}^E$ satisfies all of them. Given a subset S of $T_1 \cup R$ such that $T_1 \subseteq S$, we have

$$\begin{aligned}
x(\delta(S)) &= \sum_{v \in S} x(\delta(v)) - 2x(\gamma(S))\\
&= 2\,|S \cap R| + r - 2x(\gamma(S))\\
&\geq 2\,|S \cap R| + r - 2\,|S \cap R|\\
&= r,
\end{aligned}$$

so x satisfies (2). A similar argument shows that x satisfies (3). Trivially, x also satisfies (1) and (5)–(8). Therefore, by Theorem 1.4, $x \in \mathrm{conv}(\mathcal{K})$, as required. ∎

The independent path-matching polytope

We now prove Theorem 1.7 as a consequence of Corollary 3.2. For the special case of the Matching Polyhedron Theorem, the proof reduces to one due to Schrijver [18].

Proof of Theorem 1.7: It is clear that inequalities (9)–(15) are valid for conv($\mathcal{K}^*$). Now suppose that $x \in \mathbf{R}^E$ satisfies all of these inequalities. Create a copy $\tilde{v}$ of each $v \in V$, and for $S \subseteq V$, denote by $\tilde{S}$ the corresponding copy of S. Similarly, for a subset F of E, denote by $\tilde{F}$ the set $\{\tilde{v}\tilde{w} : vw \in F\}$. Now construct a graph $G' = (V', E')$ such that $V' = V \cup \tilde{V}$ and $E' = E \cup \tilde{E} \cup \{v\tilde{v} : v \in V\}$, and let $T_1' = T_1 \cup \tilde{T}_2$, $T_2' = T_2 \cup \tilde{T}_1$, and $R' = R \cup \tilde{R}$. Let $\tilde{M}_i$ denote a copy of M_i on the set $\tilde{T}_i$ for $i = 1$ and 2. Let M_1' be the direct sum of M_1 with $\tilde{M}_2$, and M_2' be the direct sum of M_2 with $\tilde{M}_1$. In what follows, we use $\mathcal{K}^*$ to denote $\mathcal{K}^*(G, M_1, M_2)$ and $\mathcal{K}$ to denote $\mathcal{K}(G', M_1', M_2')$.

Claim *If $z' \in conv(\mathcal{K})$ and z is the restriction of z' to E, then $z \in conv(\mathcal{K}^*)$.*
It suffices to prove the claim when z' is an extreme point. Thus, assume that $z' = \psi^L$ for some $L \in \mathcal{K}$. Let $K = L \cap E$, and let F be the set of matching edges of K that are not matching edges of L. Then, clearly, $z = \frac{1}{2}(\psi^K + \psi^{K \setminus F})$. Hence, $z \in$ conv($\mathcal{K}^*$), which proves the claim.

By the claim, we can prove the theorem by constructing $x' \in$ conv($\mathcal{K}$), such that x is the restriction of x' to E. By the Matroid Polyhedron Theorem [9], the vector $t = (x(\delta(v)) : v \in T_1)$ is a convex combination of incidence vectors of independent sets of M_1. Therefore, there exists, for each $v \in T_1$, a number $y_v \geq x(\delta(v))$ such that $(y_v : v \in T_1)$ is a convex combination of incidence vectors of bases of M_1. (To get y, we simply extend to a basis each of the independent sets in the expression for t, and use the same coefficients.) We similarly define y_v for $v \in T_2$, using M_2 instead of M_1. Now we define $x' \in \mathbf{R}^{E'}$ by:

- for $vw \in E$, $x'_{vw} = x_{vw}$, and $x'_{\tilde{v}\tilde{w}} = x_{vw}$;
- for $v \in R$, $x'_{v\tilde{v}} = 2 - x(\delta_G(v))$;
- for $v \in T_1 \cup T_2$, $x_{v\tilde{v}} = y_v - x(\delta_G(v))$.

By Corollary 3.2, conv($\mathcal{K}$) is defined by (24)–(31) (applied to G'). The proof will be finished if we can show that x' satisfies all of these inequalities. We show that it satisfies inequalities (25) and (26); it obviously satisfies the others.

Let $S' \subseteq T_1' \cup R'$ such that $T_1' \subseteq S'$. Define $S, U \subseteq V$ such that $S' = S \cup \tilde{U}$. Thus $T_1 \subseteq S \subseteq T_1 \cup R$ and $T_2 \subseteq U \subseteq T_2 \cup R$. Then

$$\begin{aligned}
& x'(\gamma_{G'}(S')) \\
= \;& x(\gamma(S)) + x(\gamma(U)) + 2\,|S \cap U| - \sum_{v \in S \cap U} x(\delta(v)) \\
= \;& x(\gamma(S)) + x(\gamma(U)) - 2x(\gamma(S \cap U)) - x(\delta(S \cap U)) + 2\,|S \cap U| && (32) \\
\leq \;& x(\gamma(S \backslash U)) + x(\gamma(U \backslash S)) + 2\,|S \cap U| && (33) \\
\leq \;& |(S \backslash U) \cap R| + |(U \backslash S) \cap R| + 2\,|S \cap U| && (34) \\
= \;& |S \cap R| + |U \cap R| \\
= \;& |S' \cap R'|,
\end{aligned}$$

where we get (33) from (32) by nonnegativity, and we get (34) from (33) by inequalities (10) and (11). Thus x' satisfies the inequalities (25).

Now, let $S' \subseteq R'$ such that $|S'|$ is odd. Define $S, U \subseteq V$ by $S' = S \cup \tilde{U}$. Thus $S, U \subseteq R$, and exactly one of $|S|, |U|$ is odd. Therefore exactly one of $|S \backslash U|, |U \backslash S|$ is odd. Then, by the inequalities (12) and (9),

$$x(\gamma(S \backslash U)) + x(\gamma(U \backslash S)) \leq |S \backslash U| + |U \backslash S| - 1. \qquad (35)$$

Now,

$$\begin{aligned}
& x'(\gamma_{G'}(S')) \\
= \;& x(\gamma(S)) + x(\gamma(U)) + 2\,|S \cap U| - \sum_{v \in S \cap T} x(\delta(v)) \\
= \;& x(\gamma(S)) + x(\gamma(U)) - 2x(\gamma(S \cap U)) - x(\delta(S \cap U)) + 2\,|S \cap U| && (36) \\
\leq \;& x(\gamma(S \backslash U)) + x(\gamma(U \backslash S)) + 2\,|S \cap U| && (37) \\
\leq \;& |S \backslash U| + |U \backslash S| - 1 + 2\,|S \cap U| && (38) \\
= \;& |S| + |U| - 1 \\
= \;& |S'| - 1,
\end{aligned}$$

where we get (37) from (36) by nonnegativity, and we get (38) from (37) by inequality (35). Therefore, x' satisfies the inequalities (26). ∎

4 Total dual integrality

By the Independent Path-Matching Polyhedron Theorem 1.7, the polyhedron defined by inequalities (9)–(15) has integral extreme points. Therefore, for any objective function $c \in \mathbf{R}^E$, the linear programming problem (P) below has an integral optimal solution

$$(P) \qquad \begin{cases} \max c^T x \\ \text{subject to} \\ (9) - (15). \end{cases}$$

Given a partition T_1, T_2, R of the vertices of G, we define

$$\begin{aligned} \Omega_1 &= \{S : T_1 \subseteq S \subseteq T_1 \cup R\}, \\ \Omega_2 &= \{S : T_2 \subseteq S \subseteq T_2 \cup R\}, \\ \Omega_{12} &= \{S \subseteq R : |S| \text{ is odd}\}, \\ \Omega_1' &= \{S : S \subseteq T_1\}, \\ \Omega_2' &= \{S : S \subseteq T_2\}. \end{aligned}$$

Let $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_{12}$, and $\Omega' = \Omega_1' \cup \Omega_2'$. For a set $S \in \Omega$, define $f(S) \in \{0,1\}$ such that $f(S) = 1$ exactly when $S \in \Omega_{12}$. For a set $S \in \Omega'$, define $g(S)$ to be $r_1(S)$ if $S \subseteq T_1$, and to be $r_2(S)$ otherwise. For variables $y \in \mathbf{R}^V$, $z \in \mathbf{R}^\Omega$, and $w \in \mathbf{R}^{\Omega'}$, the dual (D) of (P) is given by

$$(D) \qquad \begin{cases} \min \sum\limits_{v \in R} 2y_v + \sum\limits_{S \in \Omega} (|S \cap R| - f(S)) z_S + \sum\limits_{A \in \Omega'} g(A) w_A \\ \text{subject to} \\ y_u + y_v + \sum\limits_{\substack{S \in \Omega \\ u,v \in S}} z_S + \sum\limits_{\substack{A \in \Omega' \\ uv \in \delta(A)}} w_A \geq c_{uv} \quad (uv \in E) \\ y \geq 0, z \geq 0, w \geq 0. \end{cases}$$

We will prove that, whenever c is integral, there exists an integral optimal solution to (D). In other words, we will show that the system of inequalities (9)–(15) is *totally dual integral*; see Schrijver [18]. Cunningham and Marsh [5] proved that the system of inequalities in Edmonds' characterization of the matching polyhedron is totally dual integral, and Edmonds [9] proved that the system of inequalities in his description of the matroid intersection polyhedron is totally dual integral. Our theorem generalizes these theorems. Our proof uses ideas from Schrijver's proof [17] of the matching result and from Edmonds' proof of the matroid intersection result.

Let $\mathcal{S}$ be a collection of subsets. We call $\mathcal{S}$ a *laminar family* if, for each $S, T \in \mathcal{S}$, either $S \subseteq T$, $T \subseteq S$ or $S \cap T = \emptyset$. We call $\mathcal{S}$ a *chain* if, for each $S, T \in \mathcal{S}$, either $S \subseteq T$ or $T \subseteq S$.

Theorem 4.1 *For all integral c, there exists an integral optimal solution (y, z, w) to (D) such that supp(z) is laminar, and supp$(w) \cap \Omega'_1$ and supp$(w) \cap \Omega'_2$ are chains.*

Proof: It suffices to prove the theorem for nonnegative c. Suppose that the result fails, and let G, M_1, M_2, c form a counterexample with $|V| + |E| + c(E)$ as small as possible. For each edge e of G, $c_e \geq 1$, since otherwise we can delete e.

Claim 1 *For every optimal solution (y, z, w) to (D), $y = 0$.*

Let $\mathcal{F}$ denote the set of independent path-matchings that attain the optimum of (P). Suppose that there exists $v \in R$ such that $\psi^K(\delta(v)) = 2$ for each K in $\mathcal{F}$. We decrease the weight of each edge incident with v by 1 to get c'. Then, by our choice of c, there exists an integral optimal solution (y', z', w') to (D), with respect to c', with the required properties. By increasing y'_v by 1, we obtain an integral optimal solution to (D), with respect to c, having these properties, and this is a contradiction. So, for all $v \in R$, there exists $K \in \mathcal{F}$ such that $\psi^K(\delta(v)) < 2$. Thus, by complementary slackness, $y_v = 0$, proving Claim 1.

Claim 2 *There exists an optimal solution to (D) such that supp(z) is laminar.*

Let (y, z, w) be an optimal solution to (D) that minimizes $\sum(z_S\,|S|\,|V \backslash S| : S \in \Omega)$. Suppose that supp$(z)$ is not laminar, and let $S, U \in$ supp(z) such that $|S \backslash U|, |U \backslash S|, |S \cap U| > 0$. By a simple case analysis, we find that either $S \backslash U$ and $U \backslash S$ are both in Ω, or $S \cap U$ and $S \cup U$ are both in Ω. We consider these cases separately.

Case 1: $S \backslash U$ and $U \backslash S$ are both in Ω. Let ε be the minimum of z_S and z_U. We construct $z' \in \mathbf{R}^\Omega$ from z by decreasing z_S and z_U by ε, and increasing $z_{S \backslash U}$ and $z_{U \backslash S}$ by ε. Now, construct $y' \in \mathbf{R}^V$, by increasing y_v by ε for all $v \in S \cap U$. One easily checks that (y', z', w) is an optimal solution to (D). However, $y' \neq 0$, which contradicts Claim 1.

Case 2: $S \cap U$ and $S \cup U$ are both in Ω. Let ε be the minimum of z_S and z_U. We construct $z' \in \mathbf{R}^\Omega$ from z by decreasing z_S and z_U by ε, and increasing $z_{S\cap U}$ and $z_{S\cup U}$ by ε. One easily checks that (y, z', w) is an optimal solution to (D), and that the choice of (y, z, w) is contradicted. This proves Claim 2.

Now choose (y, z, w) as in Claim 2, so that $\sum(w_A|A||V\backslash A| : A \in \Omega')$ is minimized. Suppose that there exist $A, B \in \text{supp}(w)$ such that $A\backslash B$ and $B\backslash A$ are both nonempty, with $A, B \in \Omega_1'$ or $A, B \in \Omega_2'$. Let ε be the minimum of w_A and w_B. We construct $w' \in \mathbf{R}^{\Omega'}$ from w by decreasing w_A and w_B by ε, and increasing $w_{A\cap B}$ and $w_{A\cup B}$ by ε. One easily checks, using the submodularity of the rank functions, that is, $g(A) + g(B) \geq g(A \cup B) + g(A \cap B)$, that (y, z, w') is an optimal solution to (D), and that the choice of (y, z, w) is contradicted. Therefore, there exists an optimal solution (y, z, w) of (D) such that $y = 0$, $\text{supp}(z)$ is laminar, and $\text{supp}(w) \cap \Omega_1'$ and $\text{supp}(w) \cap \Omega_2'$ are chains.

We know that (D) has an optimal solution that is an optimal solution of the linear program (D') obtained from (D) by deleting all the variables other than those for which z, w take positive values. To show that (D) has an integral optimal solution, it suffices to show that (D') does. Let F be the constraint matrix of (D'). Now the families

$$\mathcal{C} = \{\gamma(S) : S \in \mathit{supp}(z) \cap (\Omega_{12} \cup \Omega_1)\} \cup \{\delta(A) : A \in \text{supp}(w) \cap \Omega_2'\}$$

and

$$\mathcal{D} = \{\gamma(S) : S \in \mathit{supp}(z) \cap \Omega_2)\} \cup \{\delta(A) : A \in \text{supp}(w) \cap \Omega_1'\}$$

are laminar families of subsets of E, and the columns of F are the incidence vectors of the elements of $\mathcal{C} \cup \mathcal{D}$. By a result of [9], F is totally unimodular; since c is integral, (D'), and therefore (D), has an integral optimal solution. This solution has all the required properties, so this is a contradiction. ∎

Theorem 4.1 has the following consequences. The first one is just a specialization to ordinary path-matching.

Corollary 4.2 *The system of inequalities (16)–(21) is totally dual integral.*

Proof: This is almost immediate from the theorem. The dual variables w_A corresponding to the constraints (13) and (14) must be replaced by dual variables w_v corresponding to the constraints (16). But this is quite easy to do. We simply set $w_v = \sum(w_A : v \in A)$ for each $v \in T_1 \cup T_2$. It is easy to check that the resulting dual solution has the desired properties. ▮

The second consequence gives a totally dual integral description for the basic path-matching polyhedron. This is a bit subtle. First, note that the system (1)–(8) of Theorem 1.4, is not totally dual integral. This is already demonstrated in the case of perfect matchings of a graph. We do not know whether the system of inequalities (24)–(31) of Corollary 4.3 is totally dual integral, but we believe that it is. We *can* show that a closely related system describing the basic path-matching polyhedron is totally dual integral. The proof uses a simple trick (adding a large even integer to each objective coefficient in (P) and applying Theorem 4.1).

Corollary 4.3 *The system of inequalities (11), (24)–(31) is totally dual integral.* ▮

We remark that it is quite easy to show from Corollary 4.3 that, as in the matching case, the system (1)–(8) is "totally dual half-integral", that is, that an optimal dual solution can be required to be half-integral if c is integral.

Finally, we use Theorem 4.1 to prove the min-max formula of Theorem 1.9.

Proof of Theorem 1.9: By an argument similar to that used to prove the necessity of the condition in Theorem 1.1, we can show that the maximum is at most the minimum. We omit the details. It remains to prove that there is a stable pair making the second expression at most the maximum of $\psi^K(E)$ over independent path-matchings K. This maximum is the optimal value of the linear programming problem (P) when $c = (1, 1, \ldots, 1)$. Therefore, by the duality theorem, it is the optimal value of problem (D) for this c. We apply Theorem 4.1 to obtain an optimal solution (y, z, w) with the stated properties. It is easy to see that this solution is $\{0, 1\}$-valued. We can arrange that it has the following additional properties:

- $y_v = 0$ for all $v \in T_1 \cup T_2$. (Proof: If $y_v = 1$, we can instead put $w_{\{v\}} = 1$.)
- If $S \neq T$ and $z_S = z_T = 1$, then $S \cap T = \emptyset$. (Proof: Since it is easy to see that neither $S \subseteq T$ nor $T \subseteq S$ is possible, this follows from the laminar property.)

- If $z_S = 1$, then $y_v = 0$ for all $v \in S$. (Proof: Suppose not. If $S \in \Omega_1 \cup \Omega_2$, then put $z_{S \setminus \{v\}} = 1$ instead of z_S. If $S \in \Omega_{12}$, take $u \in S \setminus \{v\}$ and put $y_u = 1$ and $z_{S \setminus \{v,u\}} = 1$ and $z_S = 0$.)

- There exist sets A, B such that $\mathrm{supp}(w) \cap \Omega_1 = \{A\}$ and $\mathrm{supp}(w) \cap \Omega_2 = \{B\}$. (Proof: If $w_U = w_Q = 1$ for $U, Q \in \Omega_1$, then we can instead put $w_{U \cup Q} = 1$. If no such set exists, we can take $A = T_1$. The same argument works for Ω_2.)

- There exist sets A', B' such that $\mathrm{supp}(z) \cap \Omega'_1 = \{A'\}$ and $\mathrm{supp}(z) \cap \Omega'_2 = \{B'\}$. (Proof: We can use the same argument as for the previous property, except that if there is no such set we take $A' = \emptyset$.)

Let $C = \{v : y_v = 1\}$, and let $S_1, \ldots, S_k$ be the sets $S \in \Omega_{12}$, such that $z_S = 1$. Then the maximum of $\psi^K(E)$ over independent path-matchings is

$$\alpha = r_1(A') + |A \cap R| + r_2(B') + |B \cap R| + 2|C| + \sum_{i=1}^{k}(|S_i| - 1).$$

We define D_1 to be $((T_1 \setminus A') \cup A \cup (\cup(S_i : 1 \leq i \leq k))$, and similarly for D_2. Then no edge of G joins a vertex in $D_1 \setminus D_2$ to a vertex in D_2 or a vertex in $D_2 \setminus D_1$ to a vertex in D_1, since such an edge would violate the corresponding feasibility constraint in problem (D). Moreover,

$$r_1(T_1 \setminus D_1) + r_2(T_2 \setminus D_2) + |R \setminus (D_1 \cup D_2)| + |R| - \mathrm{odd}(G[D_1 \cap D_2]) \leq \alpha,$$

as required. ∎

Acknowledgments. We are grateful to L. Lovász and A. Sebő for helpful conversations. This work was supported in part by a grant from NSERC of Canada and by a Canadian Commonwealth Scholarship.

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