Delta-matroids, Jump Systems and Bisubmodular Polyhedra *

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Abstract

We relate an axiomatic generalization of matroids, called a jump system, to polyhedra arising from bisubmodular functions. Unlike the case for usual submodularity, the points of interest are not all the integral points in the relevant polyhedron, but form a subset of them. However, we do show that the convex hull of the set of points of a jump system is a bisubmodular polyhedron, and that the integral points of an integral bisubmodular polyhedron determine a (special) jump system. We also prove addition and composition theorems for jump systems, which have several applications for delta-matroids and matroids.

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1 Introduction

Matroids are important as a unifying structure in pure combinatorics, as well as a useful model in the theory of algorithms and in combinatorial optimization. (See Bixby and Cunningham [1] for a survey of the latter aspects.) Delta-matroids constitute an interesting generalization, and have been introduced only recently. Many of the nice properties associated with matroids (greedy algorithm, polyhedral description, interesting examples) extend to delta-matroids. In the present paper we begin by reviewing some of this work. Then we prove a new composition theorem for delta-matroids. It has several applications, including constructions for matroids. An important theme is to identify in which of the applications the composition is algorithmically constructible.

The polyhedral aspects of matroids, developed more than twenty years ago by Edmonds [11], led him to a different generalization, integral polymatroids. In a certain sense there are two views of an integral polymatroid; first, it is a polyhedron P, and second, it is a set $\mathcal{F}$ of integral points. There is a simple relation between the two views—P is the convex hull of $\mathcal{F}$, and $\mathcal{F}$ is the set of integral points in P. In this paper we introduce jump systems, a common generalization of delta-matroids and (the second view of) integral polymatroids. A jump system is defined by a set $\mathcal{F}$ of integral points, but it is not generally true that it is the set of integral points in its convex hull. We present some examples of jump systems and prove an addition theorem, which implies the composition theorem for delta-matroids.

Although jump systems cannot be defined via polyhedra, there is an important subclass that can. These arise from (integral) bisubmodular polyhedra, introduced by Dunstan and Welsh [10] in 1973 in a paper that seems to have been fully appreciated only recently. We prove that the integral points in such a polyhedron determine a jump system. Moreover, there is a partial converse—if $\mathcal{F}$ is a set of integral points determining a jump system, then the convex hull of $\mathcal{F}$ is an integral bisubmodular polyhedron. So it is true that a polyhedron with integral vertices is bisubmodular if and only if the integral points in it form a jump system.

Throughout this paper S, with or without subscripts, is a finite set. We use $\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{Z}$, and $\mathbf{Z}_+$ to denote the sets of real numbers, non-negative real numbers, integers, and non-negative integers, respectively. For $x \in \mathbf{R}^S$ and $A \subseteq S$, we often use $x(A)$ as an abbreviation for $\sum(x_j : j \in A)$. For $c, x \in \mathbf{R}^S$ we write cx to mean $\sum(c_j x_j : j \in S)$. For $x \in \mathbf{R}^S$ and $A \subseteq S$, we use $x_{|A}$ to denote the restriction of x to A, that is, the vector $x' \in \mathbf{R}^A$ such that $x'_j = x_j$ for all $j \in A$. Finally, we use the symbol A also to denote the incidence vector of A, that is, the vector $x \in \mathbf{R}^S$ such that $x_j = 1$ if $j \in A$ and $x_j = 0$ if $j \notin A$.

2 Delta-matroids

Let $\mathcal{F}$ be a family of subsets of a finite set S. Then $(S, \mathcal{F})$ is a *delta-matroid* if the following *symmetric exchange axiom* is satisfied:

(SEA) *If $F_1, F_2 \in \mathcal{F}$ and $j \in F_1 \Delta F_2$ then there is $k \in F_1 \Delta F_2$ such that $F_1 \Delta \{j, k\} \in \mathcal{F}$.*

(Here and elsewhere Δ denotes symmetric difference.) These structures have been introduced by Bouchet [2]. Essentially equivalent structures were independently considered by Dress and Havel [8] and by Chandrasekaran and Kabadi [5]. A main motivation for their study is that, if $\mathcal{F}$ is the family of bases of a matroid on S, then $(S, \mathcal{F})$ is a delta-matroid. In fact, matroids are precisely the delta-matroids for which all members of $\mathcal{F}$ have the same cardinality. (We remark that throughout the paper we use "matroid" to mean a matroid defined by its family of bases.) In addition to these examples, we mention a few others.

Matching delta-matroids. Let $G = (V, E)$ be a graph, let $S = V$, and let $F \in \mathcal{F}$ if and only if there is a matching of G covering precisely the elements of F. Then $(S, \mathcal{F})$ is a delta-matroid. This can be proved using augmenting path arguments.

Twisting. Let $(S, \mathcal{F})$ be a delta-matroid, and let $N \subseteq S$. Let $\mathcal{F} \Delta N$ denote $\{F \Delta N : F \in \mathcal{F}\}$. Then $(S, \mathcal{F} \Delta N)$ is a delta-matroid. For example, we can get delta-matroids by applying twisting to a matroid. In one case we get again a matroid; namely, when $N = S$, we get the dual matroid.

Linear delta-matroids. Let $M = (m_{ij} : i \in S, j \in S)$ be a skew-symmetric matrix over a field. Define $\mathcal{F}$ by $F \in \mathcal{F}$ if and only if the principal submatrix $(m_{ij} : i \in F, j \in F)$ is non-singular. Then $(S, \mathcal{F})$ is a delta-matroid. The proof of this result is not trivial; see Bouchet [3], where it is also generalized. (For example, a symmetric matrix can also be used.)

Another basic fact is that, if $(S, \mathcal{F})$ is a delta-matroid and $\mathcal{F}'$ is the family of maximal members of $\mathcal{F}$, then $(S, \mathcal{F}')$ is a matroid. This and twisting can be used to justify a greedy algorithm for optimizing any linear function over $\mathcal{F}$. Namely, $|c|(F \Delta N) = c(F) - c(N)$, where $N = \{j : c_j < 0\}$. Therefore, we can apply the matroid greedy algorithm to the maximal members of $\mathcal{F} \Delta N$ with weight function $|c|$. Translating that algorithm into one operating directly on $(S, \mathcal{F})$ and c, we get the following procedure. It appears in [2] and [5], but a similar kind of greedy algorithm can be found in Dunstan and Welsh [10].

Greedy Algorithm for Delta-Matroids

Input: Delta-matroid $(S, \mathcal{F})$ and weight vector $c \in \mathbf{R}^S$.
Objective: To find $F \in \mathcal{F}$ such that $c(F)$ is maximum.

begin
order S as $\{e_1, e_2, \dots, e_n\}$ so that $|c_{e_1}| \geq |c_{e_2}| \geq \dots \geq |c_{e_n}|$;
for $i = 1$ **to** $n+1$ let $T_i = \{e_i, \dots, e_n\}$;
$J \leftarrow \emptyset$;
for $i = 1$ **to** n
if $c_{e_i} \geq 0$ and there exists $F \in \mathcal{F}$ with $J \cup \{e_i\} \subseteq F \subseteq J \cup T_i$
then $J \leftarrow J \cup \{e_i\}$;
if $c_{e_i} < 0$ and there does not exist $F \in \mathcal{F}$ with $J \subseteq F \subseteq J \cup T_{i+1}$
then $J \leftarrow J \cup \{e_i\}$;
end.

Notice that to implement this algorithm we need to be able to answer the question, given disjoint subsets A, B of S,

(2.1) *Does there exist $F \in \mathcal{F}$ with $A \subseteq F \subseteq S \setminus B$?*

A more general question is to ask for the value $f(A, B)$, defined to be $\max_{F \in \mathcal{F}}(|F \cap A| - |F \cap B|)$, since the answer to (2.1) is "yes" exactly when $f(A, B) = |A|$. However, the two questions are algorithmically equivalent because $f(A, B)$ can be computed by the greedy algorithm with $c_j = 1$ for $j \in A$, -1 for $j \in B$, and 0 otherwise. We consider the existence of an efficient subroutine to evaluate the function f (or answer the question (2.1)) to be the measure of algorithmic tractability of the delta-matroid. (If $(S, \mathcal{F})$ is a matroid with rank function r, a simple argument shows that $f(A,B) = r(A) + r(S \setminus B) - r(S)$. Since $r(A) = f(A, \emptyset)$, it follows that this oracle is available for a matroid exactly when the usual one is available.)

Composition of delta-matroids

Our main result on delta-matroids is a composition theorem. We define the *composition* of delta-matroids $(S_0, \mathcal{F}_0)$, $(S_1, \mathcal{F}_1)$ to be $(S, \mathcal{F})$ where $S = S_0 \Delta S_1$ and $\mathcal{F} = \{F_0 \Delta F_1 : F_0 \in \mathcal{F}_0,\ F_1 \in \mathcal{F}_1,\ F_0 \cap S_1 = F_1 \cap S_0\}$. That is, each feasible set is a symmetric difference of two feasible sets, one from each of the initial delta-matroids, that agree on $S_0 \cap S_1$. The proof that this construction gives a delta-matroid is our original one, which we include because of its algorithmic flavour. However, the reader is warned that the next section contains an easier proof of a more general result, so he may want to skip this prove on a first reading.

(2.2) Theorem. *The composition of delta-matroids is a delta-matroid.*

Proof. We consider $F, G \in \mathcal{F}$, $j \in F \Delta G$, and we search for $k \in F \Delta G$ such that $F \Delta \{j,k\} \in \mathcal{F}$. There exist $F_0, G_0 \in \mathcal{F}_0$, $F_1, G_1 \in \mathcal{F}_1$ such that $F = F_0 \Delta F_1$ and $G = G_0 \Delta G_1$. We also consider $S' = S_0 \cap S_1$, $F' = F_0 \cap S' = F_1 \cap S'$, $G' = G_0 \cap S' = G_1 \cap S'$. For any integer i we let F_i, G_i, $\mathcal{F}_i$ be respectively equal to F_0, G_0, $\mathcal{F}_0$ if i is even, F_1, G_1, $\mathcal{F}_1$ if i is odd.

The element j belongs to $F_0 \Delta G_0 \Delta F_1 \Delta G_1$. By symmetry we may assume that $j \in F_1 \Delta G_1$. Applying (SEA) to $F_1, G_1 \in \mathcal{F}_1$ and $j \in F_1 \Delta G_1$ we can find $z \in F_1 \Delta G_1$ such that $F_1 \Delta \{j,z\} \in \mathcal{F}_1$. If $z \notin S'$ we have $F_1 \Delta \{j,z\} \Delta F_0 = F \Delta \{j,z\} \subseteq S$, and the property is proved with $k = z$. From now on we assume that $z \in S'$, so that $z \in F' \Delta G'$.

We consider a sequence $U = (j_1, j_2, \ldots, j_r)$ of pairwise distinct elements belonging to $F' \Delta G'$ with $j_1 = z$. For $0 \leq i \leq r$ we let $U_i = \{j, j_1, j_2, \ldots, j_i\}$ if i is odd, $U_i = \{j_1, j_2, \ldots, j_i\}$ if i is even, and we suppose that $\Phi_i = F_i \Delta U_i \in \mathcal{F}_i$. The conditions are satisfied if $U = (j_1)$ because $\Phi_0 = F_0$ and $\Phi_1 = F_1 \Delta \{j,z\}$. From now on we suppose that the length of U is maximal.

We have $(\Phi_{r-1} \Delta G_{r-1}) \cap S' = (F_{r-1} \Delta U_{r-1} \Delta G_{r-1}) \cap S' = F' \Delta G' \Delta U_{r-1}$. The element j_r belongs to $F' \Delta G'$ and it does not belong to U_{r-1}. Therefore $j_r \in (\Phi_{r-1} \Delta G_{r-1}) \cap S' \subseteq \Phi_{r-1} \Delta G_{r-1}$. We apply (SEA) to $\Phi_{r-1}, G_{r-1} \in \mathcal{F}_{r-1}$ and $j_r \in \Phi_{r-1} \Delta G_{r-1}$. This yields an element $j_{r+1} \in \Phi_{r-1}\Delta G_{r-1}$ such that $\Phi_{r-1}\Delta\{j_r, j_{r+1}\} \in \mathcal{F}_{r-1}$. We let $U_{r+1} = U_{r-1}\Delta\{j_r, j_{r+1}\}$ and $\Phi_{r+1} = \Phi_{r-1}\Delta\{j_r, j_{r+1}\}$.

We claim that either $j_{r+1} \notin S'$ or $j_r = j_{r+1}$. If this is not true, we have $j_{r+1} \in (\Phi_{r-1}\Delta G_{r-1}) \cap S' = F'\Delta G'\Delta U_{r-1}$. Since $U_{r-1} \subseteq F'\Delta G'$, this implies that j_{r+1} is distinct from j_1, j_2, …, j_r. Therefore $(j_1, j_2, \ldots, j_{r+1})$ satisfies the same properties as U, which contradicts the maximality of U.

If either $j_{r+1} \notin S'$ or $j_{r+1} = j_r$, we have $\Phi_{r+1} \cap S' = \Phi_r \cap S' = U_r\Delta F'$. Since $\Phi_{r+1} \in \mathcal{F}_{r+1}$ and $\Phi_r \in \mathcal{F}_r$, we have $\Phi_r\Delta\Phi_{r+1} \in \mathcal{F}$. If $j_{r+1} \notin S'$ we verify that $\Phi_r\Delta\Phi_{r+1} = F\Delta\{j, j_{r+1}\}$ and $j_{r+1} \in F\Delta G$, which proves the property with $k = j_{r+1}$. If $j_{r+1} = j_r$ we have $\Phi_r\Delta\Phi_{r+1} = F\Delta\{j\}$, which proves the property with $k = j$. □

Given a set l of disjoint pairs of S and a subset $F \subseteq S$ we abuse the notation $F\Delta l$ to represent the symmetric difference of F with the union of the pairs that belong to l. Let $(S, \mathcal{F})$ be a delta-matroid. For $F, F' \in \mathcal{F}$, a *linking* L of (F, F') is a partition of $F\Delta F'$ into pairs such that $F\Delta l \in \mathcal{F}$ for all $l \subseteq L$. We say that $(S, \mathcal{F})$ is *linkable* if there exists a linking of (F, F') for all $F, F' \in \mathcal{F}$. This generalizes the notion of strong base orderability (see Welsh [25]) for matroids.

(2.3) Theorem. *The composition of linkable delta-matroids is a linkable delta-matroid.*

Proof. The notation is the same as in the proof of Theorem (2.2). For $i = 0, 1$, let L_i be a linking of (F_i, G_i). Let H be the graph defined over the vertex-set $S_0 \cup S_1$ and the edge-set $L_0 \cup L_1$. Each vertex of H has degree 0, 1, or 2, and no vertex in $S_0\Delta S_1$ has degree 2. Hence the components of H are paths and circuits, and each path ends in $S_0\Delta S_1$. Let $\mathcal{P}$ be the set of the components of H that are paths. Let $L = \{\{s, t\} : s \text{ and } t \text{ are the ends of a path in } \mathcal{P}\}$. We prove that L is a linking of $F\Delta G$. Let $l = \{(s^1, t^1), (s^2, t^2), \ldots, (s^k, t^k)\} \subseteq L$. Let P^j be the path in $\mathcal{P}$ that ends at s^j and t^j, for $1 \leq j \leq k$. Let $l_i^j = L_i \cap P^j$, for $i = 0, 1$. Since L_i is a linking of (F_i, G_i), we have

(i) $F_i' = F_i\Delta(l_i^1\Delta l_i^2 \ldots l_i^k) \in \mathcal{F}_i$.

Notice that $X\Delta l_0^j\Delta l_1^j = X\Delta\{s^j, t^j\}$ holds for all $X \subseteq S$ and $1 \leq j \leq k$. Hence it follows from (i) that $F\Delta l = F_0'\Delta F_1'$, and so $F\Delta l \in \mathcal{F}$. □

Remark. Matching delta-matroids are examples of linkable delta-matroids. But for matching delta-matroids an even stronger property holds. For $F, F' \in \mathcal{F}$, there is a partition of $F \Delta F'$ that is a linking of both (F, F') and (F', F).

Composition of Matroids

If we apply the composition to two matroids, it is clear that the composed delta-matroid is not necessarily a matroid. However, composition can be combined with twisting to provide a matroid construction.

(2.4) Theorem. *If $(S_0, \mathcal{F}_0)$, $(S_1, \mathcal{F}_1)$ are matroids, then the composition $(S, \mathcal{F})$ of $(S_0, \mathcal{F}_0)$ with $(S_1, \mathcal{F}_1 \Delta (S_0 \cap S_1))$, is a matroid (provided $\mathcal{F}$ is non-empty).*

Proof. $(S, \mathcal{F})$ is a delta-matroid by Theorem (2.2), so we need only show that the members of $\mathcal{F}$ all have the same cardinality. But

$$\mathcal{F} = \{(F_0 \cup F_1) \setminus (S_0 \cap S_1) : F_0 \in \mathcal{F}_0,\ F_1 \in \mathcal{F}_1,\ F_0 \cap F_1 = \emptyset,\ F_0 \cup F_1 \supseteq S_0 \cap S_1\}.$$

Thus $F \in \mathcal{F}$ implies $|F| = |F_0| + |F_1| - |S_0 \cap S_1|$, and we are done. $\square$

In fact, this matroid composition can be obtained from standard constructions: $(S, \mathcal{F}) = ((S, \mathcal{F}_0) + (S, \mathcal{F}_1))/(S_0 \cap S_1)$, where $+$ denotes matroid union [25] and $/$ denotes contraction. This composition was investigated in [6] and [23]. It is easy to derive a formula for its rank function r in terms of the rank functions r_0, r_1 of $(S, \mathcal{F}_0)$, $(S, \mathcal{F}_1)$, namely

$$r(A) = \min_{X \subseteq S_0 \cap S_1} (r_0(X \cup (A \cap S_0)) + r_1(X \cup (A \cap S_1)) - |X|).$$

The research in [6, 23] concentrated on cases where $|S| > |S_0|, |S_1|$ and treated the resulting decomposition, which has some nice properties based on connectivity. But the composition also yields constructions for smaller matroids, as follows.

(2.5) Corollary. *Let $M_0 = (S_0, \mathcal{B}_0)$ and $M_1 = (S_1, \mathcal{B}_1)$ be matroids with $S_1 \subseteq S_0$. Then $\{B \setminus S_1 : B \in \mathcal{B}_0,\ B \cap S_1 \in \mathcal{B}_1\}$, if non-empty, is the family of bases of a matroid on $S = S_0 \setminus S_1$. Its rank function is given by*

$$r(A) = \min_{X \subseteq S_1} (r_0(A \cup X) + r_1(X) - |X|).$$

$\square$

(2.6) Corollary. *Let $M_0 = (S_0, \mathcal{B}_0)$ be a matroid, let $S \subseteq S_0$, let $S_1 = S_0 \setminus S$, and let k be an integer. Then $\{B \cap S : B \in \mathcal{B}_0,\ |B \cap S_1| = k\}$, if non-empty, is the basis family of a matroid on S. Its rank function is given by*

$$r(A) = \min(r_0(A),\ r_0(A \cup S_1) - |S_1| + k).$$

Proof. We apply (2.5), taking M_1 to be the uniform matroid of rank k on S_1. This matroid has rank function r_1 defined by $r_1(X) = \min(|X|,\ k)$. In the expression for $r(A)$, we see that if $|X| \geq k$, then we may as well take $X = S_1$, and if $|X| < k$, we may as well take $X = \emptyset$. This leads to the required expression for the rank function. $\square$

We observe that the last construction contains as special cases both contraction and deletion.

Efficient realization of composed delta-matroids

Another application of Theorem (2.2) is the following result of Bouchet [4]. We use it and its further corollary to make an important point about the availability of the oracle for a composition of delta-matroids.

(2.7) Corollary. *Let G be a bipartite graph with bipartition $\{S, S'\}$, let $(S, \mathcal{F})$ be a delta-matroid, and let $\mathcal{F}' = \{F' \subseteq S' : F'$ is matched in G to a member of $\mathcal{F}\}$. Then $(S', \mathcal{F}')$ is a delta-matroid.*

Proof. $(S', \mathcal{F}')$ is the composition of $(S, \mathcal{F})$ with the matching delta-matroid of G. $\square$

In the special case of (2.7) in which $(S, \mathcal{F})$ is a matroid, we get that $(S', \mathcal{F}')$ is also a matroid; this is a classical result (see Welsh [25]). A further specialization gives a "partition" construction for delta-matroids. This is also from [4].

(2.8) Corollary. *Let $(S, \mathcal{F}_0)$, $(S, \mathcal{F}_1)$ be delta-matroids, and let $\mathcal{F} = \{F_0 \cup F_1 : F_0 \in \mathcal{F}_0,\ F_1 \in \mathcal{F}_1,\ F_0 \cap F_1 = \emptyset\}$. Then $(S, \mathcal{F})$ is a delta-matroid.* $\square$

We refer to this construction as the "union" of delta-matroids. Corollary (2.8) can be used to show that the composition theorem (2.2) is not necessarily algorithmically realizable, in the sense that an oracle for $(S, \mathcal{F})$ may not be available from oracles for $(S_0, \mathcal{F}_0)$, $(S_1, \mathcal{F}_1)$. In the applications (2.5), (2.6), oracles can be constructed efficiently, essentially by means of the matroid partition algorithm, and of course (2.4) is even easier. We show that in (2.8) (and hence in (2.7), (2.2)), in general, they cannot.

Suppose we are given a graph $G = (V, E)$ and a matroid $M = (V, \mathcal{B})$. Consider the union $(V, \mathcal{F})$ (as in (2.8)) of the matching delta-matroid of G with the dual matroid M^* of M. Suppose that we have an oracle for $(V, \mathcal{F})$. Then we can apply the greedy algorithm to find a largest member of $\mathcal{F}$, and in particular to decide whether $V \in \mathcal{F}$. But $V \in \mathcal{F}$ if and only if it is partitionable into a matchable set and a basis of M^*, that is, if and only if there is a basis of M that is matchable in G. It is well known that deciding whether this is true ("the matroid matching problem" [17]) is not generally solvable in polynomial time. Hence an oracle for the union of $(S, \mathcal{F}_0)$, $(S, \mathcal{F}_1)$ is not constructible in polynomial time from oracles for $(S, \mathcal{F}_0)$, $(S, \mathcal{F}_1)$. The composition is a useful construction, but it is important to distinguish the cases where it is efficiently constructible from those where it is not.

We conclude the section by deriving a new class of delta-matroids from the composition theorem, and constructing the relevant oracle. A *red-blue graph* is a graph each of whose edges is coloured either red or blue. A vertex v of a red-blue graph is *bichromatic* or *monochromatic* according to whether v is incident to edges of both colours or not. An *alternating path* of a red-blue graph is a path of length at least one whose edges alternate in colour. Here is a class of delta-matroids arising from red-blue graphs. Notice that the matching delta-matroids form a subclass, arising from the case where there are no blue edges.

(2.9) Proposition. *Let $G = (V, E)$ be a red-blue graph, S be the set of monochromatic vertices, and $\mathcal{F} = \{F \subseteq S : F$ is the set of end vertices of a set of vertex-disjoint simple alternating paths$\}$. Then $(S, \mathcal{F})$ is a delta-matroid.*

Proof. Let $(S_0, \mathcal{F}_0)$ be the matching delta-matroid of the graph (S_0, E_0), where $S_0 = \{v \in V : v$ is incident to a red edge$\}$ and E_0 is the set of red edges. Similarly define $(S_1, \mathcal{F}_1)$ with red replaced by blue. It is easy to see that $(S, \mathcal{F})$ is the composition of $(S_0, \mathcal{F}_0)$ and $(S_1, \mathcal{F}_1)$. $\square$

We describe an efficient construction of the oracle for this class of delta-matroids, due to John Vande Vate. Given disjoint subsets A, B of S, delete the vertices of B from G. For each bichromatic vertex w, split w into two vertices w_1, w_2 such that w_1 is incident to the red edges previously incident to w, and w_2 is incident to the blue edges previously incident to w. Also join w_1, w_2 by a new "white" edge. Let G' be the new graph. Let P be the set of edges of a set of alternating paths determining a feasible set F, $A \subseteq F \subseteq S \backslash B$, and let M be P together with the set of white edges corresponding to bichromatic vertices not in any of the paths. Then M is a matching of G' covering all vertices not in $S \backslash (A \cup B)$. Conversely, any such matching of G' determines such a set of alternating paths. Hence the oracle is provided by a matching algorithm. In the next section we will see another example based on red-blue graphs, but allowing the alternating paths to be nonsimple.

3 2-step axiom and jump systems

For vectors $x, y \in \mathbf{Z}^S$, we use the norm $\|x\| = \sum(|x_j| : \ j \in S)$ and the distance $d(x, y) = \|x - y\|$. For $x, y \in \mathbf{Z}^S$ a *step from x to y* is a vector $s \in \mathbf{Z}^S$ such that $\|s\| = 1$ and $d(x + s, y) = d(x, y) - 1$. We denote the set of steps from x to y by $St(x, y)$. A *jump system* is a pair $(S, \mathcal{F})$ where $\mathcal{F} \subseteq \mathbf{Z}^S$ satisfies the following *2-step axiom*:

(2-SA) If $x, y \in \mathcal{F}$, $s \in \mathrm{St}(x, y)$, and $x + s \notin \mathcal{F}$, then there exists $t \in \mathrm{St}(x + s, y)$ with $x + s + t \in \mathcal{F}$.

We begin by considering some simple examples of jump systems.

Low dimensional jump systems. In Figure 1 we illustrate two choices of $\mathcal{F}$ for the case where $|S| = 2$. In both we denote members of $\mathcal{F}$ with solid dots and nonmembers by hollow or non-existent dots. It is easy to see that in the first case, we have a jump system, whereas in the second case a pair x, y violating the 2-SA is indicated. It is interesting also to consider the case $|S| = 1$, that is, to ask which subsets of the integers satisfy 2-SA. These are the sets having no gap of size bigger than one, that is, there do not exist two consecutive integers not in $\mathcal{F}$, unless either all elements of $\mathcal{F}$ are bigger than both or all are smaller than both.

Figure 1: The 2-step axiom

Hyperplanes. Let $a \in \{0, 1, -1\}^S$, let $b \in \mathbf{Z}$, and let $\mathcal{F} = \{x \in \mathbf{Z}^S : \ ax = b\}$. It is easy to check that $(S, \mathcal{F})$ is a jump system.

Delta-matroids. It is an easy exercise to prove that a pair $(S, \mathcal{F})$ such that $\mathcal{F} \subseteq \{0, 1\}^S$ is a jump system if and only if it is a delta-matroid. (Here, of course, we are identifying subsets of S with their characteristic vectors.)

Simple operations on jump systems

Here we mention a few elementary operations that preserve 2-SA.

Translation. Let $(S, \mathcal{F})$ be a jump system and let $a \in \mathbf{Z}^S$. Then the translation $(S, \mathcal{F}')$ of $(S, \mathcal{F})$ by a is defined by $\mathcal{F}' = \{x + a : \ x \in \mathcal{F}\}$, and is clearly a jump system.

Cartesian Product. Let S_0, S_1 be disjoint sets, and let $(S_i, \mathcal{F}_i)$ be a jump system for $i = 0, 1$. Define $S = S_0 \cup S_1$ and $\mathcal{F} = \{F_0 \cup F_1 : \ F_0 \in \mathcal{F}_0, F_1 \in \mathcal{F}_1\}$. Then $(S, \mathcal{F})$ is a jump system.

Reflection. Let $(S, \mathcal{F})$ be a jump system and let $N \subseteq S$. For each $x \in \mathbf{R}^S$, let x' be the vector obtained by reflecting x in the co-ordinates indexed by N, that is, $x'_j = x_j$ if $j \notin N$ and $x'_j = -x_j$ otherwise. Then, where $\mathcal{F}' = \{x' : \ x \in \mathcal{F}\}$, it is easy to see that $(S, \mathcal{F}')$ is a jump system. We observe that the twisting operation on delta-matroids is a combination of reflection and translation; more precisely, twisting by N is equivalent to reflecting in the co-ordinates indexed by N followed by translating by the characteristic vector of N.

Minors. Let $(S, \mathcal{F})$ be a jump system, let $S' \subseteq S$, let $x \in \mathbf{Z}^{S \setminus S'}$, and let $\mathcal{F}' = \{x' \in \mathbf{Z}^{S'} : \ (x', x) \in \mathcal{F}\}$. Then $(S', \mathcal{F}')$ is a jump system.

Intersection with a box. A *box* is a set of the form $\{x \in \mathbf{R}^S : \ l \leq x \leq u\}$, where $l \in (\mathbf{R} \cup \{-\infty\})^S$ and $u \in (\mathbf{R} \cup \{\infty\})^S$. It is easy to see that the intersection of a jump system with a box is again a jump system.

Restriction or projection. Let $(S, \mathcal{F})$ be a jump system and let $S' \subseteq S$. Then $(S', \mathcal{F}')$ is a jump system, where $\mathcal{F}' = \{x_{|S'} : \ x \in \mathcal{F}\}$. We remark that this is not completely obvious, but we leave the (easy) proof to the reader. Also, we point out that the minor operation is now redundant, in the sense that it can be obtained as an intersection with a box followed by a projection. (Namely, intersect with the box defined by $l_j = -\infty$, $u_j = \infty$, $j \in S'$, and $l_j = u_j = x_j$ otherwise, and then restrict to S'.)

Integral polymatroids

Now we introduce a less trivial example. An *integral polymatroid* is a polyhedron $P = \{x \in \mathbf{R}^S_+ : \ x(A) \leq f(A) \text{ for all } A \subseteq S\}$, where $f : \{0,1\}^S \to \mathbf{Z}_+$ is normalized ($f(\emptyset) = 0$) and submodular ($f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq S$).

(3.1) Proposition. *If P is an integral polymatroid in $\mathbf{R}^S$, then $P \cap \mathbf{Z}^S$ satisfies 2-SA.*

The proof uses a well-known result, from [11]. Given $x \in P$, where P is determined by f, we say that a set $A \subseteq S$ is *x-tight* or just *tight* if $x(A) = f(A)$.

(3.2) Lemma. *The union and intersection of tight sets are tight.* □

Proof of Proposition (3.1). Let x, y be integral points of P and s a step from x to y such that $x + s \notin P$. Then it is easy to see that s must be non-negative, so $s = \{e\}$ for some $e \in S$ such that $x_e < y_e$. It follows that there is an x-tight set A such that $e \in A$. Now if $y_j \geq x_j$ for all $j \in A$, then $y(A) > x(A) = f(A)$, a contradiction. So there exists $j \in A$ with $x_j > y_j$. If $x + \{e\} - \{j\} \in P$, we are done, so we may assume that for every such j there is an x-tight set A_j with $e \in A_j$ and $j \notin A_j$. The intersection of all these A_j with A is, by (3.2), an x-tight set B such that $e \in B$ and $x_j \leq y_j$ for all $j \in B$. But then $y(B) > x(B) = f(B)$, a contradiction. □

Sum of jump systems

The *sum* of jump systems $(S, \mathcal{F}_0)$ and $(S, \mathcal{F}_1)$, defined on the same set S, is the pair $(S, \mathcal{F})$ where $\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1 = \{x + y : \ x \in \mathcal{F}_0, \ y \in \mathcal{F}_1\}$. The simple proof of the following theorem was suggested to us by András Sebő.

(3.3) Theorem. *The sum of two jump systems is a jump system.*

Proof. We use the above notation. Let $x, y \in \mathcal{F}_1 + \mathcal{F}_2$ and let s be a step from x to y. We have to prove that $x + s \in \mathcal{F}_1 + \mathcal{F}_2$ or there exists a step t from $x + s$ to y such that $x + s + t \in \mathcal{F}_1 + \mathcal{F}_2$. We assume that $x + s \notin \mathcal{F}_1 + \mathcal{F}_2$ and we search for t. Let $x = x_1 + x_2$ and $y = y_1 + y_2$ with $x_1, y_1 \in \mathcal{F}_1$ and $x_2, y_2 \in \mathcal{F}_2$. We have $x_1 + s \notin \mathcal{F}_1$ and $x_2 + s \notin \mathcal{F}_2$ (for example if $x_1 + s \in \mathcal{F}_1$ then $(x_1 + s) + x_2 = x + s \in \mathcal{F}_1 + \mathcal{F}_2$, a contradiction).

We claim that we can find $x_1' \in \mathcal{F}_1$, $x_2' \in \mathcal{F}_2$ and a step t satisfying $x + s + t = x_1' + x_2'$. Since s is a step from $x_1 + x_2$ to $y_1 + y_2$, s is a step from x_1 to y_1 or a step from x_2 to y_2. By symmetry we may assume the former. Apply 2-SA to $x_1, y_1 \in \mathcal{F}_1$ and the step s from x_1 to y_1. Since $x_1 + s \notin \mathcal{F}_1$ there exists a step t from $x_1 + s$ to y_1 such that $x_1 + s + t \in \mathcal{F}_1$. Then $(x_1 + s + t) + x_2 = x + s + t \in \mathcal{F}_1 + \mathcal{F}_2$, which implies the existence of x_1' and x_2'.

Choose a triple (x_1', x_2', t) that minimizes $d(x_1', y_1) + d(x_2', y_2)$. We show that, under this assumption, t is a step from $x + s$ to y, proving the theorem. Assume not for a contradiction.

Then $-t$ is a step from $x'_1 + x'_2 = x + s + t$ to $y_1 + y_2 = y$. This implies that $-t$ is a step from x'_1 to y_1 or a step from x'_2 to y_2. By symmetry we assume the former. Apply 2-SA to $x'_1, y_1 \in \mathcal{F}_1$ and the step $-t$. The point $x'_1 - t$ does not belong to $\mathcal{F}_1$ because, if so, we should have $(x'_1 - t) + x'_2 = x + s \in \mathcal{F}_1 + \mathcal{F}_2$, which contradicts the initial assumption. Thus we can find a step r from $x'_1 - t$ to y_1 such that $x'_1 - t + r \in \mathcal{F}_1$. This implies $x + s + r = (x'_1 - t + r) + x'_2 \in \mathcal{F}_1 + \mathcal{F}_2$, where $x'_1 - t + r$ is closer to y_1 than x'_1. So the triple $(x'_1 - t + r, x'_2, r)$ contradicts the choice of (x'_1, x'_2, t). □

Bidirected graphs

We consider finite graphs that may have loops and multiple edges. In order to define bidirections, it is convenient to let each edge be incident to two half-edges. Formally a graph G is defined by three pairwise disjoint finite sets: a set of vertices V, a set of edges E, and a set of half-edges H. There is an incidence relation between H and V, as well as between H and E. These incidence relations are such that each half-edge is incident to precisely one vertex and one edge. Further an edge is incident to precisely two half-edges. We denote by $\mathrm{h}v$ the set of the half-edges incident to a vertex v. The degree of v is $\mathrm{d}(v) = |\mathrm{h}v|$.

A *biorientation*, or *bidirection*, over G is a function $\epsilon : H \to \{-1, +1\}$. For $f \in \mathbf{Z}^E$ and $v \in V$, the *excess* of f at v is $\mathrm{ex}(f)_v = \sum(\epsilon(h)f(e) : h \in \mathrm{h}v, e \text{ is the edge incident to } h)$, and the *excess* of f is the vector $\mathrm{ex}(f) = (\mathrm{ex}(f)_v : v \in V)$. Given $c_1, c_2 \in \mathbf{Z}^E$, with $c_1 \leq c_2$, we denote by $[c_1, c_2]$ the set $\{f \in \mathbf{Z}^E : \ c_1 \leq f \leq c_2\}$.

(3.4) Proposition. *Let $c_1, c_2 \in \mathbf{Z}^E$, such that $c_1 \leq c_2$. Then $(V, \{\mathrm{ex}(f) : f \in [c_1, c_2]\})$ is a jump system.*

Proof. For $h \in H$ let $x(h) \in \mathbf{Z}^V$ be defined by $x(h)_v = \epsilon(h)$ if the vertex v is incident to h, $x(h)_v = 0$ otherwise. For $e \in E$ let $\mathcal{F}_e = \{\lambda(x(h') + x(h'')) : \lambda \in [c_1(e), c_2(e)]\}$, where h' and h'' are the half-edges of G incident to e. We easily verify that $(V, \mathcal{F}_e)$ is a jump system. (One way is to observe that it is a hyperplane jump system intersected with a box and then extended by zeroes, but it is perhaps as easy to check directly.) We have

$(V, \{\mathrm{ex}(f) : f \in [c_1, c_2]\}) = \sum((V, \mathcal{F}_e) : e \in E)$, where the summation stands for the sum operation considered in Theorem (3.3). The result follows from that theorem. $\square$

The special case in which we take the bidirection to be trivial, that is, all the values of ϵ to be $+1$, is already quite interesting. If we also define $c_1(e) = 0$ and $c_2(e) = 1$ for each edge e, then $\{\mathrm{ex}(f) : f \in [c_1, c_2]\}$ is the set of degree sequences of subgraphs of G. If we now intersect this set with the unit cube, we get the matching delta-matroid of G. More general sets of this type are investigated in [7].

Suppose that we consider again the red-blue graph example of Proposition (2.9), but this time we allow the alternating paths to repeat vertices, but not edges. We show that we obtain another delta-matroid. We form a bidirected graph, by assigning to each red edge two positive half-edges, and assigning to each blue edge two negative half-edges; we define again $c_1(e) = 0$ and $c_2(e) = 1$ for each edge e. Now consider the resulting jump system, and reflect it in the co-ordinates corresponding to the vertices incident only to blue edges. Next, intersect it with the box determined by $0, u$, where $u_j = 1$ if j is monochromatic, and $u_j = 0$ otherwise. Finally, project the jump system to the co-ordinates corresponding to the monochromatic vertices. The resulting jump system is a delta-matroid, and it is easy to see that it is precisely the desired one. Moreover, an oracle for this delta-matroid can be realized in polynomial time by methods of bidirected matching; see [12].

Composition of jump systems

Let $(S_0, \mathcal{F}_0)$ and $(S_1, \mathcal{F}_1)$ be two jump systems. The *composition* of $(S_0, \mathcal{F}_0)$ and $(S_1, \mathcal{F}_1)$ is the pair $(S, \mathcal{F})$, where $S = S_0 \Delta S_1$ and $\mathcal{F} \subseteq \mathbf{Z}^S$ is defined by $x \in \mathcal{F}$ if and only if there exists $x_0 \in \mathcal{F}_0$ and $x_1 \in \mathcal{F}_1$ satisfying $x_{0|S_0 \cap S_1} = x_{1|S_0 \cap S_1}$, $x_{|S_0 \setminus S_1} = x_{0|S_0 \setminus S_1}$, $x_{|S_1 \setminus S_0} = x_{1|S_1 \setminus S_0}$. (We may also speak of the composition $\mathcal{F}$ of $\mathcal{F}_0$ and $\mathcal{F}_1$.) Notice that this definition, in the case of $\{0, 1\}$-valued vectors, corresponds to the composition of delta-matroids introduced in Section 2. Hence the next result generalizes (2.2).

(3.5) Proposition. *The composition of two jump systems is a jump system.*

Proof. For $i = 1, 2$ we extend each vector in $\mathcal{F}_i$ to an element of $\mathbf{Z}^{S_0 \cup S_1}$ by filling it out with zeroes. Then we reflect $\mathcal{F}_1$ in the components corresponding to $S_0 \cap S_1$, then we take the sum, and then we take the minor associated with the vector $0 \in \mathbf{Z}^{S_0 \cap S_1}$. $\square$

Conversely, Theorem (3.3) can be easily derived from the preceding proposition. (In fact, the original version of this paper proved the proposition directly and used it to prove Theorem (3.3).) Consider two sets $\mathcal{F}', \mathcal{F}'' \subseteq \mathbf{Z}^S$ that satisfy 2-SA. We first notice that $\Phi = \{(x, y, x + y) : x, y \in \mathbf{Z}\}$ is a subset of $\mathbf{Z}^3$ which satisfies 2-SA. (For example, it is an instance of the hyperplane systems introduced earlier.) Let us consider a family $(T_v = \{v', v'', v\} : v \in S)$ of pairwise disjoint 3-element sets. For each $v \in S$ let $\Phi_v = \{(x_{v'}, x_{v''}, x_v) : x_{v'}, x_{v''} \in \mathbf{Z}, x_v = x_{v'} + x_{v''}\} \subseteq \mathbf{Z}^{T_v}$, and consider the cartesian product $\Phi = \times(\Phi_v : v \in S) \subseteq \times(\mathbf{Z}^{T_v} : v \in S) = \mathbf{Z}^{S' \cup S'' \cup S}$, with $S' = \{v' : v \in S\}$ and $S'' = \{v'' : v \in S\}$. Then Φ satisfies 2-SA. We make a copy $\mathcal{G}' \subseteq \mathbf{Z}^{S'}$ of $\mathcal{F}'$ and a copy $\mathcal{G}'' \subseteq \mathbf{Z}^{S''}$ of $\mathcal{F}''$. The cartesian product $\mathcal{G} = \mathcal{G}' \times \mathcal{G}'' \subseteq \mathbf{Z}^{S' \cup S''}$ satisfies 2-SA. Finally we notice that the composition of $\Phi \subseteq \mathbf{Z}^{S' \cup S'' \cup S}$ with $\mathcal{G} \subseteq \mathbf{Z}^{S' \cup S''}$ is equal to $\mathcal{F}' + \mathcal{F}''$.

4 Bisubmodular polyhedra and jump systems

Here we describe a generalization of (integral) polymatroids, called (integral) bisubmodular polyhedra. We show that the integral points of an integral bisubmodular polyhedron satisfy the 2-SA. In the next section, we show a partial converse: The convex hull of a set satisfying the 2-SA is an integral bisubmodular polyhedron.

A function f from pairs (A, B) of disjoint subsets of S to $\mathbf{R} \cup \{\infty\}$ is called *bisubmodular* if it satisfies, for all such pairs (A, B), (A', B'),

$$f(A, B) + f(A', B') \geq f((A, B) \wedge (A', B')) + f((A, B) \vee (A', B')).$$

Here $(A, B) \wedge (A', B')$ denotes $(A \cap A', B \cap B')$, and we call it the *intersection* of (A, B) and (A', B'); $(A, B) \vee (A', B')$ denotes $((A \cup A') \setminus (B \cup B'), (B \cup B') \setminus (A \cup A'))$, and we call it the *reduced union* of (A, B) and (A', B'). (Notice that the operation $\vee$ is not associative.) It is convenient to assume throughout that $f(\emptyset, \emptyset) = 0$. The bisubmodular inequality (on real-valued functions) has been introduced by Kabadi and Chandrasekaran [5, 16], by Nakamura [18, 19], and by Qi [22]. The term "bisubmodular" was introduced by Nakamura [20].

The *bisubmodular polyhedron* associated with f is $P(f) = \{x \in \mathbf{R}^S : x(A) - x(B) \leq f(A,B),\ A, B \subseteq S,\ A \cap B = \emptyset\}$. These polyhedra, again with the exception that the function values are finite, were introduced by Dunstan and Welsh [10] and studied in [16, 18, 22]. Nakamura showed the equivalence of the Dunstan-Welsh definition and the bisubmodular one. The function f associated in Section 2 with a delta-matroid $(S, \mathcal{F})$ is bisubmodular and the associated bisubmodular polyhedron is the convex hull of the elements of $\mathcal{F}$. This result appears in [5] and [2]. We say that f is *integral* if its finite values are integral, and that $P(f)$ is *integral* if f is integral.

A number of more familiar classes of polyhedra fall into this class. If f' is submodular on subsets of S, and $f'(\emptyset) = 0$, then f defined by $f(A, \emptyset) = f'(A)$ for $A \subseteq S$ and $f(A, B) = \infty$ for $B \neq \emptyset$, is bisubmodular. The associated $P(f)$ is $\{x \in \mathbf{R}^S : x(A) \leq f'(A)$ for all $A \subseteq S\}$, the *submodular polyhedron* associated with f'. If we take $f(A, B) = f'(A)$ for all pairs A, B of disjoint subsets of S, then it is easy to check that f is bisubmodular if and only f' is also *monotone*: if $A_1 \subseteq A_2$, then $f'(A_1) \leq f'(A_2)$. In this case $P(f)$ is $\{x \in \mathbf{R}^S_+ : x(A) \leq f'(A)$ for all $A \subseteq S\}$, the *polymatroid* associated with f'. (Although $P(f')$ is a polymatroid even without the assumption of monotonicity, it is known that every polymatroid is determined by a monotone submodular function, so every polymatroid is a bisubmodular polyhedron.) Finally, the *base polyhedron* $\{x \in \mathbf{R}^S : x(A) \leq f'(A)$ for all $A \subseteq S,\ x(S) = f'(S)\}$ associated with f' is obtained by taking $f(A, B) = f'(A) + f'(S \setminus B) - f'(S)$, and f is bisubmodular.

Another, more general, class of bisubmodular polyhedra consists of Frank's generalized polymatroids. Here we suppose that g, h are submodular functions on S, which are allowed to take the value ∞, and that they also satisfy

$$g(A) + h(B) \geq g(A \setminus B) + h(B \setminus A)$$

for all pairs of subsets A, B of S. Then $Q(g, h) = \{x \in \mathbf{R}^S : -h(A) \leq x(A) \leq g(A)$ for all $A \subseteq S\}$ is the *generalized polymatroid* determined by g and h. If we define $f(A, B)$ to be $g(A) + h(B)$ for disjoint pairs A, B of subsets of S, then $P(f) = Q(g, h)$, and f is bisubmodular. The class of generalized polymatroids contains all the special classes mentioned earlier, but the class of bisubmodular polyhedra is even larger.

(4.1) Proposition. *Let f be bisubmodular and let $C \subseteq S$. Then reflecting $P(f)$ in C gives a bisubmodular polyhedron $P(f')$, where f' is defined by*

$$f'(A, B) = f((A \setminus C) \cup (B \cap C), (B \setminus C) \cup (A \cap C)).$$

Proof. It is clear that $x' \in P(f)$ if and only if $x'(A') - x'(B') \leq f(A', B')$ for all pairs (A', B'), where x' is obtained by reflecting x in C. But this is equivalent to

$$x(A' \setminus C) - x(A' \cap C) - x(B' \setminus C) + x(B' \cap C) \leq f(A', B')$$

or, taking $A = (A' \setminus C) \cup (B' \cap C)$, $B = (B' \setminus C) \cup (A' \cap C)$,

$$x(A) - x(B) \leq f((A \setminus C) \cup (B \cap C), (B \setminus C) \cup (A \cap C)).$$

Hence, the reflection of $P(f)$ is $P(f')$, and it remains only to prove that f' is bisubmodular. This can be done by a straightforward computation, which we omit. □

(4.2) Proposition. *Let f be bisubmodular and $S' \subseteq S$. Then the projection of $P(f)$ onto the co-ordinates indexed by S' is a bisubmodular polyhedron $P(f')$, where f' is the restriction of f to S'.*

Proof. It is obvious that f' is bisubmodular, so we need only show that $P(f')$ is the projection. It is enough to prove this in the case in which $S' = S \setminus \{e\}$ for some $e \in S$. By Fourier elimination of x_e, the projection is determined by two classes of inequalities. The first consists of inequalities

(i) $x(A) - x(B) \leq f(A, B)$, where $e \notin A \cup B$.

The second consists of inequalities each of which is obtained by adding an inequality for $P(f)$ in which x_e has coefficient 1, to one in which x_e has coefficient -1. So each such inequality has the form

(ii) $x(A') - x(B') + x(A'') - x(B'') \leq f(A', B') + f(A'', B'')$, where $e \in A' \cap B''$.

We need to show that each inequality of type (ii) is implied by those of type (i). In fact, we add the inequality for $(A', B') \wedge (A'', B'')$ to the one for $(A', B') \vee (A'', B'')$. These inequalities are both of type (i). Their sum has the same left-hand side as (ii) and its right-hand side is no larger than the right-hand side of (ii), by bisubmodularity. $\square$

We remark that it follows that every bisubmodular polyhedron is non-empty: since $f(\emptyset, \emptyset) = 0$, this is true by induction. Besides projection, several other operations that preserve 2-SA also preserve bisubmodular polyhedra, and the corresponding bisubmodular function can be explicitly constructed. For cartesian product, translation, and minors this is easy to show, and we do not bother to state the results. On the other hand, for intersection with a box, it is not obvious, and the formula for the defining function is not easy to establish. This result will be discussed elsewhere.

The proof that integral bisubmodular functions yield jump systems uses a basic lemma, the analogue for bisubmodular polyhedra of Lemma (3.2). Given $x \in P(f)$, we say that a pair (A, B) is *x-tight* (or just *tight*) if $x(A) - x(B) = f(A, B)$.

(4.3) Lemma. *The intersection and the reduced union of x-tight pairs are x-tight.*

Proof. Suppose that (A, B) and (A', B') are x-tight.

$$\begin{aligned} x((A \cup A') \setminus (B \cup B')) &- x((B \cup B') \setminus (A \cup A')) + x(A \cap A') - x(B \cap B') \\ &= x(A) - x(B) + x(A') - x(B') \\ &= f(A, B) + f(A', B') \\ &\geq f((A, B) \wedge (A', B')) + f((A, B) \vee (A', B')). \end{aligned}$$

Since $x \in P(f)$, the inequality also holds in the other direction, so we have equality throughout. $\square$

We remark that the above lemma, or similar arguments as in the proof, can be used to obtain further results. For example, if (A, B) and (A', B') are x-tight, then so is $(A \setminus B', B \setminus$

$A')$, by applying the lemma again to (A, B) and $(A, B) \vee (A', B')$. This was pointed out in [16]. Also if f and x are integral, (A, B) is x-tight, and $x(A') - x(B') = f(A', B') - 1$, then one of the intersection and reduced union is x-tight.

(4.4) Theorem. *Let f be bisubmodular and integral. Then $\mathcal{F} = \mathbf{Z}^S \cap P(f)$ satisfies the 2-SA.*

Proof. Let $x, y \in \mathcal{F}$ and $s \in \mathrm{St}(x, y)$ and suppose 2-SA fails. By reflecting $P(f)$ in $\{j : x_j > y_j\}$, we may assume that $x \leq y$. Suppose that $s = \{e\}$. Let $Q = \{j \in S \setminus \{e\} \ : x_j < y_j\}$, and let $Q' = \{j \in Q : \text{there exists } x\text{-tight } (A, B) \text{ with } e \in A,\, j \notin B\}$.

Claim. There exists x-tight (A, B) with $e \in A$ and $Q \setminus B = Q'$.

Suppose first that $Q' = \emptyset$. Choose x-tight (A', B') with $e \in A'$. (Such exists, because $x + \{e\} \notin P(f)$.) Then by the definition of Q', $Q \setminus B' = \emptyset = Q'$, as required. So suppose that $Q' \neq \emptyset$. Now take the intersection of all the x-tight pairs (A'', B'') with $e \in A''$ and $Q \setminus B'' \neq \emptyset$. This gives an x-tight pair (A, B) with $B \cap Q' = \emptyset$ and so, by the definition of Q', $Q \setminus B = Q'$. This completes the proof of the claim.

Suppose there exists $j \in Q \setminus Q'$. Notice that $j \in B$. Since $x + \{e\} + \{j\} \notin P(f)$ and $j \notin Q'$, there exists (A', B') such that

(i) $x(A') - x(B') = f(A', B')$, $j \in A'$, $e \notin B'$;

or

(ii) $x(A') - x(B') \geq f(A', B') - 1$, $j, e \in A'$.

In case (i) the reduced union of (A, B) and (A', B') is an x-tight pair (A'', B'') with $e \in A''$, $j \notin B''$, so $j \in Q'$, a contradiction. In case (ii) both the reduced union and the intersection of (A, B), (A', B') are pairs (A'', B'') with $e \in A''$, $j \notin B''$. Moreover, at least one of the two pairs is x-tight, so $j \in Q'$, a contradiction. It follows that $Q \setminus Q' = \emptyset$, so $y(A) - y(B) > x(A) - x(B) = f(A, B)$, again a contradiction. □

We use Lemma (4.3) to prove another basic fact about bisubmodular polyhedra, that each such polyhedron has a unique defining function. For the case where all function values are finite, this result is proved in [14], page 94.

(4.5) Theorem. *If f is bisubmodular on pairs of disjoint subsets of S, then for each such pair A, B we have $f(A,B) = \max_{x \in P(f)}(x(A) - x(B))$. Moreover, if f is integral, then the maximum is achieved by an integral x.*

Proof. By applying reflection and projection, using Propositions (4.1) and (4.2), we can assume that $A = S$ and $B = \emptyset$. Obviously, the maximum is at most $f(S, \emptyset)$, so if it is ∞, we are done. We choose $\bar{x} \in P(f)$ maximizing $x(A) - x(B)$. Fix $e \in S$. Since $\bar{x}_e$ cannot be increased, there is a tight pair (A', B') with $e \in A'$.

Claim. For each $j \notin A'$ there is a tight pair (A_j, B_j) such that $e \in A_j$ and $j \notin B_j$.

Proof of Claim. If not, then there is a tight pair (A'', B'') with $j \in A''$ and $e \notin B''$. (Otherwise we could increase both $\bar{x}_e$ and $\bar{x}_j$). Now take $(A_j, B_j) = (A', B') \vee (A'', B'')$. This pair is tight by Lemma (4.3), and it is easy to see that it satisfies the conditions of the claim.

Now take the intersection over all $j \notin A'$ of the pairs (A_j, B_j). We get a tight pair (A_e, B_e) with $e \in A_e$ and $B_e \subset A'$. The intersection of (A_e, B_e) with (A', B') is a tight pair $(A'_e, \emptyset)$ with $e \in A'_e$. Finally, the union over all e of these tight pairs is the tight pair $(S, \emptyset)$, so $\bar{x}(S) = f(S, \emptyset)$, as required. It is straightforward to check that if f is integral, then the whole argument applies to integral points, so the second part is proved also. $\square$

5 Jump systems and bisubmodular polyhedra

We say that $x \in \mathcal{F}$ is (A,B)-*maximal* in $\mathcal{F}$ if $y \in \mathcal{F}$, $y_j \geq x_j$ for all $j \in A$, $y_j \leq x_j$ for all $j \in B$ imply $y_{|A \cup B} = x_{|A \cup B}$.

(5.1) Lemma. *If $\mathcal{F}$ satisfies 2-SA and $y, x \in \mathcal{F}$ with $y(A) - y(B) > x(A) - x(B)$, then x is not (A,B)-maximal.*

Proof. By twisting at B, we may assume that $B = \emptyset$. Suppose that x is $(A, \emptyset)$-maximal and there exists $y \in \mathcal{F}$ and $y(A) > x(A)$. Subject to this, choose y so that $\sum_{j \in A} |x_j - y_j|$ is minimum. By the maximality of x, there exists $e \in A$ with $x_e > y_e$. Either $y' = y + \{e\} \in \mathcal{F}$, or $y'' = y + \{e\} + s \in \mathcal{F}$ for some step s from $y + \{e\}$ to x. But this contradicts the choice of y. $\square$

Given $\mathcal{F}$, define f by $f(A,B) = \max_{x\in\mathcal{F}}(x(A) - x(B))$, if the maximum exists, and to be ∞ otherwise.

(5.2) Lemma. *If $\mathcal{F}$ satisfies 2-SA, then f is bisubmodular.*

Proof. Let (A,B), (A',B') be pairs of disjoint sets. First, we consider the case where the right-hand side of the bisubmodular inequality is not ∞. We may assume that $f(A,B) \neq \infty$, since otherwise the inequality holds trivially. Choose $x \in \mathcal{F}$ to be $(A\cap A', B\cap B')$-maximal. Now by increasing x_j for $j \in (B\Delta B') \setminus (A \cup A')$, we can find an x' that is $(A\vee A', B\vee B')$-maximal. Then by ((5.1)), we get

$$\begin{aligned}
&f(A,B) + f(A',B') \\
&\quad\geq\ x'(A) - x'(B) + x'(A') - x'(B') \\
&\quad=\ x(A\cap A') - x(B\cap B') + x'((A\cup A')\setminus(B\cup B')) - x'((B\cup B')\setminus(A\cup A')) \\
&\quad=\ f((A,B)\wedge(A',B')) + f((A,B)\vee(A',B')),
\end{aligned}$$

as required.

Now suppose that the right-hand side of the bisubmodular inequality is infinity. Let $\mathcal{F}^k$ be $\mathcal{F} \cap \{x \in \mathbf{Z}^S : -k \leq x_j \leq k \text{ for all } j \in S\}$. Then $\mathcal{F}^k$ satisfies 2-SA; let f^k be the corresponding (bisubmodular!) function. Then the right-hand side of the bisubmodular inequality for f^k goes to ∞ with k, and so the left-hand side must also, and we are finished. $\square$

(5.3) Theorem. *If $\mathcal{F}$ satisfies 2-SA, then* $\mathrm{conv}(\mathcal{F})$ *is an integral bisubmodular polyhedron.*

Proof. Define f by $f(A,B) = \max_{x\in\mathcal{F}}(x(A) - x(B))$. Then f is integral and bisubmodular, by (5.2), and $\mathcal{F} \subseteq P(f)$. If $P(f) \neq \mathrm{conv}(\mathcal{F})$, then there exists $c \in \mathbf{R}^S$ and $y \in P(f)$ such that $cy > cx$ for every $x \in \mathcal{F}$. By a straightforward perturbation argument we can choose c so that there does not exist $j \in S$ with $c_j = 0$ and there do not exist distinct elements j, k of S with $|c_j| = |c_k|$. By reflection in $N = \{j : c_j < 0\}$, we may assume that $c_j > 0$ for all $j \in S$. Now $\max_{x\in\mathcal{F}} cx$ exists, so $\max_{x\in\mathcal{F}} x(S)$ exists, by (5.1) with $A = S$, $B = \emptyset$.

Therefore, we can form the polyhedron $B(f) = \{x \in P(f) : x(S) = f(S, \emptyset)\}$. The set $\mathcal{B}$ of maximal members of $\mathcal{F}$ is contained in $B(f)$. Since $c_j > 0$, $j \in S$, there exists $y \in B(f)$ with $cy > cx$ for every $x \in \mathcal{B}$. This again follows from (5.1). We need the following

Claim. If $y, x \in \mathcal{B}$, $A \subseteq S$, and $y(A) > x(A)$, then x is not A-maximal over $\mathcal{B}$.

Proof of Claim. If possible, choose y and x violating the statement with $\sum_{j \in A} |x_j - y_j|$ as small as possible. Clearly there exists $e \in A$ with $x_e > y_e$. Apply the 2-SA to get $j \in S$ with $y + \{e\}$ or $y + \{e\} + \{j\}$ or $y + \{e\} - \{j\} \in \mathcal{F}$. By the definition of $\mathcal{B}$, the only possibility is the last one. But then $y'' = y + \{e\} - \{j\} \in \mathcal{B}$, and this contradicts the choice of y, and the claim is proved.

Let us relabel the elements of S as $e_1, e_2, \ldots, e_n$ so that $c_{e_1} > c_{e_2} > \ldots > c_{e_n}$, and let T_i denote $\{e_1, e_2, \ldots, e_i\}$ for $0 \leq i \leq n$. For $T \subseteq S$, $z \in \mathbf{R}^T$, and $Q \subseteq \mathbf{R}^S$ we write $z \hat{\in} Q$ to mean that there exists $\bar{z} \in Q$ such that $\bar{z}_{|T} = z$. Choose $\bar{x} \in \mathcal{B}$ as follows:

For $i = 1$ to n

Choose $\bar{x}_{e_i}$ to be $\max(\alpha : (\bar{x}_{e_1}, \ldots, \bar{x}_{e_{i-1}}, \alpha) \hat{\in} \mathcal{B}$, if the maximum exists.

Otherwise, stop.

Now suppose the procedure runs to completion and delivers $\bar{x} \in \mathcal{B}$. By the claim, $\bar{x}(T_i) = f(T_i, \emptyset) < \infty$, $1 \leq i \leq n$. Now for any $x \in B(f)$, we have

$$\begin{aligned} cx &= \sum_{i=1}^{n} c_{e_i} \left(x(T_i) - x(T_{i-1})\right) \\ &= \sum_{i=1}^{n-1} (c_{e_{i+1}} - c_{e_i}) x(T_i) + c_{e_n} x(T_n) \\ &\leq \sum_{i=1}^{n-1} (c_{e_{i+1}} - c_{e_i}) \bar{x}(T_i) + c_{e_n} \bar{x}(T_n) = c\bar{x}. \end{aligned}$$

This shows that a point of $\mathcal{B}$ maximizes cx over $B(f)$, a contradiction. Now suppose that the procedure stops early; say that j is the first index for which the maximum does not exist and $\bar{x}$ is the point constructed after step $j-1$. Then by the claim, we have $\bar{x}(T_i) = f(T_i, \emptyset)$ for $1 \leq i \leq j-1$. Moreover, $\bar{x}$ is not A-maximal in $\mathcal{B}$, so there exists $x' \in \mathcal{B}$ with $x'_{e_i} = \bar{x}_{e_i}$, $1 \leq i \leq j-1$ and $x'_j = \bar{x}_j + \alpha$ for some positive integer α. But then, $x'(S \setminus T_j) = \bar{x}(S \setminus T_j) - \alpha$, so $cx' \geq c\bar{x} + \alpha\epsilon$, where $\epsilon = \min(c_{e_i} - c_{e_{i+1}} : 1 \leq i \leq n-1)$. But the same argument can be applied to x', and so on, so cx is unbounded on $\mathcal{B}$, a contradiction. $\square$

We remark that the proof of (5.3) contains the basis of a greedy algorithm for optimizing a linear function over a set satisfying 2-SA. However, we have managed to avoid many of the awkward parts of such an algorithm (such as those dealing with unboundedness and equal cost coefficients). These difficulties are handled for some classes of polyhedra in [13], [14], [15].

It is a consequence of (5.3) that the convex hull of a set satisfying 2-SA is given by inequalities having coefficients $0, 1, -1$. This result can be applied to the bidirected-graph example of Section 3 to conclude that the resulting polyhedra can be described in this way. For the case of trivial bidirections, these results, and somewhat more, were proved in [7].

We can also prove that an integral bisubmodular polyhedron, that is, the polyhedron determined by a bisubmodular polyhedron that is integral, is indeed an integral polyhedron. This result, in a slightly less general setting, appears in [16], [18], and [22].

(5.4) Corollary. *Every integral bisubmodular polyhedron is the convex hull of its integral points.*

Proof. Let f be an integral bisubmodular function defined on pairs of disjoint subsets of S. By Theorem (4.4), $\mathcal{F} = P(f) \cap \mathbf{Z}^S$ satisfies 2-SA, and so by Theorem (5.3), $\mathrm{conv}(\mathcal{F})$ is a bisubmodular polyhedron $P(f')$, where f' is defined by $f'(A, B) = \max_{x \in \mathcal{F}}(x(A) - x(B))$. Now by Theorem (4.5), we have that $f = f'$, and we are done. $\square$

A *gap* of a set $\mathcal{F} \subseteq \mathbf{Z}^S$ is an integral point in $\mathrm{conv}(\mathcal{F}) \setminus \mathcal{F}$. The examples of Figure 1 show that adding a gap to a set satisfying 2-SA can create a set violating 2-SA. On the other hand we have the following result.

(5.5) Corollary. *Suppose that $\mathcal{F}$ satisfies 2-SA and $\mathcal{F}'$ is obtained from $\mathcal{F}$ by adding all of the gaps of $\mathcal{F}$. Then $\mathcal{F}'$ satisfies 2-SA.*

Proof. By (5.3), $\mathrm{conv}(\mathcal{F})$ is an integral bisubmodular polyhedron P. By (4.4), the integral points in P satisfy 2-SA. $\square$

The following related results have been obtained recently. Duchamp [9] has proved that the "delta-sum" $(S,\mathcal{F})$, of delta-matroids $(S,\mathcal{F}_0)$ and $(S,\mathcal{F}_1)$, is again a delta-matroid. Here $\mathcal{F}=\{F_0\Delta F_1 : F_0\in\mathcal{F}_0, F_1\in\mathcal{F}_1\}$. This result implies the composition theorem for delta-matroids. Payan [21] has proved that the mod 2 reduction of a jump system is a delta-matroid. (That is, each component of a vector in $\mathcal{F}$ is replaced by 0 or 1 according to its parity.) This result together with Theorem (3.3) implies the result of Duchamp. Although adding an arbitrary gap can violate 2-SA, there is a notion intermediate between this and adding all gaps. If $j\in S$, a *gap in direction* j is a point $x\notin\mathcal{F}$ such that $x+\{j\}$ and $x-\{j\}$ are both in $\mathcal{F}$. We originally conjectured that adding all gaps in the same direction preserves the 2-SA. Payan [21] has proved this conjecture.

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