Wheel Inequalities for Stable Set Polytopes

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Abstract

We introduce new classes of valid inequalities, called wheel inequalities, for the stable set polytope P_G of a graph G. Each "wheel configuration" gives rise to two such inequalities. The simplest wheel configuration is an "odd" subdivision W of a wheel, and for these we give necessary and sufficient conditions for the wheel inequality to be facet-inducing for P_W. Generalizations arise by allowing subdivision paths to intersect, and by replacing the "hub" of the wheel by a clique. The separation problem for these inequalities can be solved in polynomial time.

1 Introduction

Let $G = (V, E)$ be a simple connected graph with $|V| = n \geq 2$ and $|E| = m$. A subset of V is called a *stable set* if it does not contain adjacent vertices of G. Let N be a stable set. The *incidence vector* of N is $x \in \{0,1\}^V$ such that $x_v = 1$ if and only if $v \in N$. The *stable set polytope* of G, denoted by P_G, is the convex hull of incidence vectors of stable sets of G. Some well-known valid inequalities for P_G include the *trivial inequalities* ($x_v \geq 0$ for $v \in V$), the *cycle inequalities* ($\sum_{v\in C} x_v \leq k$ where C is the vertex-set of a cycle of length $2k+1$), and the *clique inequalities* ($\sum_{v\in S} x_v \leq 1$ where S induces a clique). A clique inequality is called an *edge inequality* if the clique has just two vertices. Papers that studied stable set

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polytopes include Chvátal [9], Fonlupt and Uhry [11], Gerards [12], Giles and Trotter [13], Mahjoub [16], Nemhauser and Trotter [17], Padberg [19], Tesch [20], Trotter [21] and Wolsey [22]; see also Grötschel, Lovász and Schrijver [14].

In this paper, we introduce a large class of valid inequalities, which we call "wheel inequalities." Our most basic class of inequalities are inequalities that have "simple 1-wheel configurations" (that is, subdivisions of wheels in which each face-cycle is odd) as their support graphs. In fact, each 1-wheel configuration gives two such valid inequalities; they are related in a simple way. Each of them is derived using the Chvátal-Gomory procedure [8]. Moreover, necessary and sufficient conditions for these inequalities to be facet-inducing when the underlying graph is a simple 1-wheel are given.

Our attack on the separation problem for the simple 1-wheel inequalities leads to a surprising phenomenon, namely, that we can solve the problem only by extending the class to a larger one that includes "non-simple 1-wheels." (Loosely speaking, non-simple 1-wheels are obtained from simple ones by identifying vertices.) We also consider the generalization of 1-wheels to p-wheels (wheels with a "hub" of size p). For one of the two classes, the necessary and sufficient conditions for p-wheel inequalities to be facet-inducing when the underlying graph is a simple p-wheel are the same as for $p = 1$, but for the other class this is not so. The separation problems for p-wheel inequalities can also be solved. Separation algorithms will be discussed only briefly here; for a more detailed discussion, we refer readers to Cheng and Cunningham [6]. The 1-wheel inequalities can be viewed as a special case of inequalities previously introduced in a polyhedral approach to the maximum 2-satisfiability problem; see Cheriyan *et al.* [7] and Cunningham *et al.* [10].

Throughout this paper, graph terminology follows that of Bondy and Murty [3]. In particular, a *walk* is a finite non-empty sequence $(v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$, whose terms are alternately vertices and edges such that the ends of e_i are v_{i-1} and v_i for $1 \le i \le k$. A *path* is a walk for which all the vertices are distinct. In a simple graph, we will list only the vertices of a walk. The degree of a vertex v is denoted by $\deg(v)$ and the set of vertices adjacent to v is denoted by $N(v)$.

2 1-Wheel Inequalities

Let k be a positive integer, and let $G_1 = (V_1, E_1)$ be a graph with $V_1 = \{v_0, v_1, v_2, \ldots, v_{2k+1}\}$ and $E_1 = \{(v_0, v_i), (v_i, v_{i+1}) : 1 \le i \le 2k+1\}$. (We take $v_{2k+2} = v_1$.) Consider a subdivision of G_1. Let $P_{0,i}$ and $P_{i,i+1}$ denote the paths obtained from (v_0, v_i) and (v_i, v_{i+1}) respectively through the subdivision. This graph is a *simple 1-wheel of size* $2k+1$ if the cycle C_i consisting of $P_{0,i}, P_{i,i+1}, P_{0,i+1}$ is odd for each i. We denote this simple 1-wheel by $W = W(v_0; v_1, v_2, \ldots, v_{2k+1})$. We call v_0 the *hub*, $P_{0,1}, P_{0,2}, \ldots, P_{0,2k+1}$ the *spokes*, $P_{1,2}, P_{2,3}, \ldots, P_{2k,2k+1}, P_{2k+1,1}$ the *rim-paths*, $v_1, v_2, \ldots, v_{2k+1}$ the *spoke-ends*, and the cycle consisting of $P_{1,2}, P_{2,3}, \ldots, P_{2k+1,1}$ the *rim*. Figure 1 gives an example of a simple 1-wheel where a is the hub and $\{b, c, d, g, i\}$ is the set of spoke-ends. We partition $\{v_1, v_2, \ldots, v_{2k+1}\}$ into two sets $\mathcal{E} = \mathcal{E}(W)$ and

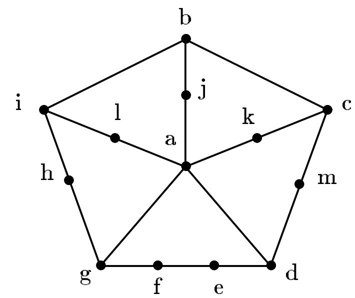

Figure 1: A simple 1-wheel

$\mathcal{O} = \mathcal{O}(W)$ where $v_i \in \mathcal{E}$ ($\mathcal{O}$) if $P_{0,i}$ is an even (odd) path. In Figure 1, $\mathcal{O} = \{d, g\}$ and $\mathcal{E} = \{b, c, i\}$. Let $S = S(W)$ be the set of internal vertices of the spokes and $R = R(W)$ be the set of internal vertices of the rim-paths. In Figure 1, $S = \{j, k, l\}$ and $R = \{m, e, f, h\}$.

An equivalent definition for simple 1-wheels is the following. Let $G_1 = (V_1, E_1)$ be a graph with $V_1 = \{v_0, v_1, v_2, \ldots, v_{2k+1}\}$ and $E_1 = \{(v_0, v_i), (v_i, v_{i+1}) : 1 \leq i \leq 2k+1\}$. Let $(\mathcal{E}, \mathcal{O})$ be a partition of $\{v_1, v_2, \ldots, v_{2k+1}\}$. Consider a subdivision of G_1. Let $P_{0,i}$ and $P_{i,i+1}$ denote the paths obtained from (v_0, v_i) and (v_i, v_{i+1}) respectively through the subdivision. This graph is a *simple 1-wheel of size* $2k+1$ if $v_i \in \mathcal{E}$ ($\mathcal{O}$) implies $P_{0,i}$ is an even (odd) path, and $P_{i,i+1}$ is an odd path if and only if v_i and v_{i+1} belong to the same class.

Let $W(v_0; v_1, \ldots, v_{2k+1})$ be a simple 1-wheel. Let C_i be the odd cycle consisting of $P_{0,i}, P_{i,i+1}, P_{0,i+1}$ for $i = 1, 2, \ldots, 2k+1$. Corresponding to a partition $(\mathcal{A}, \mathcal{B})$ of the edges of the rim (defined below), we derive valid inequalities for P_W as follows: (Note that for simplicity, we write $x_0, x_1, x_2, \ldots, x_{2k+1}$ instead of $x_{v_0}, x_{v_1}, x_{v_2}, \ldots, x_{v_{2k+1}}$.)

1. Calculate the sum of the sequence of cycle inequalities for C_i, for $i = 1, 2, \ldots, 2k+1$:

$$(2k+1)x_0 + 2\sum_{i=1}^{2k+1} x_i + 2\sum_{v \in S} x_v + \sum_{v \in R} x_v \leq |S| + |R|/2 + 2k + 1. \tag{1}$$

2. Add either $\sum_{(u,v)\in\mathcal{A}}(x_u + x_v \leq 1)$ or $\sum_{(u,v)\in\mathcal{B}}(x_u + x_v \leq 1)$ to (1).

3. Add either $-x_0 \leq 0$ or $x_0 \leq 1$ so that every coefficient is even and the right-hand side is odd.

4. Divide the resulting inequality by 2 and round down the right-hand side.

We define the partition $(\mathcal{A}, \mathcal{B}) = (\mathcal{A}(W), \mathcal{B}(W))$ of the edges of the rim according to the following rules for the path $P_{i,i+1}$ where $i \in \{1, 2, \ldots, 2k+1\}$:

1. If one end is in $\mathcal{E}$ and the other is in $\mathcal{O}$, then the path is even say $(u_1, \ldots, u_{2s+1})$ where $u_1 \in \mathcal{E}$; we put the edges $(u_1, u_2), (u_3, u_4), \ldots, (u_{2s-1}, u_{2s})$ into $\mathcal{A}$ and the edges $(u_2, u_3), (u_4, u_5), \ldots, (u_{2s}, u_{2s+1})$ into $\mathcal{B}$.

2. If both ends are in $\mathcal{E}$, then the path is odd say $(u_1, u_2, \ldots, u_{2s})$; we put $(u_1, u_2), (u_3, u_4), \ldots, (u_{2s-1}, u_{2s})$ into $\mathcal{A}$ and $(u_2, u_3), (u_4, u_5) \ldots, (u_{2s-2}, u_{2s-1})$ into $\mathcal{B}$.

3. If both ends are in $\mathcal{O}$, then the path is odd say $(u_1, u_2, \ldots, u_{2s})$; we put $(u_2, u_3), (u_4, u_5), \ldots, (u_{2s-2}, u_{2s-1})$ into $\mathcal{A}$ and $(u_1, u_2), (u_3, u_4), \ldots, (u_{2s-1}, u_{2s})$ into $\mathcal{B}$.

We note that $|R|$ is even since $\sum_{i=1}^{2k+1} |C_i| \equiv 1 \pmod 2$. Moreover, $|\mathcal{A}| = |\mathcal{E}| + |R|/2$ and $|\mathcal{B}| = |\mathcal{O}| + |R|/2$. Every $v \in \mathcal{E}$ is covered by exactly two elements in $\mathcal{A}$ but none in $\mathcal{B}$; every $v \in \mathcal{O}$ is covered by exactly two elements in $\mathcal{B}$ but none in $\mathcal{A}$. In addition, every vertex in R is covered by exactly one element in $\mathcal{A}$ and exactly one element in $\mathcal{B}$. (For example, in Figure 1, $\mathcal{A} = \{(b,i), (b,c), (c,m), (e,f), (h,i)\}$ and $\mathcal{B} = \{(d,m), (d,e), (f,g), (g,h)\}$.)

Now we get the two inequalities as follows:

- **Using $\mathcal{A}$:**

  We add $\sum_{e=(u,v)\in\mathcal{A}} (x_u + x_v \leq 1) \equiv 2\sum_{v\in\mathcal{E}} x_v + \sum_{v\in R} x_v \leq |\mathcal{E}| + |R|/2$ to (1). Hence we obtain

$$(2k+1)x_0 + 2\sum_{i=1}^{2k+1} x_i + 2\sum_{v\in\mathcal{E}} x_v + 2\sum_{v\in S\cup R} x_v \leq |S| + |R| + |\mathcal{E}| + 2k + 1. \tag{2}$$

  From the definition of $\mathcal{E}$, we see that $|S| + |\mathcal{E}|$ is even. Hence the right-hand side of (2) is odd. Therefore, adding $-x_0 \leq 0$ to (2), dividing by 2, and rounding down the right-hand side, we have

$$I_{\mathcal{A}}^W : \; kx_0 + \sum_{i=1}^{2k+1} x_i + \sum_{v\in\mathcal{E}} x_v + \sum_{v\in S\cup R} x_v \leq k + (|S| + |R| + |\mathcal{E}|)/2.$$

- **Using $\mathcal{B}$:**

  We add $\sum_{e=(u,v)\in\mathcal{B}} (x_u + x_v \leq 1) \equiv 2\sum_{v\in\mathcal{O}} x_v + \sum_{v\in R} x_v \leq |\mathcal{O}| + |R|/2$ to (1). One can check that the right-hand side of the resulting inequality is even; therefore, by adding $x_0 \leq 1$ to it and then dividing it by 2 (with rounding), we have

$$I_{\mathcal{B}}^W : \; (k+1)x_0 + \sum_{i=1}^{2k+1} x_i + \sum_{v\in\mathcal{O}} x_v + \sum_{v\in S\cup R} x_v \leq k + (|S| + |R| + |\mathcal{O}| + 1)/2.$$

We use $\mathcal{I}_{\mathcal{E}}^W$ instead of $\mathcal{I}_{\mathcal{A}}^W$ and $\mathcal{I}_{\mathcal{O}}^W$ instead of $\mathcal{I}_{\mathcal{B}}^W$. (The reason is that the coefficients of elements of $\mathcal{E}$ ($\mathcal{O}$) in $\mathcal{I}_{\mathcal{A}}^W$ ($\mathcal{I}_{\mathcal{B}}^W$) are 2.) These are the *1-wheel inequalities*. Moreover, we will use $\mathcal{I}_{\mathcal{E}}$ and $\mathcal{I}_{\mathcal{O}}$ instead of $\mathcal{I}_{\mathcal{E}}^W$ and $\mathcal{I}_{\mathcal{O}}^W$ if it is clear from the context what W is. For example, with respect to Figure 1, we have $\mathcal{I}_{\mathcal{E}} : 2x_a + 2x_b + 2x_c + 2x_i + \sum_{v\notin\{a,b,c,i\}} x_v \leq 7$ and $\mathcal{I}_{\mathcal{O}} : 3x_a + 2x_d + 2x_g + \sum_{v\notin\{a,d,g\}} x_v \leq 7$.

Theorem 2.1 *Let G be a graph and W be a simple 1-wheel that is a subgraph of G. Then the inequalities $\mathcal{I}_{\mathcal{E}}^W$ (that is, $\mathcal{I}_{\mathcal{A}}^W$) and $\mathcal{I}_{\mathcal{O}}^W$ (that is, $\mathcal{I}_{\mathcal{B}}^W$) are both valid for P_G.* □

Instances of 1-wheel inequalities have occurred repeatedly in the literature; see, for example, Chvátal [9], Grötschel, Lovász and Schrijver [14], and Barahona and Mahjoub [1]. Although up to now no general classes seem to have been defined, [14] page 301, does refer (without definition) to such a class. For $k = 1$, the 1-wheel reduces to an "odd-K_4" which is an important structure in the study of the stable set problem. For example, Barahona and Mahjoub [2] gave a complete polyhedral description for P_G where G is a simple 1-wheel of size 3. Gerards [12] gave a min-max relation for stable sets in graphs that do not contain odd-K_4 as a subgraph.

3 Separation Algorithms and General 1-Wheel Inequalities

In this section, we describe the basic approach to solving the separation problem for the class consisting of trivial, edge, cycle and 1-wheel inequalities. As noted above, this approach was introduced in the setting of the maximum 2-satisfiability problem by Cheriyan *et al.* [7]. Our purpose here is mainly to motivate the introduction of the "general 1-wheel inequalities", which arise naturally in the separation procedure. The full details of the procedure can be found in Cheng and Cunningham [6] or Cheng [5].

Given $G = (V, E)$ and $x \in \mathbf{R}^V$, the following notation is used throughout this section. $W = W(v_0; v_1, v_2, \ldots, v_{2k+1})$ denotes a 1-wheel of size $2k + 1$ and $w_e = (1 - x_u - x_v)/2$ for all $e = (u, v) \in E$. We also use w^* to denote w when x is replaced by x^*.

We first briefly describe a well-known algorithm for finding a minimum-weight odd cycle in graph having non-negative edge-weights c_e. (The first such algorithm is due to Grötschel and Pulleyblank [15].) Let $H = (V_H, E_H)$ be a simple graph and $c : E_H \longrightarrow \mathbf{R}_+$ be a vector of edge-weights. We can find a minimum-weight odd cycle in H as follows: Construct a bipartite graph $H' = (V^1 \cup V^2, E')$ where V^1 and V^2 are copies of V_H with (u^1, v^2) and (u^2, v^1) in E' if and only if (u, v) in E_H; moreover, $c'(u^1, v^2) = c'(u^2, v^1) = c(u, v)$. Hence a minimum-weight path (with respect to c') from v^1 to v^2 in H' corresponds to a minimum-weight odd closed walk (with respect to c) containing v^1 in H. Hence we can find a minimum-weight odd closed walk in H. However since c is nonnegative, we can find a minimum-weight odd cycle in H, since every odd closed walk contains a subsequence that is an odd cycle. Moreover, such an odd cycle can be found in $O(|V_H|^3)$ time. Furthermore, the same trick enables us to establish the following result. (We remark that a minimum-weight non-empty even walk from u to u can be assumed to be of length 2.)

Proposition 3.1 *Let $H = (V_H, E_H)$ be a graph. Then minimum-weight non-empty odd and even walks from u to v for every $u, v \in V_H$ ($u = v$ is possible) can be found in $O(|V_H|^3)$ time.* □

We now describe the well-known technique to solve the separation problem for the class of trivial, edge and cycle inequalities. Let $x^* \in \mathbf{R}^V$. We may assume x^* satisfies the trivial and edge inequalities. (They are easy to check, and knowing that they are not violated is important in what follows.) Suppose $C = (v_1, v_2, \ldots, v_{2k+1})$ is an odd cycle in G. Then $w^*(C) = k + 1/2 - \sum_{i=1}^{2k+1} x_i^*$. (Recall that $w_e^* =$

$(1 - x_u^* - x_v^*)/2$ for all $e = (u, v) \in E$.) Hence x^* violates the cycle inequality corresponding to C if and only if $w^*(C) < 1/2$. Therefore a most-violated cycle inequality corresponds to an odd cycle in G having minimum weight (with respect to w^*). Thus the separation problem for the class consisting of the trivial, edge and cycle inequalities can be solved in $O(n^3)$ time. Hence to solve our problem in polynomial time, it is sufficient to answer the following question in polynomial time: Given $x^* \in \mathbf{R}^V$ satisfying the trivial, edge and cycle inequalities, does x^* violate an inequality of the form $\mathcal{I}_{\mathcal{E}}$ or $\mathcal{I}_{\mathcal{O}}$? (We use this terminology to mean does x^* violate $\mathcal{I}_{\mathcal{E}}^W$ or $\mathcal{I}_{\mathcal{O}}^W$ for some 1-wheel W.) So throughout this section, we assume x^* satisfies the trivial, edge, and cycle inequalities.

Let C be an odd cycle. Then the cycle inequality can be written as

$$\sum_{e \in C} w_e \geq \frac{1}{2}.$$

(To see this, observe that $\sum_{i=1}^{2k+1} x_i \leq k$ can be rewritten as $(x_1 + x_2)/2 + (x_2 + x_3)/2 + \cdots + (x_{2k} + x_{2k+1})/2 + (x_{2k+1} + x_1)/2 \leq k$ and the left-hand side has $2k+1$ terms.) Let W be a 1-wheel of size $2k+1$. Then following the procedure in Section 2 in deriving $\mathcal{I}_{\mathcal{E}}^W$ and using the fact that $\sum_{e=(u,v)\in\mathcal{A}}(x_u + x_v \leq 1)$ is $2\sum_{e\in\mathcal{A}} w_e \geq 0$, we have

$$\mathcal{I}_{\mathcal{E}}^W : \frac{1}{2}\sum_{i=1}^{2k+1}\left\{\sum_{e\in C_i} w_e - \frac{1}{2}\right\} + \sum_{e\in\mathcal{A}} w_e + \frac{1}{2}x_0 \geq +\frac{1}{2}$$

where C_i is the odd cycle defined earlier. Hence $\mathcal{I}_{\mathcal{E}}^W$ can be rewritten as

$$\mathcal{I}_{\mathcal{E}}^W : \left(2\sum_{i=1}^{2k+1} w(P_{0,i}) + \sum_{i=1}^{2k+1} w(P_{i,i+1}) - \frac{1}{2}(2k+1)\right) + 2\sum_{e\in\mathcal{A}} w_e \geq 1 - x_0. \quad (3)$$

For simplicity, the notation $g(v) = 1/4 - x_v/2$ and

$$f(v) = \begin{cases} 1/4 - x_v/2 & \text{if } v \in \mathcal{E} \\ -1/4 + x_v/2 & \text{if } v \in \mathcal{O} \end{cases}$$

for $v \in \mathcal{E} \cup \mathcal{O}$ is used in the rest of this section. We also use g^*, f^* to denote g, f when x is replaced by x^*.

Lemma 3.2

$$2\sum_{e\in\mathcal{A}\cap P_{i,i+1}} w_e = \sum_{e\in P_{i,i+1}} w_e + f(v_i) + f(v_{i+1}) \textit{ for } i = 1, 2, \ldots, 2k+1.$$

Proof: This follows from the definitions of w_e and $\mathcal{A}$. □

By applying Lemma 3.2 to (3), we have the following result.

Theorem 3.3 *Suppose $F_i = f(v_i) + f(v_{i+1})$. Then $\mathcal{I}_{\mathcal{E}}^W$ can be rewritten as*

$$\mathcal{I}_{\mathcal{E}}^W : \left(2\sum_{i=1}^{2k+1} w(P_{0,i}) + 2\sum_{i=1}^{2k+1} w(P_{i,i+1}) - \frac{1}{2}(2k+1)\right) + \sum_{i=1}^{2k+1} F_i \geq 1 - x_0. \quad (4)$$

□

Suppose that we know the hub is v_0 and the set of spoke-ends is $\{v_1, v_2, \ldots, v_{2k+1}\}$, and we want to find a most-violated inequality $\mathcal{I}_{\mathcal{E}}^{W}$ corresponding to them. Then we know the parities of the paths $P_{0,1}, P_{0,2}, \ldots, P_{0,2k+1}, P_{1,2}, P_{2,3}, \ldots, P_{2k+1,1}$. So by Theorem 3.3, we need to find paths of the specified parities having minimum *total* weight such that they are *internally disjoint*. This problem is difficult. We relax the conditions by allowing the paths to be walks, and allowing them to intersect. Then the problem becomes easy, because each of the walks can be chosen to be of least weight. It is fortunate that this relaxation corresponds to a generalization of simple 1-wheel inequalities. In fact, this class can be introduced in another way, as inequalities obtainable from simple 1-wheel inequalities by a sequence of vertex-identifications.

Proposition 3.4 *Let $\sum_{i=1}^{n} a_i x_i \leq b$ be a valid inequality for P_G and let v_1 and v_2 be two nonadjacent vertices of G. If H is obtained from G by identifying v_1 and v_2 where the vertex v_2 of H is obtained from the identification of v_1 and v_2 of G, then $(a_1 + a_2)x_2 + \sum_{i=3}^{n} a_i x_i \leq b$ is a valid inequality for P_H.*

Proof: This follows from the fact that $(x_1^*, x_2^*, x_3^*, \ldots, x_n^*)^T$ is an incidence vector of a stable set of G whenever $(x_2^*, x_3^*, \ldots, x_n^*)^T$ is an incidence vector of a stable set of H. □

We assume that after an identification, any duplicate edge is deleted. From now on, whenever we identify two vertices, we assume, without saying so explicitly, that the two vertices are not adjacent. Let H be a graph and H' be the graph obtained from H by a sequence of identifications of vertices. By applying Proposition 3.4 repeatedly, a valid inequality for P_H provides a valid inequality for $P_{H'}$. It is clear that such a sequence of identifications of vertices induces a partition of the vertices of H such that H' is obtained from H by identifying vertices in the same class of the partition.

Suppose $W(v_0; v_1, v_2, \ldots, v_{2k+1})$ is a simple 1-wheel and $\mathcal{P}$ is a partition of $V(W)$. Then we define a *(general) 1-wheel* $W'(v_0; v_1, v_2, \ldots, v_{2k+1})$ to be the graph obtained from W by identifying the vertices in the same class of $\mathcal{P}$. We note that a spoke or a rim-path may actually be a walk in a general 1-wheel. More importantly, the spokes and rim-paths may intersect other than at their ends. Figure 2 is a 1-wheel obtained from Figure 1 by identifying j and l, and by identifying g and k . For a general 1-wheel, S, R, $\mathcal{E}$ and $\mathcal{O}$ are treated as multisets. (For example, $S = \{j, j, g\}$ and $R = \{m, e, f, h\}$ for Figure 2, so $|S| = 3$ and $|R| = 4$.) We also remark that it is clear that none of the rim-paths and spokes is empty.

Given a general 1-wheel $W(v_0; v_1, \ldots, v_{2k+1})$, we assume that the hub, the spoke-ends, the spokes and the rim-paths are explicitly given, so there is no confusion as to which simple 1-wheel W is obtained from. Of course, a hub or spoke-end may also be an internal vertex of some spoke or some rim-path. For example, in the graph of Figure 2, a is the hub, b, c, d, g and i are the spoke-ends, $(a, j, b), (a, g, c), (a, d), (a, g), (a, j, i)$ are the spokes and $(b, c), (c, m, d), (d, e, f, g), (g, h, i), (i, b)$ are the rim-paths. Moreover, g serves as a spoke-end and as an internal vertex of the spoke (a, g, c). We remark that another way to see that general 1-wheel inequalities are valid without using Proposition 3.4 is to observe that the Chvátal-Gomory derivation works

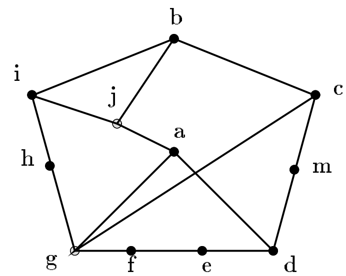

Figure 2: A non-simple 1-wheel

for general 1-wheels; of course sets are treated as multisets during this procedure. Hence Theorem 2.1 is true for general 1-wheels. Therefore, the following result follows from (4).

Theorem 3.5 *Let* W *determine an inequality of the form* $\mathcal{I}_{\mathcal{E}}$ *that is most-violated by* x^*. *Then every spoke and rim-path is a minimum-weight walk with respect to* w^* *of its parity joining its ends.* □

Motivated by Theorem 3.5, we compute, for each $u, v \in V$, the minimum weight with respect to w^* of an even (odd) walk from u to v in G. (u and v may be the same.) We denote this minimum by $w^*_E(u,v)$ ($w^*_O(u,v)$). To solve our problem for $\mathcal{I}_{\mathcal{E}}$, it is enough to find an algorithm for finding a most-violated inequality of the form $\mathcal{I}_{\mathcal{E}}$ with some specific vertex, say v_0, as the hub. We construct an auxiliary graph $H = (V_H, E_H)$ from $G = (V, E)$ as follows: H is a complete graph with loops where $V_H = V^{\mathcal{E}} \cup V^{\mathcal{O}}$, and $V^{\mathcal{E}}$ and $V^{\mathcal{O}}$ are copies of V. If $a \in V^{\mathcal{E}}$ ($V^{\mathcal{O}}$) is a copy of b, then b is denoted by α_a. A vertex in $V^{\mathcal{E}}$ represents a potential even spoke-end and a vertex in $V^{\mathcal{O}}$ represents a potential odd spoke-end. For simplicity, we let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ denote the respective statements $u, v \in V^{\mathcal{E}}$, $u, v \in V^{\mathcal{O}}$ and $u \in V^{\mathcal{E}}, v \in V^{\mathcal{O}}$. For any $(u, v) \in E_H$, we define $w_H(u,v)$, the edge-weight for (u,v), to be

$$\begin{cases} w^*_E(v_0,\alpha_u) + w^*_E(v_0,\alpha_v) + 2w^*_O(\alpha_u,\alpha_v) - 1/2 + g^*(\alpha_u) + g^*(\alpha_v) & \text{if } \mathcal{T}_1 \\ w^*_O(v_0,\alpha_u) + w^*_O(v_0,\alpha_v) + 2w^*_O(\alpha_u,\alpha_v) - 1/2 - g^*(\alpha_u) - g^*(\alpha_v) & \text{if } \mathcal{T}_2 \\ w^*_E(v_0,\alpha_u) + w^*_O(v_0,\alpha_v) + 2w^*_E(\alpha_u,\alpha_v) - 1/2 + g^*(\alpha_u) - g^*(\alpha_v) & \text{if } \mathcal{T}_3. \end{cases}$$

Then it follows from (4) that an odd closed walk of length at least 3 in H induces a 1-wheel with v_0 as the hub. Hence, it follows from Theorem 3.5 and the definition of $w_H(u,v)$ that we can find a most violated $\mathcal{I}_{\mathcal{E}}$ with v_0 as the hub by finding a minimum-weight odd closed walk of length at least 3 in H.

Lemma 3.6 $w_H(u,v) \geq 0$ *for all* $(u,v) \in E_H$.

Proof: Let $P^*_{a,b,e}$ be a minimum-weight even walk from a to b in G with respect to w^* and $P^*_{a,b,o}$ be a minimum-weight odd walk from a to b in G with respect to w^*. Then by using Lemma 3.2, we can rewrite

$w_H(u,v)$ as

$$\begin{cases} (w^*(P^*_{0,\alpha_u,e}) + w^*(P^*_{0,\alpha_v,e}) + w^*(P^*_{\alpha_u,\alpha_v,o}) - 1/2) + \left(2 \sum_{e \in \mathcal{A} \cap P^*_{\alpha_u,\alpha_v,o}} w^*_e\right) & \text{if } \mathcal{T}_1 \\ (w^*(P^*_{0,\alpha_u,o}) + w^*(P^*_{0,\alpha_v,o}) + w^*(P^*_{\alpha_u,\alpha_v,o}) - 1/2) + \left(2 \sum_{e \in \mathcal{A} \cap P^*_{\alpha_u,\alpha_v,o}} w^*_e\right) & \text{if } \mathcal{T}_2 \\ (w^*(P^*_{0,\alpha_u,e}) + w^*(P^*_{0,\alpha_v,o}) + w^*(P^*_{\alpha_u,\alpha_v,e}) - 1/2) + \left(2 \sum_{e \in \mathcal{A} \cap P^*_{\alpha_u,\alpha_v,e}} w^*_e\right) & \text{if } \mathcal{T}_3. \end{cases}$$

In each case, the second summand is nonnegative since x^* satisfies the trivial and edge inequalities. Moreover, the first summand is nonnegative since x^* satisfies the cycle inequalities. □

The above discussion suggests that in order to determine whether any 1-wheel inequality of the form $\mathcal{I}_\mathcal{E}$ is violated by x^*, we should compare $w^*_H(C)$ to $1 - x^*_0$ where C is a minimum-weight cycle in the auxiliary graph. However, a cycle of length 1 does not correspond to a 1-wheel. The next lemma shows that this does not cause a problem.

Lemma 3.7 *$w_H(e) \geq 1 - x^*_0$ for any loop e in the auxiliary graph.*

Proof: Suppose $e = (u^\mathcal{E}, u^\mathcal{E})$ where $u \in V$. (u and v_0 may be the same.) If the claim is not true, then we must have

$$2w^*_E(v_0, u) + 2w^*_O(u,u) - \frac{1}{2} + 2g^*(u) < 1 - x^*_0.$$

Since $g^*(u) = 1/4 - x^*_u/2$ and $w^*_O(u,u) \geq 1/2$ (because x^* satisfies the cycle inequalities), we have

$$2w^*_E(v_0, u) + x^*_0 - x^*_u < 0.$$

Suppose the walk $(v_0, y_1, y_2, \ldots, y_{2l+1}, u)$ has weight $w^*_E(v_0, u)$. Then

$$2w^*_E(v_0, u) + x^*_0 - x^*_u = (2l+2) - 2x^*_{y_1} - 2x^*_{y_2} - \cdots - 2x^*_u,$$

so $x^*_{y_1} + x^*_{y_2} + \cdots + x^*_{y_{2l+1}} + x^*_u > l+1$. However, if we add up the edge inequalities for the $l+1$ edges $(y_1, y_2), (y_3, y_4), \ldots, (y_{2k-1}, y_{2k}), (y_{2l+1}, u)$, then we have $x^*_{y_1} + x^*_{y_2} + \cdots + x^*_{y_{2l+1}} + x^*_u \leq l+1$, a contradiction. The case where $e = (u^\mathcal{O}, u^\mathcal{O})$ is similar. □

Theorem 3.8 *If C is a minimum-weight odd cycle in H, then x^* violates no inequality of the form $\mathcal{I}_\mathcal{E}$ with v_0 as the hub if and only if $w^*_H(C) \geq 1 - x^*_0$.* □

Corollary 3.9 *The separation problem for the class consisting of trivial, edge, cycle inequalities and the inequalities of the form $\mathcal{I}_\mathcal{E}$ (both simple and non-simple) can be solved in $O(n^4)$ time.* □

The separation problem for the class consisting of trivial, edge, cycle and general 1-wheel inequalities of the form $\mathcal{I}_{\mathcal{O}}$ can be solved similarly. For a more detailed discussion of separation problems for 1-wheel inequalities, see Cheng and Cunningham [6] or Cheng [5]. (The class of inequalities for which the separation problem is solved in [6] is slightly smaller, but the two separation problems are equivalent.)

We introduced general 1-wheels to handle difficulties arising from requiring paths to be disjoint. However, the generalization is more than a mere device for solving the separation problem. It is possible that a violated general 1-wheel inequality exists, when there is no violated simple 1-wheel inequality. This follows from the fact that there are non-simple 1-wheel inequalities that induce facets for P_G. For example, $\mathcal{I}_{\mathcal{E}}$ is facet-inducing for the graph obtained by identifying a and b in Figure 3.

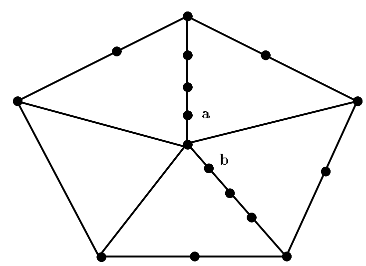

Figure 3: $\mathcal{I}_{\mathcal{E}}$ is facet-inducing for the graph obtained by identifying a and b

4 Facet-inducing 1-Wheel Inequalities

We consider the question of whether a given simple 1-wheel inequality corresponding to a simple 1-wheel W is facet-inducing for P_G. Unfortunately, the question in general seems to be difficult. (We discuss this at the end of the section.) On the other hand, for the case in which $W = G$, we have the following complete answer.

Theorem 4.1 *Let W be a simple 1-wheel. Then $\mathcal{I}_{\mathcal{E}}^W$ is facet-inducing for P_W if and only if every rim-path joining two elements of $\mathcal{E}$ has length at least 2 (and hence at least 3).*

Theorem 4.2 *Let W be a simple 1-wheel. Then $\mathcal{I}_{\mathcal{O}}^W$ is facet-inducing for P_W if and only if every rim-path joining two elements of $\mathcal{O}$ has length at least 2 (and hence at least 3) and every spoke of odd length has length at least 3.*

Theorem 4.1 and Theorem 4.2 can be combined into the following statement.

Theorem 4.3 *Let W be a simple 1-wheel. Let $a^T x \leq b$ be $\mathcal{I}_{\mathcal{E}}^W$ or $\mathcal{I}_{\mathcal{O}}^W$. Then $a^T x \leq b$ is facet-inducing for P_W if and only $\min\{a_u, a_v\} = 1$ for every edge (u, v) of W.* $\square$

It is easy to see that there exist 1-wheels for which $\mathcal{I}_{\mathcal{E}}$, $\mathcal{I}_{\mathcal{O}}$, both, or neither are facet-inducing. We prove Theorem 4.2 in this section and delay the proof of Theorem 4.1 until Section 6, when it is generalized. The proof uses the well-known approach of obtaining a facet-inducing inequality from another by replacing an edge by a path of length 3. The first step is to show that the conditions in Theorem 4.2 are necessary.

Lemma 4.4 *If $\mathcal{I}_{\mathcal{O}}^{W}$ is facet-inducing for P_W, then every rim-path joining two elements of $\mathcal{O}$ has length at least 2 (and hence at least 3) and every spoke of odd length has length at least 3.*

Proof: Suppose the first condition is not satisfied. Say the length of $P_{i,i+1}$ is 1 for some $i \in \{1, 2, \ldots, 2k+1\}$. Let C_i be the cycle consisting of $P_{i,i+1}, P_{0,i}, P_{0,i+1}$ and let I_{C_i} be the corresponding cycle inequality. Assume $k \geq 2$, and let $\overline{W}$ be the subgraph obtained from W by deleting the internal vertices and edges of $P_{0,i}$ and $P_{0,i+1}$. Then $\overline{W}$ is a 1-wheel of size $2k-1$. Its hub is v_0 and its set of spoke-ends is $\{v_1, v_2, \ldots, v_{2k+1}\} \setminus \{v_i, v_{i+1}\}$. We note that v_i and v_{i+1} are internal vertices of the rim-path $P^{\overline{W}}_{i-1,i+2}$. (Observe that in W, this is the path $(P^W_{i-1,i}, P^W_{i,i+1}, P^W_{i+1,i+2})$; that is, the path $P^W_{i-1,i}$ is followed immediately by $P^W_{i,i+1}$ and then by $P^W_{i+1,i+2}$.) Then we have

$$\mathcal{I}_{\mathcal{O}}^{W} = \mathcal{I}_{\mathcal{O}}^{\overline{W}} + I_{C_i}.$$

Suppose $k = 1$. Without loss of generality, we may assume $i = 1$. If $v_3 \in \mathcal{E}$, then let I_Q be the sum of the edge inequalities

$$x_{u_1} + x_{u_2} \leq 1, x_{u_3} + x_{u_4} \leq 1, \ldots, x_{2l+1} + x_{v_0} \leq 1$$

where $P_{0,3}$ is $(v_3, u_1, u_2, \ldots, u_{2l+1}, v_0)$; otherwise (that is, $v_3 \in \mathcal{O}$) let I_Q be the sum of the edge inequalities

$$x_{v_3} + x_{u_1} \leq 1, x_{u_2} + x_{u_3} \leq 1, \ldots, x_{2l} + x_{v_0} \leq 1$$

where $P_{0,3}$ is $(v_3, u_1, u_2, \ldots, u_{2l}, v_0)$. Let I_C be the cycle inequality for the rim (that is, the cycle consisting of $P_{1,2}, P_{2,3}$ and $P_{3,1}$) of W. Then

$$\mathcal{I}_{\mathcal{O}}^{W} = I_C + I_{C_i} + I_Q.$$

Now suppose the second condition is not satisfied. Without loss of generality, assume the length of $P_{0,1}$ is 1. Let C_i be the cycle consisting of $P_{0,i}, P_{i,i+1}$ and $P_{0,i+1}$ and I_{C_i} be its cycle inequality for $i = 1, 2, \ldots, 2k+1$. Then we let I_1 denote the inequality

$$I_{C_1} + I_{C_3} + \cdots + I_{C_{2k-1}} + I_{C_{2k+1}}.$$

We note that although the path $P_{0,1}$ is on both C_1 and C_{2k+1}, every internal vertex of the spokes still has coefficient 1 in I_1 since $P_{0,1}$ has no internal vertex. Now for each $P_{2i,2i+1}$, $1 \leq i \leq k$, we let I_2 denote the inequality

$$\sum_{i=1}^{k} \left(\sum_{e=(a,b) \in \mathcal{B}(W) \cap E(P_{2i,2i+1})} (x_a + x_b \leq 1) \right).$$

Then it is not difficult to check that

$$\mathcal{I}_{\mathcal{O}}^{W} = I_1 + I_2.$$

Hence we are done. □

We use the following strategy to prove the conditions in Theorem 4.2 are sufficient.

- *Step 1:* Given $v_0, v_1, \ldots, v_{2k+1}$, suppose we want v_0 to be the hub and $v_1, \ldots, v_{2k+1}$ to be the spoke-ends of a 1-wheel. Furthermore, suppose we fix a subset of $\{v_1, v_2, \ldots, v_{2k+1}\}$ to be $\mathcal{O}$. Then there exists a unique 1-wheel that satisfies the conditions in Theorem 4.2 with minimum number of vertices. We denote such a minimal wheel by

$$W_{\mathcal{B}}(v_0; v_1, \ldots, v_{2k+1}; \mathcal{O}).$$

  We call $W_{\mathcal{B}}$ the *base wheel for* $(v_0; v_1, \ldots, v_{2k+1})$ *and* $\mathcal{O}$ or simply the *base wheel.* For example, suppose the prescribed data are: the hub is v_0, the spoke-ends are v_1, v_2, v_3, v_4 and v_5, and $\mathcal{O} = \{v_1, v_2, v_4\}$. Then the base wheel for these data is shown in Figure 4. (Note that for simplicity, the vertices in the figure are labelled i instead of v_i.)

- *Step 2:* We prove that for any base wheel $W_{\mathcal{B}}$, $\mathcal{I}_{\mathcal{O}}^{W_{\mathcal{B}}}$ is facet-inducing for $P_{W_{\mathcal{B}}}$.

- *Step 3:* Let W be a wheel for which $\mathcal{I}_{\mathcal{O}}^{W}$ is facet-inducing for P_W and let W' be a wheel obtained from W by replacing an edge (a, b) of W by a path of length 3. Then we prove $\mathcal{I}_{\mathcal{O}}^{W}$ is facet-inducing for $P_{W'}$.

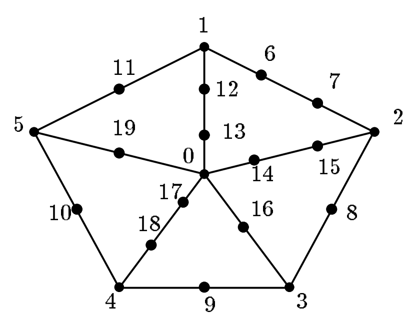

Figure 4: A base wheel

Let $W_{\mathcal{B}}$ be the base wheel for the data $(v_0; v_1, \ldots, v_{2k+1})$ and a prescribed $\mathcal{O}$. Then $W_{\mathcal{B}}$ has the following properties:

1. $v_i \in \mathcal{E}$ implies $P_{0,i}$ is of length 2 for $i \in \{1, 2, \ldots, 2k+1\}$;

2. $v_i \in \mathcal{O}$ implies $P_{0,i}$ is of length 3 for $i \in \{1, 2, \ldots, 2k+1\}$;

3. $v_i, v_{i+1} \in \mathcal{E}$ implies $P_{i,i+1}$ is of length 1 for $i \in \{1, 2, \ldots, 2k+1\}$;

4. $v_i, v_{i+1} \in \mathcal{O}$ implies $P_{i,i+1}$ is of length 3 for $i \in \{1, 2, \ldots, 2k+1\}$; and

5. $|\{v_i, v_{i+1}\} \cap \mathcal{E}| = 1$ implies $P_{i,i+1}$ is of length 2 for $i \in \{1, 2, \ldots, 2k+1\}$.

Hence, since $|S(W_{\mathcal{B}})| = |\mathcal{E}| + 2|\mathcal{O}|$ and $|R(W_{\mathcal{B}})| = 2|\mathcal{O}|$, $\mathcal{I}_{\mathcal{O}}^{W_{\mathcal{B}}}$ can be simplified to

$$\mathcal{I}_{\mathcal{O}}^{W_{\mathcal{B}}} : (k+1)x_0 + \sum_{v \in \mathcal{E}} x_v + 2\sum_{v \in \mathcal{O}} x_v + \sum_{v \in S \cup R} x_v \leq 2k + 1 + 2|\mathcal{O}|.$$

We consider two types of stable sets of $W_{\mathcal{B}}$ that satisfy $\mathcal{I}_{\mathcal{O}}^{W_{\mathcal{B}}}$ with equality.

- **Type 1:** Let $N = X \cup Y$ where X is the union of $\{v_0\}$ and a stable set of the rim, C, of $W_{\mathcal{B}}$ of size $(|C| - 1)/2$, and Y is constructed as follows: Set $Y = \emptyset$; for each $v_i \in \mathcal{O}$ where $i \in \{1, \ldots, 2k+1\}$ and $v_i \notin X$, adjoin the internal vertex of $P_{0,i}$ that is a neighbour of v_i to Y.

- **Type 2:** Let N be the union of $\mathcal{O}$ and the set of neighbours of v_0.

It is easy to see that both constructions yield stable sets satisfying $\mathcal{I}_{\mathcal{O}}^{W_{\mathcal{B}}}$ with equality. For example, let $W_{\mathcal{B}}$ be the base wheel in Figure 4. Then we have $\mathcal{I}_{\mathcal{O}}^{W_{\mathcal{B}}} : 3x_0 + 2x_1 + 2x_2 + 2x_4 + x_3 + x_5 + \sum_{i=6}^{19} x_i \leq 11$; $\{0, 1, 7, 8, 9, 10, 15, 18\}$ and $\{0, 6, 2, 3, 4, 5, 12\}$ are both Type 1 stable sets, and $\{1, 2, 4, 13, 14, 16, 17, 19\}$ is the unique Type 2 stable set.

Lemma 4.5 *Let $W_{\mathcal{B}}$ be the base wheel for the data $(v_0; v_1, \ldots, v_{2k+1})$ and a prescribed $\mathcal{O}$. Then $\mathcal{I}_{\mathcal{O}}^{W_{\mathcal{B}}}$ is facet-inducing for $P_{W_{\mathcal{B}}}$.*

Proof: Suppose the face induced by $\mathcal{I}_{\mathcal{O}}^{W_{\mathcal{B}}}$ is a subset of a facet induced by some valid inequality $a^T x \leq b$. Our goal is to show that $a^T x \leq b$ is a constant multiple of $\mathcal{I}_{\mathcal{O}}^{W_{\mathcal{B}}}$. Let (v, w) be an edge of the rim where $v, w \notin \mathcal{O}$. (It is clear that such (v, w) must exist.) Let (u_1, u_2, w, v) be the length 3 subpath of the rim, containing w and v, and having v as an end of the subpath. Let N be the Type 1 stable set of $W_{\mathcal{B}}$ with $u_1, v \in N$. Now $N' = (N \setminus \{v\}) \cup \{w\}$ is also a Type 1 stable set of $W_{\mathcal{B}}$. Since any stable set of $W_{\mathcal{B}}$ that satisfies $\mathcal{I}_{\mathcal{O}}^{W_{\mathcal{B}}}$ with equality will also satisfy $a^T x \leq b$ with equality, we can conclude that $a_v = a_w$. (For example, in Figure 4, suppose we pick v to be v_{10} and w to be v_5. Then the two Type 1 stable sets $\{0, 10, 1, 7, 8, 9, 15, 18\}$ and $\{0, 5, 1, 7, 8, 9, 15, 18\}$ imply $a_5 = a_{10}$.)

Let $v_i \in \mathcal{O}$ and v, w be the two vertices of the rim that are neighbours of v_i. We note that $v, w \notin \mathcal{O}$ since $W_{\mathcal{B}}$ is a base wheel. Let (u_1, u_2, v, v_i) be the length 3 subpath of the rim, containing v and v_i, and having v_i as an end of the subpath. Let z be the internal vertex of $P_{0,i}$ that is a neighbour of v_i. Let N be the Type 1 stable set of $W_{\mathcal{B}}$ with $u_1, v_i \in N$. Now $N' = (N \setminus \{v_i\}) \cup \{v, z\}$ is also a Type 1 stable set of $W_{\mathcal{B}}$. Hence $a_{v_i} = a_z + a_v$. (For example, in Figure 4, consider v_4 (which is in $\mathcal{O}$) and choose v to be v_{10}, so w is v_9. Then the stable sets $\{0, 4, 11, 6, 2, 3, 12\}$ and $\{0, 10, 18, 11, 6, 2, 3, 12\}$ imply $a_4 = a_{10} + a_{18}$.)

Similarly, we have $a_{v_i} = a_z + a_w$; therefore $a_v = a_w$. We can now conclude that $a_{v'} = a_{w'} = d_0$ for all v' and w' on the rim that are not in $\mathcal{O}$, for some $d_0 \in \mathbf{R}$.

Let $v_i \in \mathcal{O}$ and v, w be the two vertices of the rim that are neighbours of v_i. Let N be the Type 2 stable set of $W_\mathcal{B}$. Then $N' = (N \setminus \{v_i\}) \cup \{v, w\}$ is also a stable set of $W_\mathcal{B}$ that satisfies $\mathcal{I}_\mathcal{O}^{W_\mathcal{B}}$ with equality. Hence $a_{v_i} = 2d_0$. (For example, in Figure 4, the stable sets $\{1, 2, 4, 13, 14, 16, 17, 19\}$ and $\{6, 11, 2, 4, 13, 14, 16, 17, 19\}$ will now imply $a_1 = a_6 + a_{11} = 2d_0$.) Moreover, let $P_{0,i}$ be (v_0, y, z, v_i). We note that $y \in N'$. Then $N'' = (N' \setminus \{y\}) \cup \{z\}$ is a stable set of $W_\mathcal{B}$ that satisfies $\mathcal{I}_\mathcal{O}^{W_\mathcal{B}}$ with equality. Hence $a_y = a_z$. (For example, in Figure 4, the stable sets $\{6, 11, 2, 4, 13, 14, 16, 17, 19\}$ and $\{6, 11, 2, 4, 12, 14, 16, 17, 19\}$ imply $a_{12} = a_{13}$.) Now, we also know $a_{v_i} = a_z + d_0$; hence $a_z = d_0$. Therefore, $a_z = a_y = d_0$.

Now consider $v_i \in \mathcal{E}$. Let N be the Type 2 stable set of $W_\mathcal{B}$ and y be the internal vertex of $P_{0,i}$. Then $N' = (N \setminus \{y\}) \cup \{v_i\}$ is also a stable set of $W_\mathcal{B}$ that satisfies $\mathcal{I}_\mathcal{O}^{W_\mathcal{B}}$ with equality. Hence $a_y = a_{v_i} = d_0$. (For example, in Figure 4, the stable sets $\{1, 2, 4, 13, 14, 16, 17, 19\}$ and $\{1, 2, 4, 13, 14, 16, 17, 5\}$ will now imply $a_{19} = a_5 = d_0$.)

Now by taking a Type 1 stable set of $W_\mathcal{B}$ and the Type 2 stable set for $W_\mathcal{B}$, we have $a_0 = (k+1)d_0$. Hence $a^T x \leq b$ is a constant multiple of $\mathcal{I}_\mathcal{O}^{W_\mathcal{B}}$, and we are done. □

To extend the result to any simple 1-wheel, we use the following special form of a result of Wolsey [22]. (Actually, the proof of Lemma 4.5 can be modified to prove the general case without using Lemma 4.6.)

Lemma 4.6 *Let $G = (V, E)$ be a graph and $c^T x \leq d$ $(c \geq 0,\ d > 0)$ be facet-inducing for P_G. Let $(a, b) \in E$ with $c_a \geq c_b = \gamma$. Let T be the set of incidence vectors of stable sets of G that satisfy $c^T x \leq d$ with equality. Let G' be the graph obtained from G by replacing (a, b) by the path (a, y, z, b) where $y, z \notin V$. Suppose that*

1. *there exists $s \in T$ such that $s_a = s_b = 0$, and*
2. *there exists $s \in T$ such that $s_a = 1$, $s_b = 0$ and $s_h = 0$ for all $h \in N(b) \setminus \{a\}$.*

Then $c^T x + \gamma x_y + \gamma x_z \leq d + \gamma$ is facet-inducing for $P_{G'}$.

We note that if $c^T x \leq d$ is not an edge inequality, then condition 1 in Lemma 4.6 is automatically satisfied. Furthermore, if $\deg(b) = 2$ then condition 2 in Lemma 4.6 is also satisfied.

Now suppose that W is a 1-wheel and W' is obtained from W by replacing an edge (a, b) of W by (a, y, z, b). If $c^T x \leq d$ is the 1-wheel inequality $\mathcal{I}_\mathcal{O}^W$, then $c^T x + x_y + x_z \leq d + 1$ is the 1-wheel inequality $\mathcal{I}_\mathcal{O}^{W'}$. Moreover, given any $W(v_0; v_1, v_2, \ldots, v_{2k+1})$, W is obtainable from W_B, the corresponding base wheel, by successively replacing an edge with a path of length 3. Hence, it is enough to prove the following: If every edge in W satisfies the hypotheses of Lemma 4.6 with respect to $\mathcal{I}_\mathcal{O}^W$, then every edge in W' satisfies the hypotheses of Lemma 4.6 with respect to $\mathcal{I}_\mathcal{O}^{W'}$.

Lemma 4.7 *Let H be a graph. Suppose H' is obtained from H by replacing an edge (a, b) by (a, y, z, b). If $c^T x \leq d$ is not an edge inequality and is facet-inducing for P_H with $c \geq 0$, $c_a \geq c_b = \gamma$ and $d > 0$, and* *every edge (a', b') satisfies the conditions of Lemma 4.6 with respect to $c^Tx \leq d$, then $c^Tx + \gamma x_y + \gamma x_z \leq d + \gamma$ is facet-inducing for $P_{H'}$ and every edge in H' satisfies the conditions of Lemma 4.6 with respect to $c^Tx + \gamma x_y + \gamma x_z \leq d + \gamma$.*

Proof: It follows from Lemma 4.6 that $c^Tx+\gamma x_y+\gamma x_z \leq d+\gamma$ is facet-inducing for $P_{H'}$. Since $c^Tx \leq d$ is not an edge inequality, we only have to check condition 2. Clearly any edge of the path (a, y, z, b) satisfies condition 2 in Lemma 4.6 since $\deg(y) = \deg(z) = 2$ and $c_a \geq c_b = \gamma$. Now suppose $(a_1, b_1) \in E(H) \cap E(H')$. Since (a_1, b_1) (with respect to H) satisfies the hypotheses, there is a stable set of W whose incidence vector s satisfies $c^Tx \leq b$ with equality and $s_{a_1} = 1$, $s_{b_1} = 0$ and $s_f = 0$ for all $f \in N(b_1) \setminus \{a_1\}$. If $b_1 \notin \{a, b\}$, then we can extend s to s' by letting $s'_y = 1$ and $s'_z = 0$ if $s_a = 0$, or by letting $s'_z = 1$ and $s'_y = 0$ if $s_b = 0$. If $b_1 = a$, then we can extend s to s' by letting $s'_y = 0$ and $s'_z = 1$ (since $s_b = 0$). If $b_1 = b$, then we can extend s to s' by letting $s'_y = 1$ and $s'_z = 0$ (since $s_a = 0$). In each case s' is the incidence vector of an appropriate stable set. □

Proof of Theorem 4.2: It follows from Lemma 4.5, Lemma 4.7 and the above discussion that we have only to prove that every edge in a base wheel satisfies the hypotheses in Lemma 4.6. Let $W_{\mathcal{B}}$ be a base wheel. Since $\mathcal{I}_{\mathcal{O}}^{W_{\mathcal{B}}}$ is not an edge inequality, we have to check condition 2 in Lemma 4.6 only for edges (a, b) such that $a, b \in \mathcal{E}$. (This is because for any other edge, (y_1, y_2), one of the ends say y_1 is of degree 2 and $1 = c_{y_1} \leq c_{y_2}$. Let e be the only other neighbour of b that is a vertex of the rim. Then the Type 1 stable set N with $b, e \notin N$ has the required properties. (Note that $a \in N$ and the only other neighbour of b is not in N since v_0 is in N.) Hence we are done. □

Now we discuss whether a simple 1-wheel inequality $\mathcal{I}_{\mathcal{E}}$ corresponding to the subgraph W of G is facet-inducing for P_G. (Similar remarks apply to $\mathcal{I}_{\mathcal{O}}$.) If W is an induced subgraph of G, then this is, in principle, easy to answer. First, if $\mathcal{I}_{\mathcal{E}}$ is not facet-inducing for P_W, then it cannot be facet-inducing for P_G, so we may assume that W satisfies the conditions in Theorem 4.1. Now $\mathcal{I}_{\mathcal{E}}$ can be extended to a facet-inducing inequality I of P_G by "sequential lifting" (see, for example, Nemhauser and Wolsey [18]), and $\mathcal{I}_{\mathcal{E}}$ is facet-inducing for P_G if and only if this process results in $\mathcal{I}_{\mathcal{E}}$ being the same as I. This can be determined by solving at most n optimal stable set problems on W; it is easy to see that the latter can be done in polynomial time.

If W is not an induced subgraph of G, things are more complicated. This is quite different from the situation for a cycle inequality, for which there is a simple criterion—if the cycle is not induced, the inequality is not facet-inducing. For example, $\mathcal{I}_{\mathcal{E}}^W$ ($\mathcal{I}_{\mathcal{O}}^W$) is facet-inducing for the graph in Figure 5a (Figure 5b) where W is this graph with the edge $(4, 7)$ deleted. Even the problem of deciding whether, in a graph W' consisting of a simple 1-wheel configuration W together with a single edge, the 1-wheel inequality is facet-inducing for $P_{W'}$ is not easy. In this case the classification is known, but is not easy to describe. We do give a partial result that can be neatly stated. Suppose that W is a 1-wheel and that

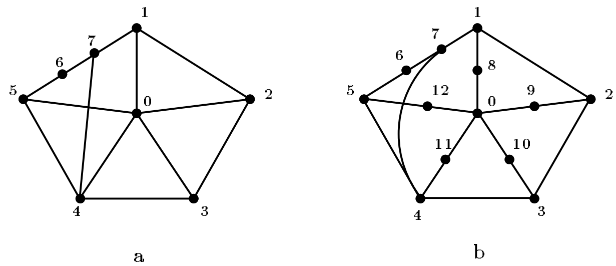

Figure 5: An example of facet-inducing wheel inequalities for wheels with chords

$(a_i, b_i) \notin E(W)$ for $i = 1, 2, \ldots, l$ where $a_i, b_i \in V(W)$. Let W' be the graph obtained from W by adding $(a_1, b_1), (a_2, b_2), \ldots, (a_l, b_l)$ to the edge-set of W. Then each (a_i, b_i), where $i \in \{1, 2, \ldots, l\}$, is a *chord* for W'. Since every stable set of W' is a stable set of W^i $(i = 0, 1, 2, \ldots, l)$ where W^0 is W and W^i is the graph obtained from W by adding (a_j, b_j) to the edge-set of W for $j \in \{1, 2, \ldots, i\}$, we may assume $\mathcal{I}_{\mathcal{E}}^{W^i}$ ($\mathcal{I}_{\mathcal{O}}^{W^i}$) is facet-inducing for P_{W^i}. Hence we can obtain necessary conditions by considering 1-wheels with only one chord. The simplest partial necessary condition is the following result.

Theorem 4.8 *Let (a, b) be a chord joining two (not necessarily internal) vertices of a spoke or rim-path, or joining two spoke-ends. Then $\mathcal{I}_{\mathcal{E}}^{W'}$ and $\mathcal{I}_{\mathcal{O}}^{W'}$ are not facet-inducing for $P_{W'}$.*

Proof: Let P be the subpath from a to b on the spoke (rim-path). If P is of odd length, then it must be of length at least 3 since (a, b) is a chord. Let P be $(a, u_1, u_2, \ldots, u_{2l}, b)$ where $l \geq 1$. Let W_1 be the 1-wheel obtained from W by replacing P with (a, b). Then

$$\mathcal{I}_{\mathcal{E}}^{W'} = \mathcal{I}_{\mathcal{E}}^{W_1} + \sum_{i=1}^{l}(x_{u_{2i-1}} + x_{u_{2i}} \leq 1) \text{ and } \mathcal{I}_{\mathcal{O}}^{W'} = \mathcal{I}_{\mathcal{O}}^{W_1} + \sum_{i=1}^{l}(x_{u_{2i-1}} + x_{u_{2i}} \leq 1).$$

If P is even, then the cycle C consisting of P and (a, b) is odd. It is easy to see that the face induced by $\mathcal{I}_{\mathcal{E}}^{W'}$ ($\mathcal{I}_{\mathcal{O}}^{W'}$) lies in the face induced by the cycle inequality of C by considering two cases, namely, a stable set containing neither a nor b and satisfying $\mathcal{I}_{\mathcal{E}}^{W'}$ ($\mathcal{I}_{\mathcal{O}}^{W'}$) with equality, and a stable set containing exactly one of a and b and satisfying $\mathcal{I}_{\mathcal{E}}^{W'}$ ($\mathcal{I}_{\mathcal{O}}^{W'}$) with equality. □

We have already seen from Figure 5 that 1-wheel inequalities can be facet-inducing even if both ends of the chords are vertices of the rim. However, the graphs in Figure 5 can be interpreted in another way; we can view them as 1-wheel configurations with different sets of spoke-ends and hubs. Figure 6 shows the same graphs as Figure 5; however, it is not true now that both ends of the chords are vertices of the rim. Clearly $\mathcal{I}_{\mathcal{E}}$ for the graph in Figure 6a is the same as $\mathcal{I}_{\mathcal{E}}$ for the graph in Figure 5a, and $\mathcal{I}_{\mathcal{O}}$ for the graph in Figure 6b is the same as $\mathcal{I}_{\mathcal{O}}$ for the graph in Figure 5b. (We observe that the set of spoke-ends for the graphs in Figure 6a and Figure 6b is $\{1, 2, 3, 4, 7\}$.) To be precise, suppose G is the underlying graph of two 1-wheels with one chord, $W(v_0; v_1, v_2, \ldots, v_{2k+1})$ and $W'(v'_0; v'_1 v'_2, \ldots, v'_{2k+1})$. If $\mathcal{I}^W_{\mathcal{E}}$ ($\mathcal{I}^W_{\mathcal{O}}$) is the same as $\mathcal{I}^{W'}_{\mathcal{E}}$ ($\mathcal{I}^{W'}_{\mathcal{O}}$), then W is said to be $\mathcal{E}$-equivalent ($\mathcal{O}$-equivalent) to W'. The above examples illustrate the following result. See Cheng [4] for a (long) proof.

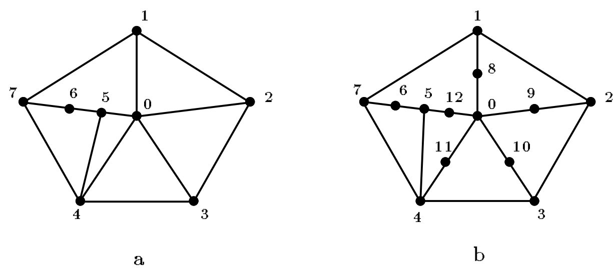

Figure 6: A redrawing of Figure 5

Theorem 4.9 *Suppose W' is obtained from a simple 1-wheel by adding a chord (a, b) on the rim. Then $\mathcal{I}^{W'}_{\mathcal{E}}$ ($\mathcal{I}^{W'}_{\mathcal{O}}$) is not facet-inducing for $P_{W'}$ unless W' is $\mathcal{E}$-equivalent ($\mathcal{O}$-equivalent) to W'', a 1-wheel with one chord such that not both ends of the chord are vertices of the rim.*

5 p-Wheel Inequalities

In the same way a cycle is "formed" by a circle of edges (K_2's), we can view a 1-wheel as being "formed" by a circle of cycles (which are homeomorphic to K_3). Figure 7 gives a configuration "formed" by a circle of 1-wheels each homeomorphic to K_4. The subgraphs induced by each of the following is a 1-wheel (K_4): $\{a, b, c, d\}$, $\{a, b, d, e\}$, $\{a, b, e, f\}$, $\{a, b, f, g\}$ and $\{a, b, c, g\}$. (Tesch [20] investigated a class of inequalities for the stable set problem based on configurations such as this. More exactly, they were based on replacing the hub of a 1-wheel in which all spokes and rim-paths were single edges by a clique.)

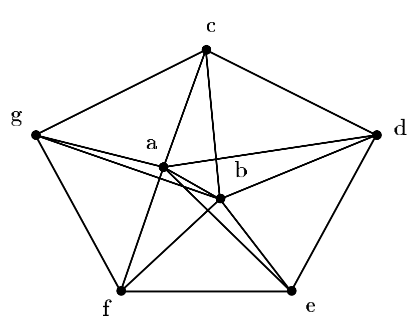

Figure 7: A configuration "formed" by 1-wheels

We define a simple p-wheel of size $2k+1$ where the hub is of size p recursively. For $p=1$, a p-wheel is, of course, a 1-wheel. Let $p \geq 2$ and $G_1 = (V_1, E_1)$ be a graph with $V_1 = \{v_{0_1}, v_{0_2}, \ldots, v_{0_p}, v_1, v_2, \ldots, v_{2k+1}\}$ and $E_1 = \{(v_{0_j}, v_{0_l}) : 1 \leq j < l \leq p\} \cup \{(v_{0_j}, v_i),\ (v_i, v_{i+1}) : 1 \leq j \leq p, 1 \leq i \leq 2k+1\}$. Consider a subdivision of G_1. Let $P_{0_j,i}$ and $P_{i,i+1}$ denote the paths obtained from (v_{0_j}, v_i) and (v_i, v_{i+1}) respectively through the subdivision. This graph is a *simple p-wheel of size* $2k+1$ if for each i, the graph W_i consisting of

$$P_{i,i+1}, P_{0_1,i}, P_{0_1,i+1}, P_{0_2,i}, P_{0_2,i+1}, \ldots, P_{0_p,i}, P_{0_p,i+1} \text{ and the clique on } v_{0_1}, v_{0_2}, \ldots, v_{0_p} \qquad (5)$$

is a $(p-1)$-wheel of size 3 for every choice of a $(p-1)$-subset of $\{v_{0_1}, v_{0_2}, \ldots, v_{0_p}\}$ being used as the hub. We denote this simple p-wheel by $W = W(v_{0_1}, v_{0_2}, \ldots, v_{0_p}; v_1, v_2, \ldots, v_{2k+1})$. We call the set $\{v_{0_1}, v_{0_2}, \ldots, v_{0_p}\}$ the *hub* of W; its elements are the *centres* of W. (For $p=1$, we have used the term "hub" to describe v_0 rather than $\{v_0\}$.) The paths $P_{1,2}, P_{2,3}, \ldots, P_{2k,2k+1}, P_{2k+1,1}$ are the *rim-paths* of W, and the paths $P_{0_j,1}, P_{0_j,2}, \ldots, P_{0_j,2k+1}$ for $1 \leq j \leq 2k+1$ are the *spokes* of W. (Note that if we let Y_j be the graph consisting of the paths $P_{1,2}, P_{2,3}, \ldots, P_{2k,2k+1}, P_{2k+1,1}$ and $P_{0_j,1}, P_{0_j,2}, \ldots, P_{0_j,2k+1}$, then Y_j is a simple 1-wheel. Moreover, a path is a rim-path of W if and only if it is a rim-path of every Y_j, and is a spoke of W if and only if it is a spoke of some Y_j.) So, loosely speaking, a p-wheel of size $2k+1$ is formed by a circle of $2k+1$ $(p-1)$-wheels such that each of them is an "odd homeomorph" of K_{p+2}. Figure 8 shows an example of a simple 2-wheel where $\{a, b\}$ is the hub, and c, d, e, f and g are the spoke-ends. The set of spokes is $\{(a,c), (a,d), (a,e), (a,m,f), (a,g), (b,i,h,c), (b,d), (b,e), (b,j,k,l,f), (b,g)\}$ and the set of rim-paths is $\{(c,d), (d,e), (e,n,f), (f,p,g), (g,c)\}$.

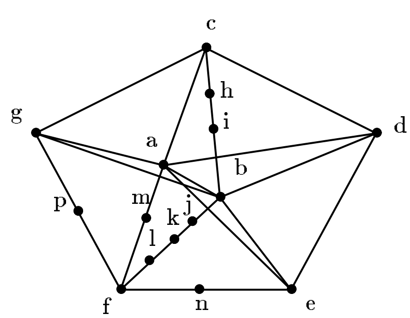

Figure 8: A simple 2-wheel

The definition has the advantage that it is symmetric, that is, the requirement that for each i, the graph consisting of (5) to be a $(p-1)$-wheel of size 3 must hold for *every* $(p-1)$-subset of $\{v_{0_1}, v_{0_2}, \ldots, v_{0_p}\}$ being used as the hub. However the definition has the disadvantage that it is not a minimal definition. Let us call the definition that we gave in the previous paragraph, *definition 1.* We call it *definition 2* if we insist only that for each i, the graph consisting of (5) be a $(p-1)$-wheel of size 3 with $\{v_{0_1}, v_{0_2}, \ldots, v_{0_{p-1}}\}$ as the hub. However, neither definition gives us an easy way to construct a p-wheel. We now offer another definition which we call *definition 3*. Let $p \geq 2$, $k \geq 1$ and $G_1 = (V_1, E_1)$ be a graph with $V_1 = \{v_{0_1}, v_{0_2}, \ldots, v_{0_p}, v_1, v_2, \ldots, v_{2k+1}\}$ and $E_1 = \{(v_{0_j}, v_{0_l}) : 1 \leq j < l \leq p\} \cup \{(v_{0_j}, v_i), (v_i, v_{i+1}) : 1 \leq j \leq p, 1 \leq i \leq 2k+1\}$. Let $\mathcal{E}$ and $\mathcal{O}$ be a partition of $\{v_1, v_2, \ldots, v_{2k+1}\}$. Consider a subdivision of G_1. Let $P_{0_j,i}$ and $P_{i,i+1}$ denote the paths obtained from (v_{0_j}, v_i) and (v_i, v_{i+1}) respectively through the subdivision. This graph is a *simple p-wheel of size* $2k+1$ if $v_i \in \mathcal{E}$ $(\mathcal{O})$ implies $P_{0_j,i}$ is an even (odd) path for every j and $P_{i,i+1}$ is an odd path if and only if v_i and v_{i+1} belong to the same class. The three definitions are equivalent. The first two definitions are used for deriving our desired inequalities. (The derivation arises naturally from these definitions.) The last definition is useful for constructing examples.

Proposition 5.1 *Definitions 1, 2 and 3 are equivalent.*

Proof: Apply induction on p. □

We partition the edges of the rim into two sets $\mathcal{A}(W)$ and $\mathcal{B}(W)$ as for a 1-wheel. For example, $\mathcal{A} = \{(f,n), (f,p)\}$ and $\mathcal{B} = \{(n,e), (e,d), (c,d), (c,g), (g,p)\}$ in Figure 8.

Suppose W is a simple p-wheel. Let $S_i = S_i(W)$ be the set of internal vertices of the spokes ending at v_{0_i} for $i = 1, 2, \ldots, p$, and let $S = S(W)$ be the set of internal vertices of all the spokes of W, that is, $S = S_1 \cup S_2 \cup \cdots \cup S_p$. Let $R = R(W)$ be the set of internal vertices of all the rim-paths of W. For example, in the graph of Figure 8, if we take $v_{0_1} = a$ and $v_{0_2} = b$, then $S_1 = \{m\}$, $S_2 = \{h, i, j, k, l\}$, $S = \{m, h, i, j, k, l\}$, and $R = \{n, p\}$.

Given a simple p-wheel W, we can derive valid inequalities for P_W by mimicking the derivation of $\mathcal{I}_{\mathcal{A}}$ and $\mathcal{I}_{\mathcal{B}}$ for a 1-wheel. Let $W(v_{0_1}, v_{0_2}, \ldots, v_{0_p}; v_1, v_2, \ldots, v_{2k+1})$ be a simple p-wheel of size $2k+1$. Moreover, let W_i be the $(p-1)$-wheel (of size 3) consisting of (5). Then our scheme is:

1. Calculate the sum of a sequence of $(p-1)$-wheel inequalities of the same form, that is, a $(p-1)$-wheel inequality of the same form for every W_i, $i = 1, 2, \ldots, 2k+1$.

2. Add either $\sum_{e=(u,v)\in\mathcal{A}}(x_u + x_v \leq 1)$ or $\sum_{e=(u,v)\in\mathcal{B}}(x_u + x_v \leq 1)$ to the sum.

3. Add either nothing or one of $\sum_{j=1}^{p}(x_{0_j} \leq 1)$ and $\sum_{j=1}^{p}(-x_{0_j} \leq 0)$ to the sum so that the resulting inequality has the property that every coefficient is even and the right-hand side is odd.

4. Divide the resulting inequality by 2 and then round down the right-hand side.

If $p = 2$, then we can use either $\mathcal{I}_{\mathcal{A}}$ or $\mathcal{I}_{\mathcal{B}}$ in Step 1; in Step 2, there are two choices. So there will be four valid inequalities for a 2-wheel (from our scheme). In a similar fashion, a p-wheel will generate 2^p inequalities. We call these *candidates for the p-wheel inequalities.* We denote the inequality that we get by using $\mathcal{I}_{\mathcal{A}}$ in Step 1 and $\mathcal{A}$ $(\mathcal{B})$ in Step 2 by $I_{\mathcal{A}^2}$ $(I_{\mathcal{A}\mathcal{B}})$. Moreover, we denote the inequality that we get by using $\mathcal{I}_{\mathcal{B}}$ in Step 1 and $\mathcal{A}$ $(\mathcal{B})$ in Step 2 by $I_{\mathcal{B}\mathcal{A}}$ $(I_{\mathcal{B}^2})$. (It is clear how to extend this notation to candidates for the p-wheel inequalities.) We will show that $I_{\mathcal{B}\mathcal{A}}$ and $I_{\mathcal{B}^2}$ are redundant and more generally, that there are just two interesting p-wheel inequalities among the 2^p candidates. We remark that one might consider a combination of $(p-1)$-wheel inequalities from different classes in Step 1; however, we don't see a natural way to perform steps similar to Steps 2 and 3 in this situation. We will show the following:

Claim: These inequalities are the only possible non-redundant inequalities for P_W generated by our scheme:

$$I^W_{\mathcal{A}^p} : k\sum_{j=1}^{p} x_{0_j} + (p+1)\sum_{v\in\mathcal{E}} x_v + \sum_{v\in\mathcal{O}} x_v + \sum_{v\in S\cup R} x_v \le k + (|S|+|R|+p|\mathcal{E}|)/2$$

and

$$I^W_{\mathcal{A}^{p-1}\mathcal{B}} : (k+1)\sum_{j=1}^{p} x_{0_j} + p\sum_{v\in\mathcal{E}} x_v + 2\sum_{v\in\mathcal{O}} x_v + \sum_{v\in S\cup R} x_v \le 2k+1+(|S|+|R|+(p-2)|\mathcal{E}|)/2.$$

We remark that $I_{\mathcal{A}^p}$ becomes $\mathcal{I}_{\mathcal{E}}$ and $I_{\mathcal{A}^{p-1}\mathcal{B}}$ becomes $\mathcal{I}_{\mathcal{O}}$ if $p=1$. So $\mathcal{E}$ and $\mathcal{O}$ no longer play symmetric roles in the general case. The reason is that the inequalities of the form $I_{\mathcal{A}^{p-1}}$ play an important role in the derivation of these two inequalities while the inequalities of the form $I_{\mathcal{A}^{p-2}\mathcal{B}}$ do not, as we will see in the proof.

We prove the claim by induction on p. Before we prove this claim, we note that Step 1 seems ambiguous because we did not specify which $(p-1)$-subset of $\{v_{0_1}, v_{0_2}, \ldots, v_{0_p}\}$ is the hub for W^i. Suppose we choose $\{v_{0_1}, v_{0_2}, \ldots, v_{0_p}\} \setminus \{v_{0_j}\}$ to be the hub. Then v_{0_j} is a spoke-end; in fact $v_{0_j} \in \mathcal{O}$. Hence, from the inequalities $I^W_{\mathcal{A}^p}$ and $I^W_{\mathcal{A}^{p-1}\mathcal{B}}$ in our claim, every v_{0_l}, $l=1,2,\ldots,p$ will have coefficient 1 in $I^{W^i}_{\mathcal{A}^p}$ and coefficient 2 in $I^{W^i}_{\mathcal{A}^{p-1}\mathcal{B}}$ since W^i is a $(p-1)$-wheel of size 3. Hence Step 1 is not ambiguous. (Of course this is provided that our claim is correct. Since we are going to use induction to prove that the claim is correct, this need no longer concern us.)

For $p=1$, the above inequalities indeed reduce to the inequalities $\mathcal{I}_{\mathcal{A}}$ and $\mathcal{I}_{\mathcal{B}}$ that we have found in Section 2. (We have also seen that our scheme gave two such inequalities.) Assume the result is true for some $p\ge 1$. Now, given $W(v_{0_1}, v_{0_2}, \ldots, v_{0_{p+1}}; v_1, v_2, \ldots, v_{2k+1})$, a $(p+1)$-wheel of size $2k+1$, we first use $I_{\mathcal{A}^p}$ for each W^i, $i=1,2,\ldots,2k+1$, in Step 1. (Recall that W^i is a p-wheel of size 3.) The sum of these inequalities is

$$(2k+1)\sum_{i=1}^{p+1} x_{0_i} + 2(p+1)\sum_{v\in\mathcal{E}} x_v + 2\sum_{v\in\mathcal{O}} x_v + 2\sum_{v\in S} x_v + \sum_{v\in R} x_v \le 2k+1+|S|+|R|/2+p|\mathcal{E}|. \quad (6)$$

According to Step 2, we have two possibilities. The first possibility is to add

$$\sum_{e=(u,v)\in\mathcal{A}} (x_u + x_v \le 1) \equiv 2\sum_{v\in\mathcal{E}} x_v + \sum_{v\in R} x_v \le |\mathcal{E}| + |R|/2$$

to (6). Hence we have

$$(2k+1)\sum_{i=1}^{p+1} x_{0_i} + 2(p+2)\sum_{v\in\mathcal{E}} x_v + 2\sum_{v\in\mathcal{O}} x_v + 2\sum_{v\in S\cup R} x_v \le 2k+1+|S|+|R|+(p+1)|\mathcal{E}|. \quad (7)$$

Since $S = S_1\cup\cdots\cup S_{p+1}$, we can conclude that $|S|\equiv 0 \pmod 2$ if $p+1$ is even and $|S|\equiv|S_1| \pmod 2$ if $p+1$ is odd. We now use the fact that $|S_1|\equiv|\mathcal{E}| \pmod 2$ to conclude $|S|+|R|+(p+1)|\mathcal{E}|$ is even. (Recall that $|R|$ is even.) Hence the right-hand side of (7) is odd. So we use $-\sum_{j=1}^{p+1}(x_{0_j} \leq 0)$ in Step 3. Applying Step 4, we have

$$I^W_{\mathcal{A}^{p+1}}: k\sum_{i=1}^{p+1} x_{0_i} + (p+2)\sum_{v\in\mathcal{E}} x_v + \sum_{v\in\mathcal{O}} x_v + \sum_{v\in S\cup R} x_v \leq k + (|S| + |R| + (p+1)|\mathcal{E}|)/2.$$

The second possibility in Step 2 is to add

$$\sum_{e=(u,v)\in\mathcal{B}} (x_u + x_v \leq 1) \equiv 2\sum_{v\in\mathcal{O}} x_v + \sum_{v\in R} x_v \leq |\mathcal{O}| + |R|/2$$

to (6). Hence we have

$$(2k+1)\sum_{i=1}^{p+1} x_{0_i} + 2(p+1)\sum_{v\in\mathcal{E}} x_v + 4\sum_{v\in\mathcal{O}} x_v + 2\sum_{v\in S\cup R} x_v \leq 2k + 1 + |S| + |R| + p|\mathcal{E}| + |\mathcal{O}|. \tag{8}$$

The right-hand side of (8) can be simplified to $2(2k+1) + |S| + |R| + (p-1)|\mathcal{E}|$. Since $p+1$ and $p-1$ have the same parity and we have already seen that $|S| + |R| + (p+1)|\mathcal{E}|$ is even, $|S| + |R| + (p-1)|\mathcal{E}|$ must be even as well. Hence the right-hand side of (8) is even. So we use $\sum_{j=1}^{p+1}(x_{0_j} \leq 1)$ in Step 3. Applying Step 4, we have

$$I^W_{\mathcal{A}^p B}: (k+1)\sum_{i=1}^{p+1} x_{0_i} + (p+1)\sum_{v\in\mathcal{E}} x_v + 2\sum_{v\in\mathcal{O}} x_v + \sum_{v\in S\cup R} x_v \leq 2k + 1 + \frac{|S| + |R| + (p-1)|\mathcal{E}|}{2}.$$

We now use $I_{\mathcal{A}^{p-1}\mathcal{B}}$ for each W^i, $i = 1, 2, \ldots, 2k+1$, in Step 1. The sum of these inequalities is

$$2(2k+1)\sum_{i=1}^{p+1} x_{0_i} + 2p\sum_{v\in\mathcal{E}} x_v + 4\sum_{v\in\mathcal{O}} x_v + 2\sum_{v\in S} x_v + \sum_{v\in R} x_v \leq 3(2k+1) + |S| + \frac{|R|}{2} + (p-2)|\mathcal{E}|. \tag{9}$$

According to Step 2, we have two possibilities. The first possibility is to add

$$\sum_{e=(u,v)\in\mathcal{A}} (x_u + x_v \leq 1) \equiv 2\sum_{v\in\mathcal{E}} x_v + \sum_{v\in R} x_v \leq |\mathcal{E}| + |R|/2$$

to (9). Hence we have

$$2(2k+1)\sum_{i=1}^{p+1} x_{0_i} + 2(p+1)\sum_{v\in\mathcal{E}} x_v + 4\sum_{v\in\mathcal{O}} x_v + 2\sum_{v\in S\cup R} x_v \leq 3(2k+1) + |S| + |R| + (p-1)|\mathcal{E}|. \tag{10}$$

Since $|S| + |R| + (p-1)|\mathcal{E}|$ is even, the right-hand side of (10) is odd. So we add nothing to (10) in Step 3. Applying Step 4, we have

$$(2k+1)\sum_{i=1}^{p+1} x_{0_i} + (p+1)\sum_{v\in\mathcal{E}} x_v + 2\sum_{v\in\mathcal{O}} x_v + \sum_{v\in S\cup R} x_v \leq 3k + 1 + (|S| + |R| + (p-1)|\mathcal{E}|)/2.$$

However, this is equal to $I_{\mathcal{A}^p\mathcal{B}} + k(\sum_{i=1}^{p+1} x_{0_i} \leq 1)$, so it is redundant. The second possibility is to add

$$\sum_{e=(u,v)\in\mathcal{B}} (x_u + x_v \leq 1) \equiv 2\sum_{v\in\mathcal{O}} x_v + \sum_{v\in R} x_v \leq |\mathcal{O}| + |R|/2$$

to (9). Hence we have

$$2(2k+1)\sum_{i=1}^{p+1} x_{0_i} + 2p\sum_{v\in\mathcal{E}} x_v + 6\sum_{v\in\mathcal{O}} x_v + 2\sum_{v\in S\cup R} x_v \le 3(2k+1) + |S| + |R| + (p-2)|\mathcal{E}| + |\mathcal{O}|. \qquad (11)$$

The right-hand side of (11) can be written as $3(2k+1) + |S| + |R| + (p-1)|\mathcal{E}| - |\mathcal{E}| + |\mathcal{O}|$. Since $|S| + |R| + (p-1)|\mathcal{E}|$ is even and $-|\mathcal{E}| + |\mathcal{O}|$ is odd (since $|\mathcal{E}| + |\mathcal{O}| = 2k+1$), the right-hand side of (11) is even. However, all the coefficients in (11) are even as well; so our scheme produces nothing here. This completes the proof of the claim.

To simplify the notation, we use $\mathcal{I}_{\mathcal{E}}^W$ instead of $I_{\mathcal{A}^p}^W$ and $\mathcal{I}_{\mathcal{O}}^W$ instead of $I_{\mathcal{A}^{p-1}\mathcal{B}}^W$ for a given p-wheel. The next result follows from our discussion.

Theorem 5.2 *Let G be a graph and $W(v_{0_1}, v_{0_2}, \ldots, v_{0_p}; v_1, v_2, \ldots, v_{2k+1})$ be a simple p-wheel of size $2k+1$ that is a subgraph of G. Then the inequalities $\mathcal{I}_{\mathcal{E}}^W$ (that is, $I_{\mathcal{A}^p}^W$) and $\mathcal{I}_{\mathcal{O}}^W$ (that is, $I_{\mathcal{A}^{p-1}\mathcal{B}}^W$) are valid for P_G.* □

The inequalities in Theorem 5.2 are called *p-wheel inequalities* or simply *wheel inequalities*. For example, we have $\mathcal{I}_{\mathcal{E}} : 2x_a + 2x_b + 3x_f + \sum_{v\notin\{a,b,f\}} x_v \le 7$ and $\mathcal{I}_{\mathcal{O}} : 3x_a + 3x_b + 2x_c + 2x_d + 2x_e + 2x_f + 2x_g + \sum_{v\notin\{a,b,c,d,e,f,g\}} x_v \le 9$ for the graph of Figure 8.

We end this section with several remarks. First, if S, R and $\mathcal{E}$ are empty and $k=1$, then $\mathcal{I}_{\mathcal{E}}$ is the clique inequality of size $p+3$. Second, if we let $p=0$ in $\mathcal{I}_{\mathcal{E}}$, even though the assumption is $p \ge 1$, we get a cycle inequality, whereas $\mathcal{I}_{\mathcal{O}}$ produces nothing; so $\mathcal{I}_{\mathcal{E}}$ seems to be a "better" generalization of cycle inequalities. Third, one can define *general p-wheel inequalities* in a way similar to general 1-wheel inequalities (so Theorem 5.2 is true even for non-simple p-wheels), and the corresponding separation problem can be solved in polynomial time; see Cheng and Cunningham [6] or Cheng [5] for details.

6 Facet-Inducing Simple p-Wheel Inequalities

$W(v_{0_1}, v_{0_2}, \ldots, v_{0_p}; v_1, v_2, \ldots, v_{2k+1})$ is assumed to be a simple p-wheel of size $2k+1$ throughout this section. We consider the following questions: When will $\mathcal{I}_{\mathcal{E}}^W$ ($\mathcal{I}_{\mathcal{O}}^W$) be facet-inducing for P_W? The answer for $p=1$ is given in Theorems 4.1 and 4.2. For $p \ge 2$, it turns out that the inequalities of the form $\mathcal{I}_{\mathcal{E}}$ are very well-behaved; namely, Theorem 4.1 remains true when we replace 1 by p. We now state the result.

Theorem 6.1 *Let W be a simple p-wheel. Then $\mathcal{I}_{\mathcal{E}}^W$ is facet-inducing for P_W if and only if every rim-path joining two elements of $\mathcal{E}$ has length at least 2 (and hence at least 3).*

On the other hand, the inequalities of the form $\mathcal{I}_{\mathcal{O}}$ are harder to handle. An example that shows that we cannot simply replace 1 by p in Theorem 4.2 appears in Figure 9. Here, we have $\mathcal{I}_{\mathcal{O}} : 3x_a + 3x_b + 2x_c + 2x_d + 2x_e + 2x_f + 2x_g + \sum_{v\notin\{a,b,c,d,e,f,g\}} x_v \le 10$. We claim that it is not facet-inducing.

(We note that $\mathcal{O} = \emptyset$, that is, all the spokes are of even length, so the conditions in Theorem 4.2 are trivially satisfied.) This is because $\mathcal{I}_{\mathcal{O}} = (3x_a + x_c + x_d + x_e + x_f + x_g + x_h + x_j + x_m + x_p + x_r \leq 5) + (3x_b + x_c + x_d + x_e + x_f + x_g + x_i + x_k + x_l + x_n + x_q \leq 5)$. We note that the two summands are $\mathcal{I}_{\mathcal{O}}^{W_1}$ and $\mathcal{I}_{\mathcal{O}}^{W_2}$, where W_1 is the subgraph induced by $\{a, c, d, e, f, g, h, j, m, p, r\}$ and W_2 is induced by $\{b, c, d, e, f, g, i, k, l, n, q\}$. However, as is the case for $p = 1$, it is possible to characterize the facet-inducing inequalities of the form $\mathcal{I}_{\mathcal{O}}$ if the underlying graphs are simple p-wheels: see Cheng [5]. This result is too complicated to state (much less prove) here, but it involves characterizing a family $\mathcal{Z}$ of wheels, having the following properties.

1. Let W be a simple p-wheel such that $W \notin \mathcal{Z}$. Then $\mathcal{I}_{\mathcal{O}}^{W}$ is facet-inducing for P_W if and only if every rim-path joining two elements of $\mathcal{O}$ has length at least 2 (and hence at least 3) and every spoke of odd length has length at least 3.

2. The elements in $\mathcal{Z}$ that satisfy the above conditions are not facet-inducing for P_W, but the elements in $\mathcal{Z}$ that violate the above conditions are facet-inducing for P_W.

In other words, $\mathcal{Z}$ is precisely the set of wheels for which Theorem 4.2 does not generalize in the obvious way. Of course, the wheel of Figure 9 is a member of $\mathcal{Z}$. Cheng [5] gives a complete classification of $\mathcal{Z}$ and proves that it has the above properties.

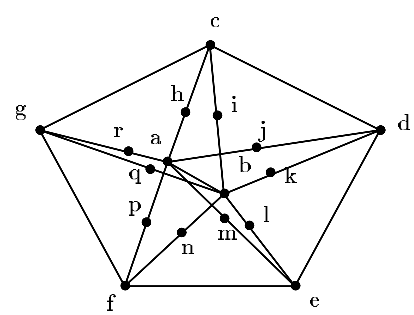

Figure 9: A wheel in which $\mathcal{I}_{\mathcal{O}}$ is not facet-inducing

We will prove Theorem 6.1, and hence Theorem 4.1. The first step is to show the condition in Theorem 6.1 is necessary. Lemmas 6.2 and 6.3 correspond to Lemmas 4.4 and 4.5 respectively. The proofs are quite similar, so we omit the proof of Lemma 6.2 and summarize the proof of Lemma 6.3. More details can be found in Cheng [5].

Lemma 6.2 *If $\mathcal{I}_{\mathcal{E}}^{W}$ is facet-inducing for P_W, then every rim-path joining two elements of $\mathcal{E}$ has length at least 2 (and hence at least 3).*

We use the same strategy as for $\mathcal{I}_{\mathcal{O}}^{W}$ when $p = 1$ to prove the condition in Theorem 6.1 is sufficient. The notion of base wheel $W_{\mathcal{B}}(v_{0_1}, \ldots, v_{0_p}; v_1, \ldots, v_{2k+1}; \mathcal{E})$ is defined as expected. For example, suppose the hub is $\{v_{0_1}, v_{0_2}\}$, the spoke-ends are v_1, v_2, v_3, v_4 and v_5; $\mathcal{E} = \{v_1, v_2, v_4\}$. Then the base wheel for these data is shown in Figure 10. The base wheel has the following properties:

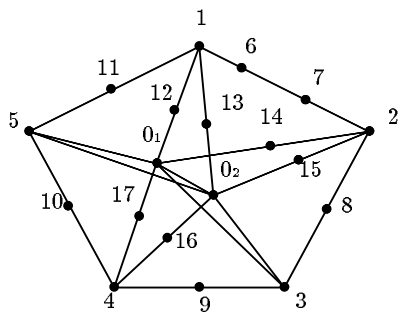

Figure 10: A base wheel

1. $v_i \in \mathcal{E}$ implies $P_{0_j,i}$ is of length 2 for $i \in \{1, 2, \ldots, 2k+1\}$ and $j \in \{1, 2, \ldots, p\}$;

2. $v_i \in \mathcal{O}$ implies $P_{0_j,i}$ is of length 1 for $i \in \{1, 2, \ldots, 2k+1\}$ and $j \in \{1, 2, \ldots, p\}$;

3. $v_i, v_{i+1} \in \mathcal{E}$ implies $P_{i,i+1}$ is of length 3 for $i \in \{1, 2, \ldots, 2k+1\}$;

4. $v_i, v_{i+1} \in \mathcal{O}$ implies $P_{i,i+1}$ is of length 1 for $i \in \{1, 2, \ldots, 2k+1\}$; and

5. $|\{v_i, v_{i+1}\} \cap \mathcal{E}| = 1$ implies $P_{i,i+1}$ is of length 2 for $i \in \{1, 2, \ldots, 2k+1\}$.

Since $|S(W_{\mathcal{B}})| = p|\mathcal{E}|$ and $|R(W_{\mathcal{B}})| = 2|\mathcal{E}|$, $\mathcal{I}_{\mathcal{E}}^{W_{\mathcal{B}}}$ can be simplified to

$$\mathcal{I}_{\mathcal{E}}^{W_{\mathcal{B}}} : k \sum_{j=1}^{p} x_{0_j} + (p+1) \sum_{v \in \mathcal{E}} x_v + \sum_{v \in \mathcal{O}} x_v + \sum_{v \in S \cup R} x_v \le k + (p+1)|\mathcal{E}|.$$

We consider two types of stable sets of $W_{\mathcal{B}}$ that satisfy $\mathcal{I}_{\mathcal{E}}^{W_{\mathcal{B}}}$ with equality.

- **Type 1:** $N = X \cup Y$ where X is a stable set of the rim, C, of $W_{\mathcal{B}}$ of size $(|C|-1)/2$ and Y is constructed as follows: If $v_i \in \mathcal{E}$ where $i \in \{1, \ldots, 2k+1\}$ and $v_i \notin X$, then adjoin the (unique) internal vertex of $P_{0_j,i}$ to Y for all $j = 1, 2, \ldots, p$.

- **Type 2:** $N = \{v_{0_j}\} \cup \mathcal{E}$ for some fixed $j \in \{1, \ldots, p\}$.

It is clear that both types of stable sets satisfy $\mathcal{I}_{\mathcal{E}}^{W_{\mathcal{B}}}$ with equality.

Lemma 6.3 *Let $W_{\mathcal{B}}$ be the base wheel for $(v_{0_1}, \ldots, v_{0_p}; v_1, \ldots, v_{2k+1})$ and a prescribed $\mathcal{E}$. Then $\mathcal{I}_{\mathcal{E}}^{W_{\mathcal{B}}}$ is facet-inducing for $P_{W_{\mathcal{B}}}$.*

Proof: Suppose the face induced by $\mathcal{I}_{\mathcal{E}}^{W_{\mathcal{B}}}$ is a subset of a facet induced by some valid inequality $a^T x \leq b$. Our goal is to show that $a^T x \leq b$ is a multiple of $\mathcal{I}_{\mathcal{E}}^{W_{\mathcal{B}}}$. Let (v, w) be an edge of the rim such that $v, w \notin \mathcal{E}$. (It is clear that such v and w must exist.) Let (u_1, u_2, w, v) be the length 3 subpath of the rim, containing w and v, and having v as an end of the subpath. Let N be the Type 1 stable set of $W_{\mathcal{B}}$ with $u_1, v \in N$. Now $N' = (N \setminus \{v\}) \cup \{w\}$ is also a Type 1 stable set of $W_{\mathcal{B}}$. Since any stable set of $W_{\mathcal{B}}$ that satisfies $\mathcal{I}_{\mathcal{E}}^{W_{\mathcal{B}}}$ with equality will also satisfy $a^T x \leq b$ with equality, we can conclude that $a_v = a_w$.

Let $v_i \in \mathcal{E}$ and v, w be the two vertices of the rim that are neighbours of v_i. We note that $v, w \notin \mathcal{E} \cup \mathcal{O}$ since $W_{\mathcal{B}}$ is a base wheel. Let (u_1, u_2, v, v_i) be the length 3 subpath of the rim, containing v and v_i, and having v_i as an end of the subpath. Let z_j be the internal vertex of $P_{0_j,i}$ for $j = 1, 2, \ldots, p$. Let N be the Type 1 stable set of $W_{\mathcal{B}}$ with $u_1, v_i \in N$. Now $N' = (N \setminus \{v_i\}) \cup \{v, z_1, \ldots, z_p\}$ is also a Type 1 stable set of $W_{\mathcal{B}}$. Hence $a_{v_i} = \sum_{j=1}^{p} a_{z_j} + a_v$. Similarly, we have $a_{v_i} = \sum_{j=1}^{p} a_{z_j} + a_w$. Therefore $a_v = a_w$. We can now conclude that $a_{v'} = a_{w'} = \alpha$ for all v' and w' on the rim that are not in $\mathcal{E}$, for some $\alpha \in \mathbf{R}$.

Let $v_i \in \mathcal{E}$ and v, w be the two vertices of the rim that are neighbours of v_i and z_j be the internal vertex of $P_{0_j,i}$ for $j = 1, 2, \ldots, p$. Let N be the Type 2 stable set of $W_{\mathcal{B}}$ with $v_{0_j} \in N$ where $j \in \{1, \ldots, p\}$. It is easy to see that $N' = (N \setminus \{v_i\}) \cup \{v, w\} \cup (\{z_1, \ldots, z_p\} \setminus \{z_j\})$ is also a stable set of $W_{\mathcal{B}}$ that satisfies $\mathcal{I}_{\mathcal{E}}^{W_{\mathcal{B}}}$ with equality. Hence $a_{v_i} = 2\alpha + \sum_{l=1}^{p} a_{z_l} - a_{z_j}$. Let us denote this value by d. Then we have $a_{v_i} = 2\alpha + (p-1)d$. Since we also have $a_{v_i} = \sum_{j=1}^{p} a_{z_j} + \alpha$ which implies $a_{v_i} = \alpha + pd$, we must have $d = \alpha$; hence $a_{v_i} = (p+1)\alpha$.

Now by taking a Type 1 stable set of $W_{\mathcal{B}}$ and a Type 2 stable set of $W_{\mathcal{B}}$ containing v_{0_j} where $j \in \{1, \ldots, p\}$, we have $a_{0_j} = k\alpha$. Hence $a^T x \leq b$ is a multiple of $\mathcal{I}_{\mathcal{E}}^{W_{\mathcal{B}}}$. □

We now complete the proof of Theorem 6.1. We first observe that, given a base wheel $W_{\mathcal{B}}$, since $\mathcal{I}_{\mathcal{E}}^{W_{\mathcal{B}}}$ is not an edge inequality, we only have to check condition 2 in Lemma 4.6 for two kinds of edges:

- If $a = v_{0_j}$ and $b \in \mathcal{O}$, then $\{v_{0_j}\} \cup \mathcal{E}$ (that is, a Type 2 stable set) satisfies condition 2.
- Suppose $a, b \in \mathcal{O}$. Let e be the only other neighbour of b that is on the rim. Then the Type 1 stable set, N, with $b, e \notin N$ will do the job. (Note that the only other neighbours of b are $v_{0_1}, \ldots, v_{0_p}$.)

So any edge (a, b) in a base wheel satisfies the hypotheses in Lemma 4.6.

Suppose W is a p-wheel. Let W' be obtained from W by replacing an edge (a, b) on the rim or on one of the spokes by (a, y, z, b). It now follows that if $c^T x \leq d$ is the p-wheel inequality $\mathcal{I}_{\mathcal{E}}^{W}$, then $c^T x + x_y + x_z \leq d + 1$ is the p-wheel inequality $\mathcal{I}_{\mathcal{E}}^{W'}$. We know that every base wheel satisfies the hypotheses in Lemma 4.6. Moreover, any p-wheel can be obtained from a corresponding base wheel, by replacing an edge in the updated graph with a path of length 3 successively. Hence by applying Lemma 4.7, the proof of Theorem 6.1 is complete.

We end this section with several remarks. First, if a simple p-wheel W is an induced subgraph of G, then one can check whether $\mathcal{I}_{\mathcal{E}}^{W}$ ($\mathcal{I}_{\mathcal{O}}^{W}$) is facet-inducing for P_G in polynomial time. Second, Theorem 4.8

is true even for p-wheels. Third, one possible way to generalize these p-wheel inequalities is to replace a centre-edge, that is, an edge of the form (v_{0_j}, v_{0_l}) by a path of odd length; it turns out that Theorem 6.1 is true for inequalities obtained in this way from the inequalities of the form $\mathcal{I}_{\mathcal{E}}$, but that inequalities obtained from the inequalities of the form $\mathcal{I}_{\mathcal{O}}$ are never facet-inducing. (See Cheng [5] for details.)

7 Non-Simple Wheel Inequalities

We have seen that the inclusion of non-simple 1-wheel inequalities makes the separation problem easier to handle. Moreover, we have observed that it is indeed possible for a non-simple 1-wheel inequality to be facet-inducing. This generates the following question: When is a non-simple 1-wheel inequality facet-inducing for its support graph? We do not know necessary and sufficient conditions for this to be true. We do present some necessary conditions and state a conjecture regarding these non-simple 1-wheel inequalities. It turns out that a non-simple wheel inequality can be facet-inducing for a very "uninteresting" reason – it can be equivalent to a cycle inequality. (It is easy to see that it cannot be equivalent to a trivial or edge inequality.) For example, consider the graph on the left in Figure 11. We have $\mathcal{I}_{\mathcal{E}} : \sum_{i=1}^{9} x_i \leq 4$. Suppose we partition the vertex-set into the following classes: $\{1,2\}, \{4,3\}, \{5,8\}, \{6,9\}, \{0,7\}$. If we identify the vertices belonging to the same class, then we obtain the graph on the right in Figure 11 where, as usual, multiple edges are deleted; moreover, $\mathcal{I}_{\mathcal{E}}$ reduces to $2(x_0 + x_1 + x_4 + x_5 + x_6 \leq 2)$ which is a positive multiple of a cycle inequality. We obtain cleaner statements of necessary conditions for non-simple 1-wheel inequalities to be facet-inducing, if we assume them to be distinct from cycle inequalities. This is a reasonable condition since we are looking for new possible facet-inducing inequalities.

Let us consider our example of a facet-inducing non-simple 1-wheel inequality again, namely, $\mathcal{I}_{\mathcal{E}}^{W}$ where W is the graph obtained by identifying a and b in Figure 3. We note that both a and b are neighbours of the hub. Another example is given in Figure 12. If W is obtained by identifying a and b for the graph on the left in Figure 12, then we have the graph on the right in Figure 12. We can see that $\mathcal{I}_{\mathcal{E}}^{W}$ is facet-inducing for P_W. We observe that the vertex obtained by identifying a and b is a spoke-end for the graph on the right in Figure 12. It is easy to construct examples for $\mathcal{I}_{\mathcal{O}}^{W}$ similar to the one we have given here. We define a *basic operation* as the identification of two vertices adjacent to the hub or the same spoke-end.

Conjecture 7.1 *Every facet-inducing non-simple 1-wheel inequality (that is not a cycle inequality) arises from applying a set of basic operations to a simple 1-wheel.*

Note that we are not claiming that every non-simple 1-wheel inequality obtained from basic operations is facet-inducing. In the rest of this section, we give some necessary conditions for $\mathcal{I}_{\mathcal{E}}^{W}$ ($\mathcal{I}_{\mathcal{O}}^{W}$) to be facet-inducing for P_W and not to be a cycle inequality where W is a non-simple 1-wheel. Before we start, we would like to give a preview of what type of results are presented in this section. The first result is that in order for a 1-wheel inequality to be facet-inducing, all rim-paths and spokes must be paths, not walks.

(This is easy to prove.) All the other results except Theorem 7.6 have the following flavour: Let W be a simple 1-wheel. Suppose P is a partition of the vertex-set of W that induces a non-simple 1-wheel. If the resulting non-simple 1-wheel inequality $\mathcal{I}_{\mathcal{E}}^{W}$ ($\mathcal{I}_{\mathcal{O}}^{W}$) is facet-inducing, then we must forbid u and v to be in the same equivalence class if there is a path P_1 from u to w and a path P_2 from v to w for some specified w such that all the internal vertices of P_1 and P_2 are of degree 2, plus a parity condition and possibly a planarity condition on W^a, the graph obtained from W by identifying u and v. The condition that all the internal vertices of P_1 and P_2 are of degree 2 is important because in this case, we only have to look at a "local" structure. Theorem 7.6 says that no two spoke-ends can be identified. Although this has a "global" structure, we are able to show that such non-simple 1-wheel inequalities are not facet-inducing by writing them as sums of other known valid inequalities. Although we have only covered a small class of non-simple 1-wheel inequalities here, we feel that there is a strong possibility that if a non-simple 1-wheel inequality is not facet-inducing and it "has" a "global" structure, then it can be written as a sum of trivial, edge, cycle and simple 1-wheel inequalities. We start with the following observation.

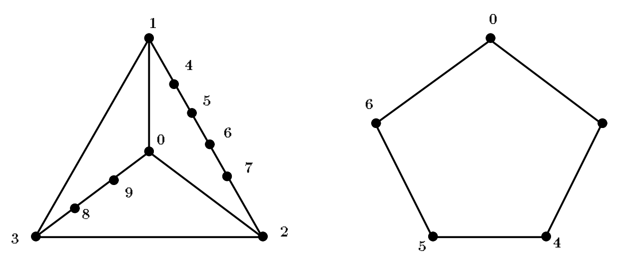

Figure 11: An example

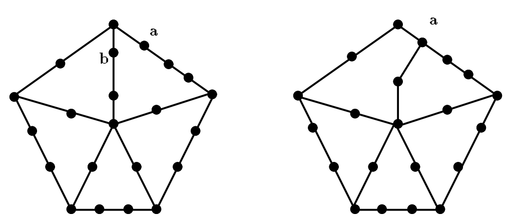

Figure 12: $\mathcal{I}_{\mathcal{E}}$ is facet-inducing for the graph obtained by identifying a and b

Proposition 7.1 *Let H be a graph and I be a valid inequality for P_H such that the support graph for I is H. Let H' be a graph obtained from H via a sequence of identifications of vertices, and I' be the inequality obtained from I by adding up the coefficients of the variables corresponding to the vertices that are identified. Suppose $I = \sum_{i=1}^{t} I_i$ where I_i is a valid inequality for P_H for $i = 1, 2, \ldots, t$. If one of the I_i's is a positive multiple of an edge inequality and H' is not the graph consisting of only one edge, then I'*

is not facet-inducing for $P_{H'}$.

Proof: Let I_i' be the inequality obtained from I_i by adding up the coefficients of the variables corresponding to the vertices that were identified to obtain H' from H. Then I_i' is a valid inequality for $P_{H'}$ where $i = 1, 2, \ldots, t$ (by Proposition 3.4) and $I' = \sum_{i=1}^{t} I_i'$. Suppose I' is facet-inducing for $P_{H'}$. Then every I_i' where $i \in \{1, 2, \ldots, t\}$ is a positive multiple of I'. (Recall that parallel edges arising from the identifications of vertices are replaced by single edges.) Without loss of generality, we may assume I_1 is a positive multiple of an edge inequality say $a(x_u + x_v \leq 1)$. Then I_1 is identical to I_1' since u and v are adjacent so they were not identified; hence $I' = \alpha(x_u + x_v \leq 1)$. Since H is the support graph for I, H' must be the graph consisting of only one edge, namely (u, v), which is a contradiction. □

We note that the assumption that H' not be the graph consisting of only one edge is important; otherwise, it is possible for I' to be facet-inducing for $P_{H'}$. For example, let $H = (V, E)$ where $V = \{v_1, v_2, v_3\}$, $E = \{(v_1, v_2), (v_2, v_3)\}$ and I denotes the inequality $x_1 + 2x_2 + x_3 \leq 2$. If v_1 and v_3 are identified, then I' will be facet-inducing for $P_{H'}$.

We explain the importance of Proposition 7.1. Suppose W is a simple 1-wheel and W' is a non-simple 1-wheel obtained from W. Suppose vertices v and w of W are identified in the process of obtaining W'. Let W^a be the non-simple 1-wheel obtained from W by identifying v and w. Then W' can be obtained from W^a via a sequence of identifications of vertices. Suppose we can show that the non-simple 1-wheel inequality, $\mathcal{I}_{\mathcal{E}}^{W^a}$ ($\mathcal{I}_{\mathcal{O}}^{W^a}$), can be written as a sum of valid inequalities for P_{W^a} such that one of them is an edge inequality. Since it is clear that W' cannot be the graph consisting of only one edge (because W is not bipartite), we can now apply Proposition 7.1 to conclude that the non-simple 1-wheel inequality, $\mathcal{I}_{\mathcal{E}}^{W'}$ ($\mathcal{I}_{\mathcal{O}}^{W'}$), is not facet-inducing for $P_{W'}$.

Suppose $\mathcal{I}_{\mathcal{E}}^{W^a}$ ($\mathcal{I}_{\mathcal{O}}^{W^a}$) can be written as a sum of distinct valid inequalities, say $\sum_{i=1}^{t} I_i$ where $t \geq 2$, for P_{W^a}, but none of the summands is an edge inequality. (We note that by distinct, we mean I_i and I_j are not multiples of each other.) Suppose I_1 is a positive multiple of a cycle inequality; let C_1 be the support graph for I_1. If we need to identify two vertices of C_1 to obtain W' from W^a, then after such an identification, the inequality corresponding to I_1 can be written as a sum of positive multiples of edge inequalities and cycle inequalities; hence we can apply Proposition 7.1. So we assume no two vertices of C_1 need to be identified in order to obtain W'. (Of course, a vertex of C_1 and a vertex not of C_1 can be identified.) If we let I_i' be the inequality obtained from I_i by adding up the coefficients of the variables corresponding to the vertices that were identified to obtain W' from W^a, then $\mathcal{I}_{\mathcal{E}}^{W'}$ ($\mathcal{I}_{\mathcal{O}}^{W'}$) $= \sum_{i=1}^{t} I_i'$ and I_1 is identical to I_1'. Therefore $\mathcal{I}_{\mathcal{E}}^{W'}$ ($\mathcal{I}_{\mathcal{O}}^{W'}$) is not facet-inducing for $P_{W'}$ unless $\mathcal{I}_{\mathcal{E}}^{W'}$ ($\mathcal{I}_{\mathcal{O}}^{W'}$) and I_i' for $i = 2, 3, \ldots, t$ are positive multiples of I_1. In this case, $\mathcal{I}_{\mathcal{E}}^{W'}$ ($\mathcal{I}_{\mathcal{O}}^{W'}$) is a positive multiple of a cycle inequality; so no new facet-inducing inequality for $P_{W'}$ is produced. (We observe that we may assume the support graph of $\mathcal{I}_{\mathcal{E}}^{W'}$ ($\mathcal{I}_{\mathcal{O}}^{W'}$) is a cycle with no chord, since otherwise the cycle inequality is not facet-inducing.) We summarize our discussion in the next result.

Proposition 7.2 *Let H be a graph and I be a valid inequality for P_H such that the support graph for I is H. Let H' be a graph obtained from H via a sequence of identifications of vertices and I' be the inequality obtained from I by adding up the coefficients of the variables corresponding to the vertices that are identified. Suppose $I = \sum_{i=1}^{t} I_i$ where $t \geq 2$ and the I_i's are distinct valid inequalities for P_H. If one of the I_i's is a positive multiple of a cycle inequality and H' is not the graph consisting of only one edge, then I' is either not facet-inducing for $P_{H'}$ or H' is a graph consisting of a cycle and I' is a positive multiple of the cycle inequality corresponding to H'.* □

Theorem 7.3 *Let W be a non-simple 1-wheel. If $\mathcal{I}_{\mathcal{E}}^{W}$ ($\mathcal{I}_{\mathcal{O}}^{W}$) is facet-inducing for P_W and $\mathcal{I}_{\mathcal{E}}^{W}$ ($\mathcal{I}_{\mathcal{O}}^{W}$) is not a positive multiple of a cycle inequality, then the rim-paths and spokes must be paths, not walks.*

Proof: By Proposition 7.1 and Proposition 7.2, it suffices to prove the following: If W^a is the non-simple 1-wheel obtained from a simple 1-wheel by identifying two vertices of a rim-path (spoke), P, then the face induced by $\mathcal{I}_{\mathcal{E}}^{W^a}$ ($\mathcal{I}_{\mathcal{O}}^{W^a}$) lies in the face induced by an edge inequality or a cycle inequality. Let u and v be the two vertices that we have identified to obtain W^a. Let $Q_{u,v}$ be the subpath of P joining u to v. Then $Q_{u,v}$ is a block (maximal 2-vertex-connected subgraph) of W^a. Moreover, $Q_{u,v}$ is either an even cycle or an odd cycle. We observe that all vertices of this cycle have coefficient 1 in $\mathcal{I}_{\mathcal{E}}^{W^a}$ ($\mathcal{I}_{\mathcal{O}}^{W^a}$) except u which has coefficient greater than 1. Therefore, if $Q_{u,v}$ is an odd cycle, then the face induced by $\mathcal{I}_{\mathcal{E}}^{W^a}$ ($\mathcal{I}_{\mathcal{O}}^{W^a}$) lies in the face induced by the cycle inequality for $Q_{u,v}$; otherwise, (that is, $Q_{u,v}$ is an even cycle) the face induced by $\mathcal{I}_{\mathcal{E}}^{W^a}$ ($\mathcal{I}_{\mathcal{O}}^{W^a}$) lies in the face induced by the edge inequality, $x_u + x_w \leq 1$, where w is a neighbour of u on the cycle. □

In the spirit of Theorem 7.3, we have the following result.

Proposition 7.4 *Let G be a graph and I be a valid inequality for P_G such that G is its support graph. Let u and w be two vertices and let P_1 and P_2 be two paths from u to w. Suppose the cycle consisting of P_1 and P_2 has odd length. If all the internal vertices of P_1 and P_2 are of degree 2 in G and their coefficients in I are the same, then the face induced by I lies in a face induced by the cycle inequality for the cycle consisting of P_1 and P_2.*

Proof: Without loss of generality, assume P_1 is $(u, y_1, y_2, \ldots, y_{2l+1}, w)$ and P_2 is $(w, z_1, z_2, \ldots, z_{2r}, u)$. Let N be a stable set of G that satisfies I with equality. Suppose $u, w \notin N$. Then exactly $l+1$ elements from $\{y_1, y_2, \ldots, y_{2l+1}\}$ and r elements from $\{z_1, z_2, \ldots, z_{2r}\}$ are in N. If $u \in N$ and $w \notin N$, then exactly l elements from $\{y_1, y_2, \ldots, y_{2l+1}\}$ and r elements from $\{z_1, z_2, \ldots, z_{2r}\}$ are in N. Finally, suppose $r \geq 1$ and $u, w \in N$. Then exactly l elements from $\{y_1, y_2, \ldots, y_{2l+1}\}$ and $r-1$ elements from $\{z_1, z_2, \ldots, z_{2r}\}$ are in N. □

The cycle that we considered in Proposition 7.4 may have two vertices with degree greater than 2, so the argument in the proof of Theorem 7.3 does not apply.

Theorem 7.5 *Let W' be a non-simple 1-wheel obtained from a simple 1-wheel $W(v_0; v_1, v_2, \ldots, v_{2k+1})$ through identification of vertices. If $\mathcal{I}_{\mathcal{E}}^{W'}$ ($\mathcal{I}_{\mathcal{O}}^{W'}$) is facet-inducing for $P_{W'}$ and is not a positive multiple of a cycle inequality, then distinct vertices u and w satisfying one of the following conditions cannot belong to the same equivalence class:*

1. *u (w) is a vertex of $P_{0,i}$ ($P_{0,r}$) not equal to v_0 such that the length of the subpath of $P_{0,i}$ from u to v_0 and the length of the subpath of $P_{0,r}$ from w to v_0 are of different parity.*
2. *u (w) is a vertex of $P_{a,i}$ ($P_{b,i}$) not equal to v_i such that a and b are distinct elements of $\{v_{i-1}, v_{i+1}, v_0\}$, and the length of the subpath of $P_{a,i}$ from u to v_i and the length of the subpath of $P_{b,i}$ from w to v_i are of different parity.*

Proof: This follows from Proposition 7.1, Proposition 7.2 and Proposition 7.4 □

For example, Theorem 7.5 tells us that we cannot identify an element of $\mathcal{E}$ with an element of $\mathcal{O}$. The next result strengthens this to any two spoke-ends.

Theorem 7.6 *Let $W(v_0; v_1, v_2, \ldots, v_{2k+1})$ be a simple 1-wheel. Let W' be a non-simple 1-wheel obtained from W. If $\mathcal{I}_{\mathcal{E}}^{W'}$ ($\mathcal{I}_{\mathcal{O}}^{W'}$) is facet-inducing for $P_{W'}$ and $\mathcal{I}_{\mathcal{E}}^{W'}$ ($\mathcal{I}_{\mathcal{O}}^{W'}$) is not a positive multiple of a cycle inequality, then no two spoke-ends can be identified.*

Proof: By Theorem 7.3, we may assume that the rim-paths are paths, not walks. Hence no two consecutive spoke-ends can be identified. We have already observed that we cannot identify an element of $\mathcal{E}$ with an element of $\mathcal{O}$. Suppose two vertices are in $\mathcal{E}$ or $\mathcal{O}$. Let W^a be the graph obtained from W by identifying these two vertices. Then it follows from our discussion that it is enough to show that $\mathcal{I}_{\mathcal{E}}^{W^a}$ ($\mathcal{I}_{\mathcal{O}}^{W^a}$) can be written as a sum of edge, cycle and simple 1-wheel inequalities; moreover, at least one summand is either an edge or a cycle inequality. Without loss of generality, assume v_1 and v_{2i} belong to $\mathcal{E}$ and that W^a is obtained from W by identifying v_1 and v_{2i}. Moreover, we may assume $2 \leq i \leq k-1$. Let $W^1(v_0; v_1, v_2, \ldots, v_{2(i-1)+1})$ be the 1-wheel with v_0 as the hub and $v_1, v_2, \ldots, v_{2(i-1)+1}$ as the spoke-ends. All spokes and rim-paths of W^1 arise from their counterparts in W in a unique way except the spoke from v_0 to v_1. We use $P_{0,1}(W)$ as the spoke $P_{0,1}(W^1)$. Let I_{2l} be the cycle inequality for the cycle consisting of the paths $P_{2i,2i+1}$, $P_{0,2i}$ and $P_{0,2i+1}$ for $l = i, i+1, \ldots, k$. Let

$$J_{2l+1} = \sum_{(u,v) \in \mathcal{A} \cap P_{2l+1,2l+2}} (x_u + x_v \leq 1) \text{ for } l = i, i+1, \ldots, k$$

and

$$K_{2l+1} = \sum_{(u,v) \in \mathcal{B} \cap P_{2l+1,2l+2}} (x_u + x_v \leq 1) \text{ for } l = i, i+1, \ldots, k.$$

Since one can show that

$$\mathcal{I}_{\mathcal{E}}^{W^a} = \mathcal{I}_{\mathcal{E}}^{W^1} + \sum_{l=1}^{k} (I_{2l} + J_{2l+1}) \text{ and } \mathcal{I}_{\mathcal{O}}^{W^a} = \mathcal{I}_{\mathcal{O}}^{W^1} + \sum_{l=1}^{k} (I_{2l} + K_{2l+1}),$$

we are done. The case for v_1 and v_{2i} belonging to $\mathcal{O}$ is similar. □

As we have already noted, Proposition 7.4 does not cover the case when identifying u and w "induces" an even cycle. Nevertheless, additional results are obtained for the case when the 1-wheel remains planar after the identification. They use the validity of yet another generalization of simple 1-wheel inequalities, which we describe now. Let T be a tree with exactly one non-leaf odd-degree vertex v_0, called the *hub*, and suppose that T is embedded in the plane. Then T has an odd number of leaves, say $v_1, v_2, \ldots, v_{2k+1}$ which we label clockwise; these are the *spoke-ends*. The path in T from v_0 to v_i is a *spoke*; it is denoted by $P_{0,i}$. We add to T a path $P_{i,i+1}$ from v_i to v_{i+1} for each i so that the resulting graph is planar, and so that the face cycles are odd. We call such a graph a *cycle-tree*; it is denoted by $H = H(v_0; v_1, v_2, \ldots, v_{2k+1})$. For example, Figure 13 is a cycle-tree where v_0 is the hub. Of course, a simple 1-wheel configuration is a special kind of cycle-tree.

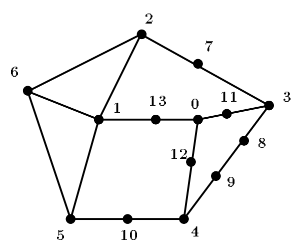

Figure 13: An example of a cycle-tree

We observe that the number of bounded faces of H is $2k + 1$. Let $L = \{v_1, v_2, \ldots, v_{2k+1}\}$. Let R be the set of vertices of the unbounded face that are not in L. Then $|R|$ is even. Let Y be the set of vertices that are not in $\{v_0\} \cup L \cup R$. We can partition L into two sets, $\mathcal{E}$ and $\mathcal{O}$, where $v_i \in \mathcal{E}$ ($\mathcal{O}$) if and only if the length of $P_{0,i}$ is even (odd). It is easy to see that $P_{i,i+1}$ is of odd length if $v_i, v_{i+1} \in \mathcal{E}$ or $\mathcal{O}$, and it is of even length otherwise. Hence we can partition the edges of the rim into two sets $\mathcal{A}$ and $\mathcal{B}$ as in Section 2. By mimicking the derivation for 1-wheel inequalities, we can obtain the following result.

Theorem 7.7 *Let G be a graph and $H(v_0; v_1, v_2, \ldots, v_{2k+1})$ be a cycle-tree that is a subgraph of G. Then*

$$\frac{d(v_0)-1}{2}x_0 + \sum_{v\in Y}\frac{d(v)}{2}x_v + 2\sum_{v\in\mathcal{E}}x_v + \sum_{v\in\mathcal{O}}x_v + \sum_{v\in R}x_v \le k + (|R|+|Y|+|\mathcal{E}|)/2, \text{ and}$$

$$\frac{d(v_0)+1}{2}x_0 + \sum_{v\in Y}\frac{d(v)}{2}x_v + 2\sum_{v\in\mathcal{O}}x_v + \sum_{v\in\mathcal{E}}x_v + \sum_{v\in R}x_v \le k + (|R|+|Y|+|\mathcal{O}|+1)/2$$

are valid inequalities for P_G.

We call the first (second) inequality in Theorem 7.7 a cycle-tree inequality of the first (second) kind. For example, if H is the graph shown in Figure 13, then $x_0 + 2x_1 + x_2 + 2x_3 + 2x_4 + x_5 + x_6 + \sum_{i=7}^{13} x_i \le 7$ is a cycle-tree inequality of the first kind; in fact, it is facet-inducing for P_T. Moreover, $2x_0+2x_1+2x_2+x_3+x_4+2x_5+2x_6+\sum_{i=7}^{13} x_i \leq 8$ is a cycle-tree inequality of the second kind; however, it is not facet-inducing for P_T. Cycle-tree inequalities perhaps deserve study in their own right, but we have used them only in our study of non-simple 1-wheel inequalities. The results of this study, which adds support to Conjecture 7.1, can be found in Cheng [5]. We state without proof one such result.

Theorem 7.8 *Let $W(v_0; v_1, v_2, \ldots, v_{2k+1})$ be a simple 1-wheel. Let u (v) be an internal vertex of $P_{0,i}$ ($P_{0,i+1}$) such that the length of the subpath of $P_{0,i}$ from u to v_0 and the length of the subpath of $P_{0,i+1}$ from v to v_0 have the same parity. Let W' be a non-simple 1-wheel obtained by identifying u and v. Then $\mathcal{I}_{\mathcal{O}}^{W'}$ can be written as a sum of edge inequalities and cycle-tree inequalities; moreover, at least one summand is an edge inequality. If, in addition, at least one of these subpaths has length greater than 1, then $\mathcal{I}_{\mathcal{E}}^{W'}$ can be written as a sum of edge inequalities and cycle-tree inequalities; moreover, at least one summand is an edge inequality.*

Finally, we remark that non-simple p-wheel inequalities can be facet-inducing. Moreover, Theorem 7.3, Proposition 7.4, Theorem 7.5 and Theorem 7.6 can be extended to results for p-wheel inequalities. (See Cheng [4].)

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