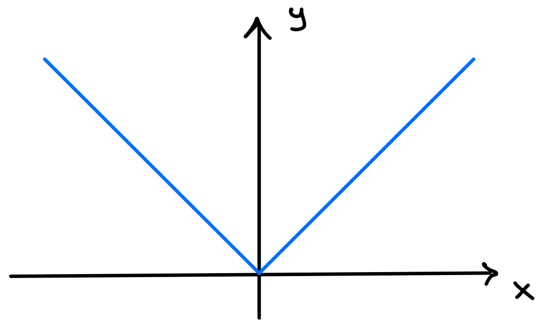
The Absolute Value Function

This is our first example of a function that is piecewise-defined

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Some useful properties:

- $|a \cdot b| = |a| \cdot |b|$
- $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

However, note that $|a \pm b| \neq |a| \pm |b|$!

e.g. $|-1+2| = |1| = 1$

$|-1| + |2| = 1 + 2 = 3$

Not equal!

Absolute values may also show up when simplifying squares and square roots:

e.g. What is $(\sqrt{x})^2$? It's x!

What is $\sqrt{x^2}$? It's $|x|$!

Indeed, $\sqrt{1^2} = \sqrt{1} = 1$ and $\sqrt{(-1)^2} = \sqrt{1} = 1 = |-1|$

This is important when solving equations.

e.g., $x^2 = 9 \Rightarrow \sqrt{x^2} = \sqrt{9}$

$\Rightarrow |x| = 3$

$\Rightarrow x = \pm 3.$

Equations and Inequalities

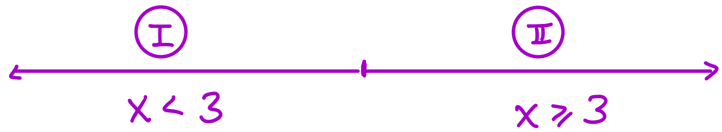
When working with absolute values, it often helps to break the problem into cases.

Ex: Find all x such that $x^2 - 3 = |x-3|$.

Solution: Note that

$$|x-3| = \begin{cases} x-3 & \text{if } x-3 \geq 0 \quad (\text{i.e., } x \geq 3) \\ -(x-3) & \text{if } x-3 < 0 \quad (\text{i.e., } x < 3) \end{cases}$$

Two cases!



I $\underline{x < 3:}$ $\quad x^2 - 3 = |x-3| \Rightarrow x^2 - 3 = -(x-3)$

$$\Rightarrow x^2 + x - 6 = 0$$

$$\Rightarrow (x+3)(x-2) = 0.$$

(Two solutions so far!) $\Rightarrow \underline{x = -3 \text{ or } x = 2}$

II $\underline{x \geq 3:}$ $\quad x^2 - 3 = |x-3| \Rightarrow x^2 - \cancel{3} = x - \cancel{3}$

$$\Rightarrow x^2 - x = 0$$

$$\Rightarrow x(x-1) = 0$$

$$\Rightarrow \underline{x = 0 \text{ or } x = 1}.$$

In II, we assumed $x \geq 3$

$\therefore$ <u>No solutions in Case II!</u>

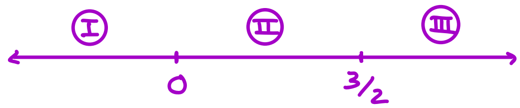
$$\boxed{\underline{\text{Solutions:}}\ \ x=-3 \text{ or } x=2}$$

Inequalities are no different, just make sure to reverse the inequality when multiplying/dividing by a negative!

<u>Ex</u>: Find all x such that $\dfrac{|2x-3|}{x} \le 1$

<u>Solution</u>: Note that $2x-3$ changes sign when $2x-3=0$, or $x = 3/2$. We have

$$|2x-3| = \begin{cases} 2x-3 & \text{if } x \geq 3/2, \\ -(2x-3) & \text{if } x < 3/2. \end{cases}$$

Furthermore, since we will multiply by x to clear the denominator, we should also consider $x>0$ vs. $x<0$ (and clearly $x=0$ isn't possible!). Thus, Three cases 

I $\underline{x < 0:}$ $\quad \dfrac{|2x-3|}{x} \le 1 \implies \dfrac{-(2x-3)}{x} \le 1$

$\overset{\cdot x}{\implies} -2x+3 \ge x$

reverse since $x<0$

$\implies 3 \ge 3x$

$\implies 1 \ge x$

(and $x<0$ in case I)

$\therefore$ All $x \in (-\infty, 0)$ are solutions

II $\underline{0 < x < 3/2:}$ $\quad \dfrac{|2x-3|}{x} \le 1 \implies \dfrac{-(2x-3)}{x} \le 1$

$\overset{\cdot x}{\implies} -2x+3 \le x$

$\implies 3 \le 3x$

$\implies 1 \le x$

(and $0 < x < 3/2$ in case II)

$\therefore$ All $x \in [1, 3/2)$ are solutions

III $\underline{x \geq 3/2:}$ $\quad \dfrac{|2x-3|}{x} \leq 1 \implies \dfrac{2x-3}{x} \leq 1$

$$\overset{\cdot x}{\implies} 2x-3 \leq x$$

$$\implies x \leq 3$$

(and $x \geq 3/2$ in case III)

$\therefore$ All $x \in [3/2, 3]$ are solutions.

$\boxed{\begin{array}{l}\underline{\text{Solution:}} \\ \text{All } x \in (-\infty, 0) \cup [1, 3/2) \cup [3/2, 3] = (-\infty, 0) \cup [1, 3].\end{array}}$

Additional Exercises:

1. Find all x such that $2|x-1| < |x| + 2$.

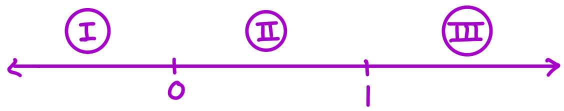
2. Find all x such that $\left|\dfrac{3x-2}{x-2}\right| \geq 1$.

3. Find all x such that $|x^2 - 2x| \leq 3$.

Solutions:

1. Note that x and $x-1$ change signs at $x=0$ and $x=1$, respectively. Thus,

Three Cases!



I $\underline{x<0:}$ $2|x-1| < |x|+2$ $\Rightarrow$ $2(-(x-1)) < -x+2$

$\Rightarrow$ $-2x+2 < -x+2$

$\Rightarrow$ $0 < x$

(impossible, since $x<0$ in Case I)

$\therefore$ No solutions with $x<0$.

II $\underline{0\le x<1:}$ $2|x-1| < |x|+2$ $\Rightarrow$ $2(-(x-1)) < x+2$

$\Rightarrow$ $-2x+\cancel{2} < x+\cancel{2}$

$\Rightarrow$ $0 < 3x$

$\Rightarrow$ $0 < x$

(and $0 \le x < 1$ in case II)

$\therefore$ All $x \in (0,1)$ are solutions!

III $\underline{x \geq 1}$: $2|x-1| < |x|+2 \Rightarrow 2(x-1) < x+2$

$\Rightarrow 2x - 2 < x+2$

$\Rightarrow x < 4$

(and $x \geq 1$ in case III)

$\therefore$ All $x \in [1,4)$ are solutions!

$$\boxed{\text{Solutions: All } x \in (0,1) \cup [1,4) = (0,4)}$$

2. $\left|\dfrac{3x-2}{x-2}\right| \geq 1$.

Note that $x \neq 2$, else $\left|\dfrac{3x-2}{x-2}\right|$ is undefined. We have

$$\left|\frac{3x-2}{x-2}\right| \geq 1 \overset{\cdot|x-2|}{\iff} \frac{|3x-2|}{|x-2|} \geq 1$$

$$\iff \underbrace{|3x-2|}_{\text{changes sign at } x = 2/3} \geq \underbrace{|x-2|}_{\text{changes sign at } x=2.}$$

$|x-2| > 0$, so multiplying by $|x-2|$ won't reverse the inequality!

Three Cases!

I $\underline{x < 2/3}$: $|3x-2| \geq |x-2| \Rightarrow -(3x-2) \geq -(x-2)$

$\Rightarrow -3x + \cancel{2} \geq -x + \cancel{2}$

$\Rightarrow 0 \geq 2x$

$\Rightarrow 0 \geq x$

$\therefore$ All $x \in (-\infty, 0]$ are solutions.

II $\underline{2/3 \leq x < 2}$: $|3x-2| \geq |x-2| \Rightarrow 3x-2 \geq -(x-2)$

$\Rightarrow 3x-2 \geq -x+2$

$\Rightarrow 4x \geq 4$

$\Rightarrow x \geq 1$

$\therefore$ All $x \in [1, 2)$ are solutions.

III $\underline{x > 2}$: $|3x-2| \geq |x-2| \Rightarrow 3x - \cancel{2} \geq x - \cancel{2}$

$\Rightarrow 2x \geq 0$

$\Rightarrow x \geq 0$

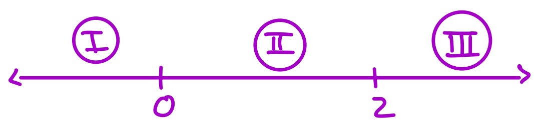
$\therefore$ All $x \in (2, \infty)$ are solutions!

$$\boxed{\underline{\text{Solutions}}: \text{ All } x \in (-\infty, 0] \cup [1, 2) \cup (2, \infty)}$$

3. $|x^2 - 2x| \leq 3$.

Note that $|x^2 - 2x| = |x(x-2)| = |x| \cdot |x-2|$, so we can restate the inequality as $|x| \cdot |x-2| \leq 3$.

Three cases!



I $\underline{x < 0}$: $|x| \cdot |x-2| \leq 3$. $\Rightarrow (-x) \cdot (-(x-2)) \leq 3$

$$\Rightarrow x^2 - 2x - 3 \leq 0$$

$$\Rightarrow (x-3)(x+1) \leq 0$$

Need $-1 \leq x \leq 3$ to achieve one positive factor and one negative factor.

$\therefore$ All $x \in [-1, 0)$ are solutions!

II $\underline{0 \leq x < 2}$: $|x| \cdot |x-2| \leq 3$. $\Rightarrow x \cdot (-(x-2)) \leq 3$

$$\Rightarrow -x^2 + 2x \leq 3$$

$f(x) = x^2 - 2x + 3$ has no real roots, so f is either always positive or always negative. $\Rightarrow$ $x^2 - 2x + 3 \geqslant 0$

Since $f(0) = 3$ (positive!), $x^2 - 2x + 3 > 0$ for all x.

$\therefore$ All $x \in [0, 2)$ are solutions!

(III) $x \geqslant 2$: $|x| \cdot |x-2| \leq 3$.

$$\Rightarrow \quad x(x-2) \leq 3$$

$$\Rightarrow \quad x^2 - 2x - 3 \leq 0$$

$$\Rightarrow \quad (x-3)(x+1) \leq 0$$

Need $-1 \leq x \leq 3$ to achieve one positive factor and one negative factor.

$\therefore$ All $x \in [2, 3]$ are solutions!

$$\boxed{\text{Solutions: All } x \in [-1, 0) \cup [0, 2) \cup [2, 3] = [-1, 3].}$$