Applications of Integration

The remainder of MATH 116 will be devoted to applications.

Big idea: Integrals arise when adding a continuum of tiny quantities. You can think of $\int_a^b f(x)\,dx$ like a big sum!

§6.6 - Average Values

Recall: The average value of real numbers $y_1, y_2, \ldots, y_n$ is

$$\boxed{y_{avg.} = \frac{y_1 + y_2 + \cdots + y_n}{n}}$$

add all quantities

divide by the number of quantities (the sample size)

Similarly, we can define the notion of an "average" for infinitely many quantities. Specifically, the average value of $y = f(x)$ for $x \in [a, b]$ is

$$\boxed{f_{avg.} = \frac{1}{b-a} \int_a^b f(x)\,dx}$$

"Add" all quantities

Divide by the length of $[a,b]$

(similar to dividing by the sample size!)

<u>Ex:</u> The average value of $f(x) = x^2$ for $x \in [0,2]$ is

$$f_{avg.} = \frac{1}{2-0} \int_0^2 f(x)\,dx$$

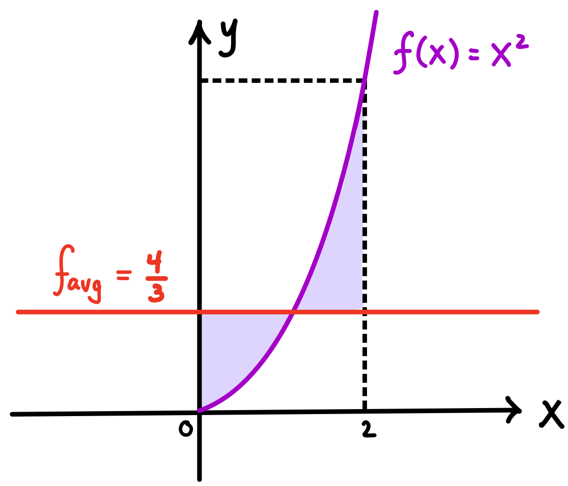
$$= \frac{1}{2} \int_0^2 x^2\,dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} \right]_0^2 = \frac{1}{2} \left(\frac{8}{3} \right) = \boxed{\frac{4}{3}}$$

<u>Geometrically:</u>

Area below f_{avg}

$=$ Area above f_{avg}



Ex: The temperature throughout the day is given by

$$T(t) = 4 - \pi \sin\left(\frac{\pi}{12} t\right) \quad {}^\circ C$$

where t is the time in hours since midnight $(t=0)$.

Determine the average temperature from midnight to noon.

Solution: For $t \in [0, 12]$, we have

$$T_{avg.} = \frac{1}{12-0} \int_0^{12} \left(4 - \pi \sin\left(\frac{\pi}{12} t\right)\right) dt$$

$$= \frac{1}{12} \int_0^{12} 4\, dt - \frac{\pi}{12} \int_0^{12} \sin\left(\frac{\pi}{12} t\right) dt$$

$$= \frac{1}{12} \left[4t\right]_0^{12} - \cancel{\frac{\pi}{12}} \cdot \left[\frac{-\cos\left(\frac{\pi}{12} t\right)}{\cancel{\frac{\pi}{12}}}\right]_0^{12}$$

$$= \frac{1}{\cancel{12}} \left[4(\cancel{12}) - 0\right] + \left[\cos(\pi) - \cos(0)\right]$$

$$= 4 + (-1) - 1$$

$$= \boxed{2\,{}^\circ C}$$

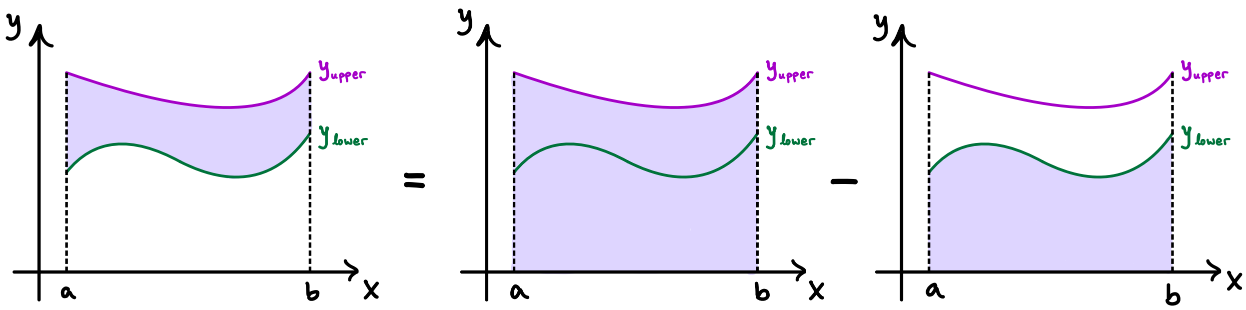
§7.1 – Area Between Curves

Recall: If $f(x) \geq 0$, then the area between the graph of f and the x-axis from $x = a$ to $x = b$ is

$$\text{Area} = \int_a^b f(x)\,dx \qquad (1)$$

More generally, the area between two curves from $x = a$ to $x = b$ can be calculated as

$$\text{Area} = \int_a^b \left(y_{upper} - y_{lower} \right) dx \qquad (2)$$



Note: Formula (1) is really a special case of formula (2) where $y_{upper} = f(x)$ and $y_{lower} = 0$, the x-axis.

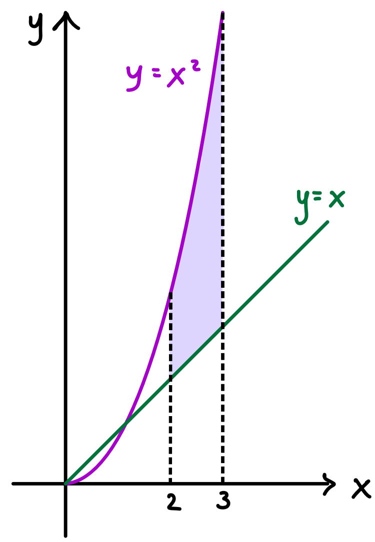
<u>Ex:</u> Find the area enclosed between $y = x$ and $y = x^2$

(a) from $x = 2$ to $x = 3$

(b) from $x = 0$ to $x = 3$

<u>Solution:</u> Start with a picture!

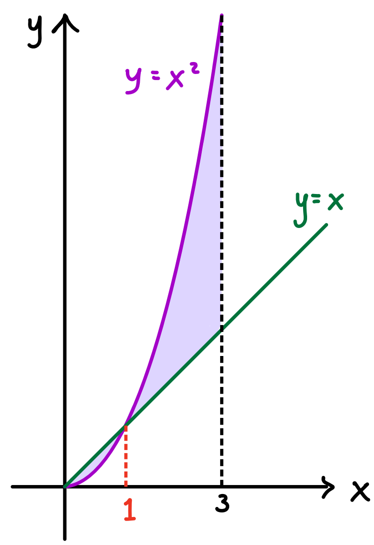
(a) For $x \in [2,3]$, we have $x \leq x^2$, hence



$$\text{Area} = \int_2^3 (y_{upper} - y_{lower})\, dx$$

$$= \int_2^3 (x^2 - x)\, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_2^3 = \dots = \boxed{\frac{23}{6}}$$



(b) The curves intersect when $x^2 = x$,

so $x = 0$ or $x = 1$.

For $x \in [0,1]$, we have $x^2 \leq x$; while

for $x \in [1,3]$, we have $x \leq x^2$, hence

$$\text{Area} = \int_0^1 (y_{upper} - y_{lower})\,dx + \int_1^3 (y_{upper} - y_{lower})\,dx$$

$$= \int_0^1 (x - x^2)\,dx + \int_1^3 (x^2 - x)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^3 = \cdots = \frac{1}{6} + \frac{14}{3} = \boxed{\frac{29}{6}}$$

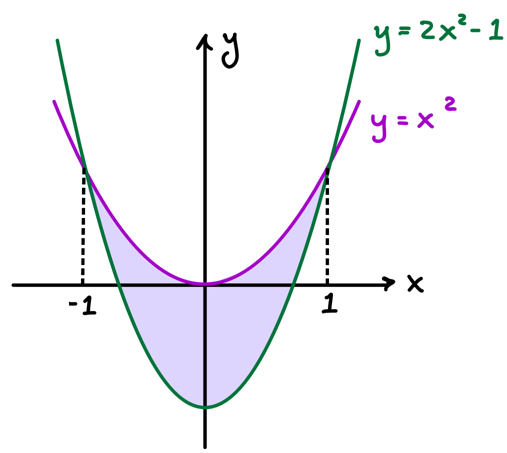
Ex: Calculate the area enclosed between $y = x^2$ and $y = 2x^2 - 1$.

Solution: The curves intersect when $x^2 = 2x^2 - 1$, or equivalently, when $x^2 = 1$, hence $x = \pm 1$. We have $y_{upper} = x^2$ and $y_{lower} = 2x^2 - 1$. Thus,

$$\text{Area} = \int_{-1}^1 \left(x^2 - (2x^2 - 1)\right)\,dx$$

$$= \int_{-1}^1 (1 - x^2)\,dx$$

$$= \left[x - \frac{x^3}{3}\right]_{-1}^1 = \cdots = \boxed{\frac{4}{3}}$$



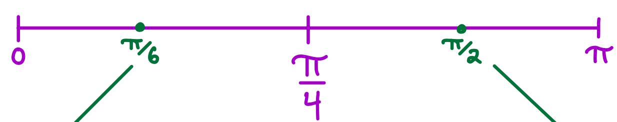
Note: It is actually possible to determine y_{upper} and y_{lower} without graphing! Instead

(i) find the point(s) where the curves cross, and

(ii) check the value of each function at a point in between to see which is y_{upper} / y_{lower}.

Ex: Calculate the area enclosed between $y = \cos x$ and $y = \sin x$ from $x = 0$ to $x = \pi$.

Solution: The curves cross when $\sin x = \cos x$, hence, when $x = \frac{\pi}{4}$.



At $x = \pi/6$, for example:

$\cos(\pi/6) = \frac{\sqrt{3}}{2}$, $\sin(\pi/6) = \frac{1}{2}$

$\Rightarrow y_{upper} = \cos x$, $y_{lower} = \sin x$

At $x = \pi/2$, for example:

$\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$

$\Rightarrow y_{upper} = \sin x$, $y_{lower} = \cos x$

Thus, we have

$$\text{Area} = \int_0^{\pi/4} \left(y_{upper} - y_{lower}\right) dx + \int_{\pi/4}^{\pi} \left(y_{upper} - y_{lower}\right) dx$$

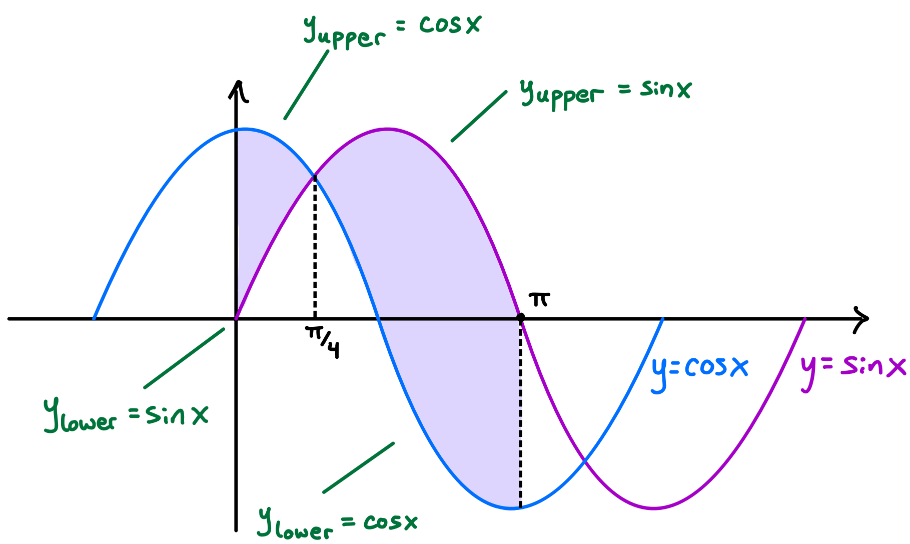
$$= \int_0^{\pi/4} \left(\cos x - \sin x\right) dx + \int_{\pi/4}^{\pi} \left(\sin x - \cos x\right) dx$$

$$= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi}$$

$$= \left[\left(\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right) - \left(\cancel{\sin 0} + \cos 0\right)\right] + \left[\left(-\cos\pi - \cancel{\sin\pi}\right) - \left(-\cos\tfrac{\pi}{4} - \sin\tfrac{\pi}{4}\right)\right]$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \cancel{1} - (\cancel{-1}) + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \boxed{2\sqrt{2}}$$

If we <u>did</u> graph, we would see the picture below:



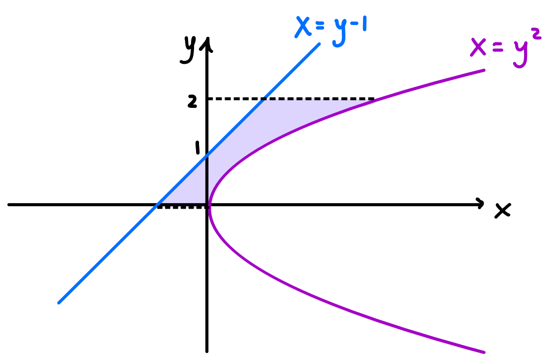
If the region is enclosed between a left curve and a right curve from $y=c$ to $y=d$, then we can get the area in between by integrating with respect to y:

$$\boxed{\text{Area} = \int_{y=c}^{y=d} \left(x_{\text{rightmost}} - x_{\text{leftmost}} \right) dy}$$

Ex: Find the area between $x = y-1$ and $x = y^2$ for $y \in [0,2]$.

Solution:

Here, the region is bounded between $x_{\text{leftmost}} = y-1$ and $x_{\text{rightmost}} = y^2$ from $y=0$ to $y=2$.



$$\therefore \text{Area} = \int_0^2 \left(y^2 - (y-1) \right) dy$$

$$= \left[\frac{y^3}{3} - \frac{y^2}{2} + y \right]_0^2 = \frac{8}{3} - \frac{4}{2} + 2 = \boxed{\frac{8}{3}}$$