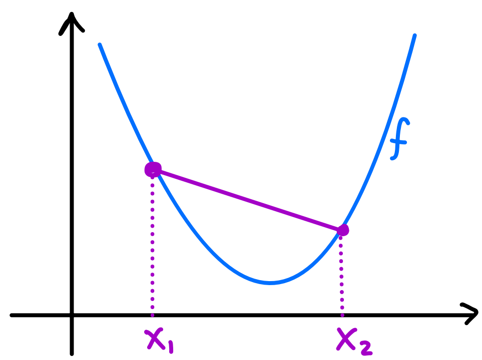
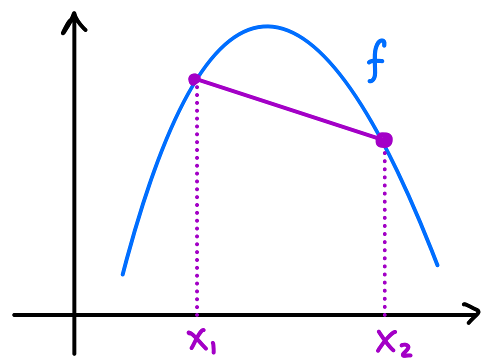
§4.4 – Concavity

Definition: f is concave up (resp. concave down) on an interval I if, whenever $x_1, x_2 \in I$, the secant line segment from $(x_1, f(x_1))$ to $(x_2, f(x_2))$ lies on or above (resp. on or below) the graph of f.

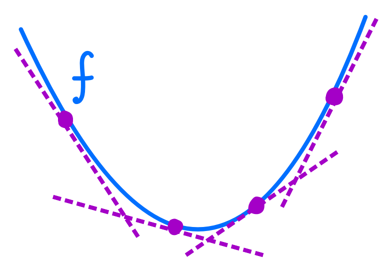


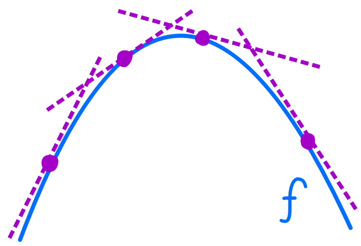
Concave Up



Concave Down

Notice that if f' (i.e., the slope of the tangent line) is increasing on I, then f is concave up on I.





Likewise, if f' is decreasing on I, then f is concave down on I.

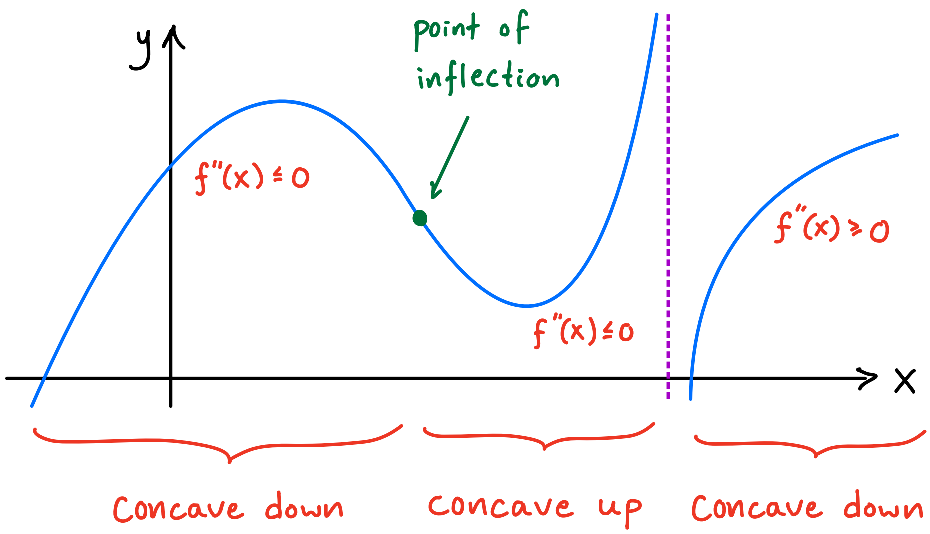
Since f'' tells us whether f' is increasing or decreasing, we can use f'' to detect the concavity of f!

Also written $\frac{d^2f}{dx^2}$ or $\frac{d^2y}{dx^2}$

Test for Concavity

If $f''(x) \geq 0$ for all $x \in (a,b)$, then f is concave up on (a,b). Likewise, if $f''(x) \leq 0$ for all $x \in (a,b)$, then f is concave down on (a,b).

Note: Intervals of concavity are separated by points where $f''(x) = 0$ or DNE. A point in the domain of f separating intervals of opposite concavity is called a point of inflection.



Remarks: As with intervals of increase / decrease,

1. we include an endpoint in an interval of concavity if f is continuous there.

2. an endpoint can belong to two intervals of different concavity.

3. don't joint intervals of concavity with unions ($\cup$).

Ex: Find the intervals of concavity and any points of inflection.

(a) $f''(x) = x^2 + 10x - 1$

Solution: $f'(x) = 2x + 10$, $f''(x) = 2$

positive everywhere

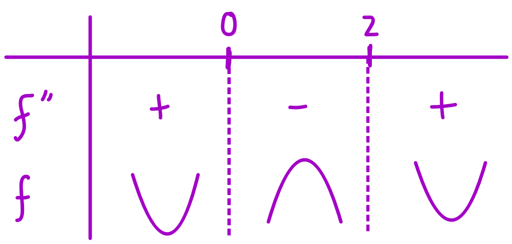
$\therefore$ $\boxed{f \text{ is concave up on } (-\infty, \infty) \text{ ; no inflection points.}}$

(b) $f(x) = x^4 - 4x^3 + 1$

exists everywhere.

Solution: $f'(x) = 4x^3 - 12x^2$, $f''(x) = 12x^2 - 24x$

$$f''(x) = 0 \implies 12x^2(x-2) = 0 \implies x = 0 \text{ or } x = 2$$



concavity changes

f is concave up on $(-\infty, 0]$ and $[2, \infty)$. f is concave down on $[0,2]$. There are inflection points at $x=0$ & $x=2$.

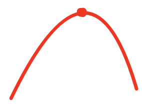
Concavity gives us a second method for classifying local maxima and minima!

The Second Derivative Test

Suppose $f'(c) = 0$ and f'' is continuous around $x = c$.

(i) $f''(c) > 0 \Rightarrow$ local min at $x = c$

(ii) $f''(c) < 0 \Rightarrow$ local max at $x = c$.



(iii) $f''(c) = 0 \Rightarrow$ the test gives no information. We may have a local max, local min, or neither.

Ex: Classify the critical point of $f(x) = x^4 - 4x^3 + 1$ at $x = 3$ as a local max, min, or neither.

Solution: $f''(x) = 12x^2 - 24x \Longrightarrow f''(3) = 36$.

Since $f''(3) > 0$, there is a $\boxed{\text{local min at } x = 3.}$