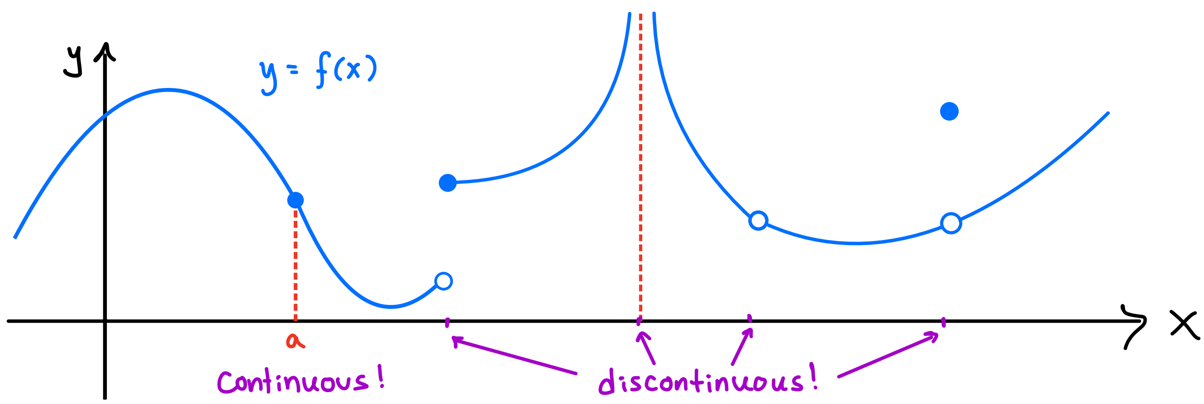
§2.4 – Continuity

Idea: $f(x)$ is continuous at $x = a$ if we can draw the graph of $f(x)$ around $x = a$ without lifting our pen.



Formally, f is continuous at $x = a$ if

(i) f is defined at $x = a$;

(ii) $\lim_{x \to a} f(x)$ exists; and

(iii) $\lim_{x \to a} f(x) = f(a)$.

Most functions we have encountered in MATH 116 are continuous at every x in their domain, including

- polynomials and rational functions
- trig and inverse trig functions
- exponential and log functions
- hyperbolic trig functions
- the absolute value function

Ex: At which points is $f(x) = \dfrac{x^2-1}{x-1}$ continuous?

Solution: $f(x) = \dfrac{x^2-1}{x-1}$ is a rational function and hence it is continuous at all points in its domain: $\{x \in \mathbb{R} : x \neq 1\}$. It is not continuous at $x=1$ since $f(1)$ is not defined.

For piecewise functions, discontinuities can sometimes be found where the function "changes pieces".

Ex: Find all constants a & b that make

$$f(x) = \begin{cases} \dfrac{b(x^2-4)}{x-2} & \text{if } x < 2, \\ 4 & \text{if } x = 2, \\ ax + 3b & \text{if } x > 2 \end{cases}$$

continuous everywhere.

Solution: Note that f will be continuous at all $x \neq 2$ since $\dfrac{b(x^2-4)}{x-2}$ is continuous for all $x<2$ and $ax+3b$ is continuous for all $x>2$. To make f continuous at $x=2$, we need

$$\lim_{x\to 2} f(x) = f(2) = 4 ,$$

or equivalently,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 4.$$

We have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{b(x^2-4)}{x-2} = \lim_{x\to 2^-} \frac{b\cancel{(x-2)}(x+2)}{\cancel{x-2}} = 4b$$

"$\frac{0}{0}$"

Hence,

$$\lim_{x\to 2^-} f(x) = 4 \Rightarrow 4b = 4 \Rightarrow \underline{b=1}$$

Also,

$b=1$!

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} ax+3b = 2a+3b = 2a+3$$

Hence,

$$\lim_{x\to 2^+} f(x) = 4 \Rightarrow 2a+3 = 4$$

$$\Rightarrow 2a = 1$$

$$\Rightarrow \underline{a = 1/2}$$

$$\boxed{\therefore\ a = \tfrac{1}{2}\ \&\ b = 1 \text{ will work.}}$$

Important fact: limits can be brought inside continuous functions! That is...

If $\lim_{x \to a} g(x) = L$ and f is continuous at L,

then $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L)$.

Ex: What is $\lim_{x \to 0} e^{x^2 \cos\left(\frac{1}{x}\right)}$?

We showed earlier that $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$.

Therefore, since e^x is continuous, we have

$$\lim_{x \to 0} e^{x^2 \cos\left(\frac{1}{x}\right)} = e^{\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)} = e^0 = \boxed{1}$$

Ex: What is $\lim_{x \to \infty} \sqrt{\frac{x+1}{2x+5}}$?

Solution: Since the square root function is continuous,

$$\lim_{x \to \infty} \sqrt{\frac{x+1}{2x+5}} = \sqrt{\lim_{x \to \infty} \frac{x+1}{2x+5}}$$

$$= \sqrt{\lim_{x \to \infty} \frac{\cancel{x}\left(1+\frac{1}{x}\right)}{\cancel{x}\left(2+\frac{5}{x}\right)}} = \sqrt{\frac{1+0}{2+0}} = \boxed{\frac{1}{\sqrt{2}}}$$