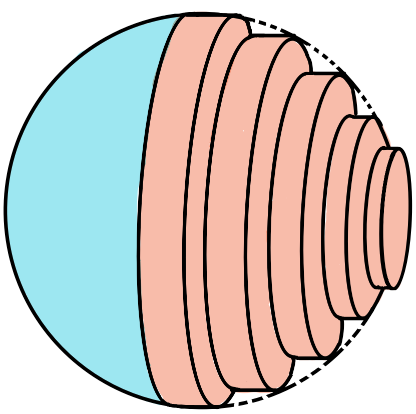
Chapter 6: Integral Calculus

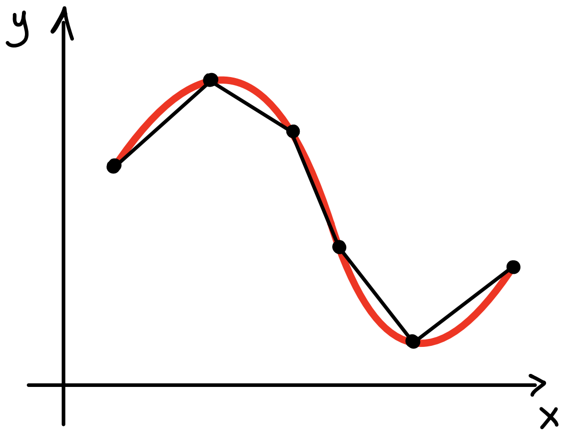
Broadly, an integral can be thought of as a way to sum up a continuum of teeny-tiny quantities.

We'll motivate integrals through the calculation of areas, but they arise in numerous other contexts.

Places Where Integrals Show Up

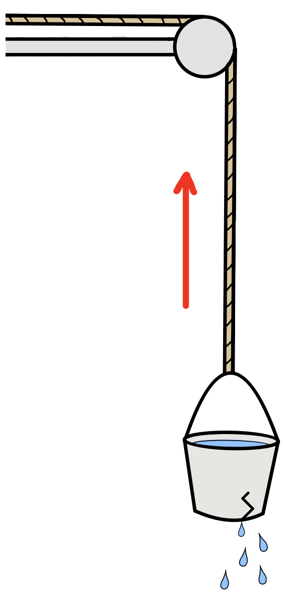
"Slicing" a sphere into thin circular disks and adding up their volumes to approximate the volume of the sphere.





Adding up the lengths of tiny straight line segments to approximate the arc length of a curve.

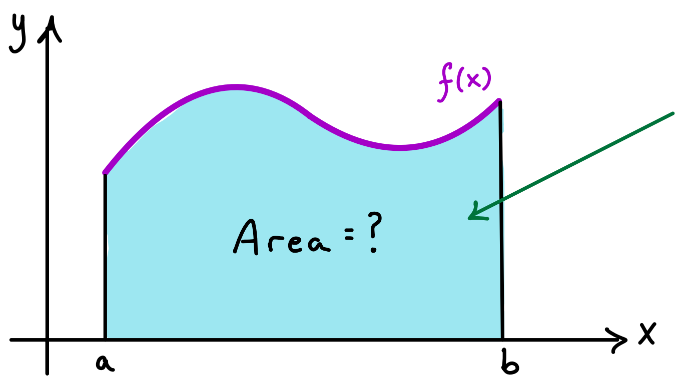
Finding the total work = force · distance to hoist a leaky bucket. The amount of force needed decreases over time. The total work can be viewed as a sum of tiny "force · distance" amounts.



We'll explore some of these applications near the end of the course!

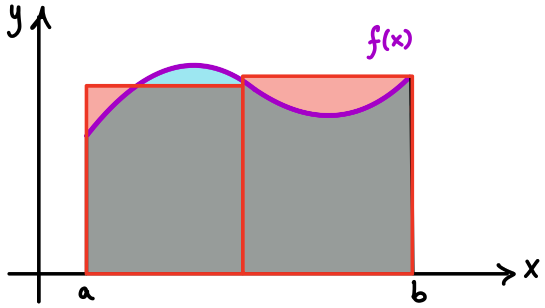
§ 6.1 – 6.3 : The Definite Integral

Motivating Question: Given a non-negative, continuous function $f(x)$, what is the area under the graph of f and above the x-axis from $x = a$ to $x = b$?

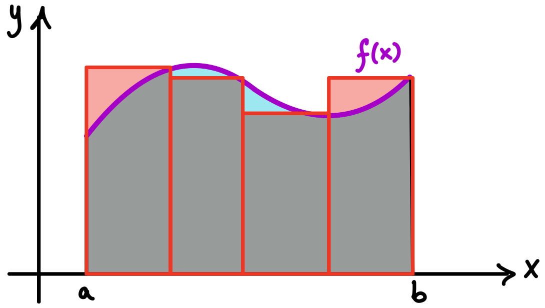


Irregular shape. No "formula" for its area...

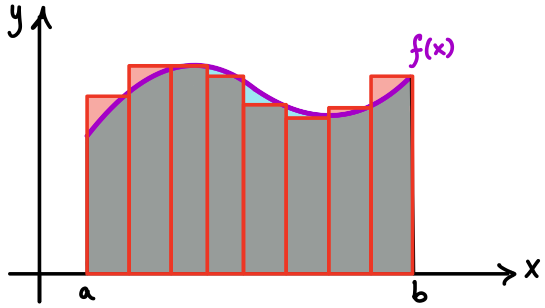
Idea: Use rectangles — a shape whose area we can easily compute — to approximate the area!



Approximating with 2 rectangles...



... 4 rectangles ...



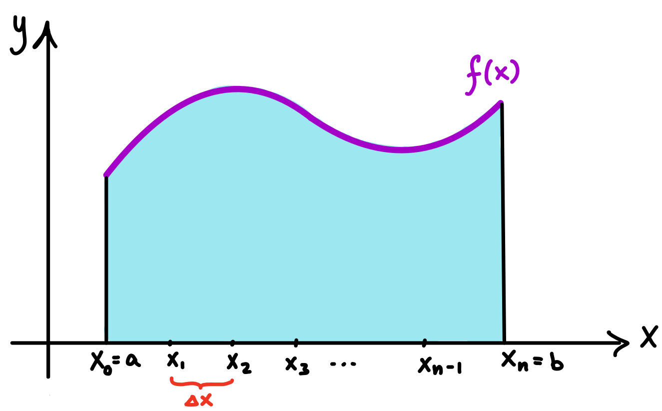
... and 8 rectangles

Observation: More rectangles $\Rightarrow$ Better approximation

To approximate with n rectangles:

Divide $[a,b]$ into n subintervals of equal length

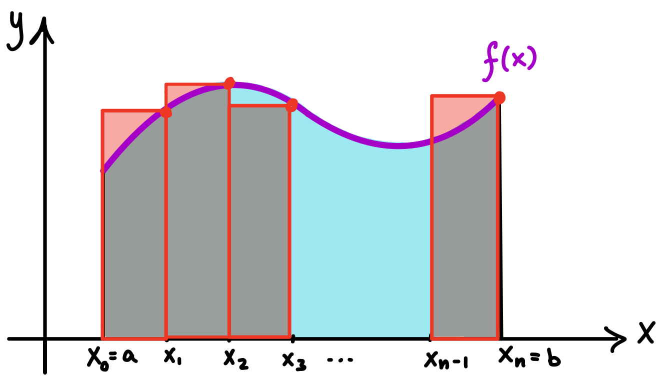
$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$



The width of each subinterval is

$$\Delta x = \frac{b-a}{n}.$$

Hence $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, ... and in general, $x_i = a + i\Delta x$.



The height of the i^{th} rectangle is $f(x_i)$.

(Here we've used the right endpoint for height)

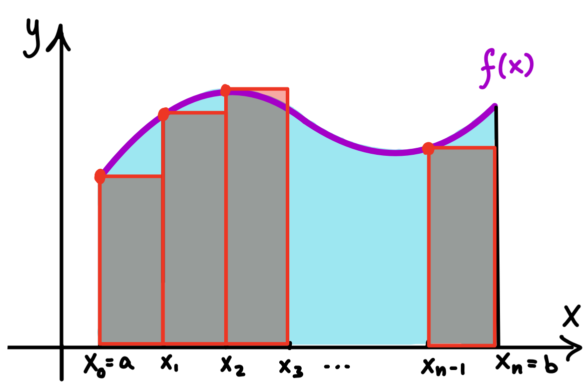
Thus, the area under the curve is approximately

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$

"sum from $i=1$ to n of $f(x_i)\Delta x$"

We call this a right-endpoint Riemann sum.

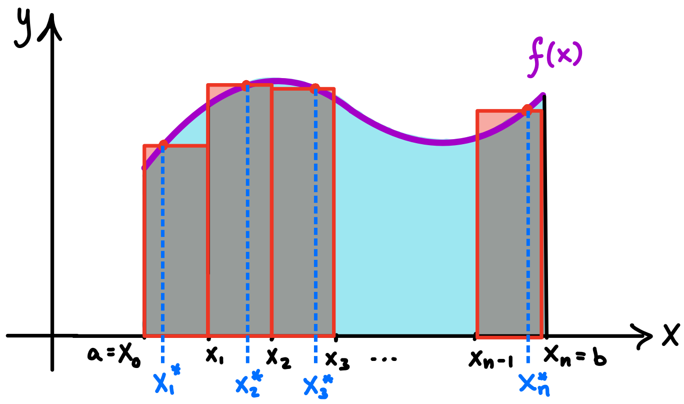
We could have alternatively used left endpoints for the height ...



$$\text{Area} \approx f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

Left-endpoint Riemann Sum

... or any other representative point $x_i^* \in [x_{i-1}, x_i]$ ...



$$\text{Area} \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

but the sum is simplest to write with right endpoints.

Useful Properties of Sums

- $\displaystyle\sum_{i=1}^{n} c\, f(i) = c\sum_{i=1}^{n} f(i)$ (c is a constant.)

- $\displaystyle\sum_{i=1}^{n} f(i) + g(i) = \sum_{i=1}^{n} f(i) + \sum_{i=1}^{n} g(i)$

Ex: Estimate the area under the graph of $f(x) = x^2$ and above the x-axis from $x = 0$ to $x = 2$. Use

(a) 4 right endpoint rectangles

(b) 4 left endpoint rectangles

Solution: $\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{4} = 0.5$

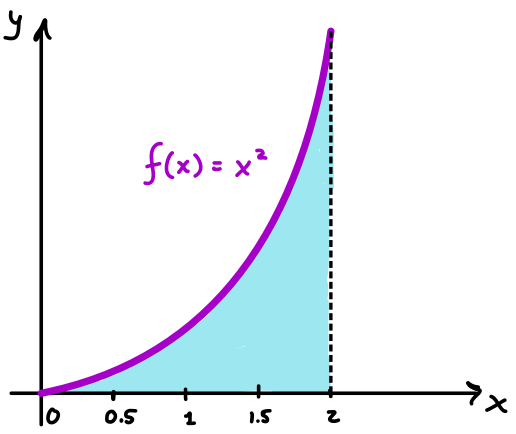
$x_0 = a = 0$

$x_1 = a + \Delta x = 0.5$

$x_2 = a + 2\Delta x = 1$

$x_3 = a + 3\Delta x = 1.5$

$x_4 = a + 4\Delta x = 2$



(a) Right endpoints

$$\text{Area} \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$= (0.5)^2 \cdot 0.5 + (1)^2 \cdot 0.5 + (1.5)^2 \cdot 0.5 + (2)^2 \cdot 0.5$$

$$= 0.125 + 0.5 + 1.125 + 2 = \boxed{3.75}$$

(b) Left endpoints

$$\text{Area} \approx f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$$

$$= (0)^2 \cdot 0.5 + (0.5)^2 \cdot 0.5 + (1)^2 \cdot 0.5 + (1.5)^2 \cdot 0.5$$

$$= 0 + 0.125 + 0.5 + 1.125 = \boxed{1.75}$$

To improve the approximation, use more rectangles!

For an exact answer, let the number of rectangles, n, approach ∞! This motivates the following:

Definition: The definite integral of $f(x)$ from $x = a$ to $x = b$ is

$$\boxed{\int_{\underbrace{a}_{\text{lower limit}}}^{\overbrace{b}^{\text{upper limit}}} \underbrace{f(x)}_{\text{integrand}}\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x}$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$.

If $f(x) \geq 0$, $\int_a^b f(x)\,dx$ represents the area under the graph of $f(x)$ from $x = a$ to $x = b$ and above the x-axis.

Ex: Use the definition of the definite integral to compute $\int_0^2 x^2\,dx$.

Solution: $\int_0^2 x^2\,dx = \lim_{x \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$.

In this case, $f(x) = x^2$. We're using n rectangles, hence

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}, \text{ and}$$

$$x_i = a + i\Delta x = 0 + i \cdot \frac{2}{n} = \frac{2i}{n}.$$

Thus, $\int_0^2 x^2\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \cdot \frac{2}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{8i^2}{n^3}$$

$$= \lim_{n\to\infty} \frac{8}{n^3} \sum_{i=1}^{n} i^2$$

What is this sum??

Useful Sums

$$\sum_{i=1}^{n} c = \underbrace{c + c + c + \cdots + c}_{n \text{ times}} = n \cdot c \qquad (c = \text{constant})$$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Okay... back to our problem.

$$\int_0^2 x^2\,dx = \lim_{n\to\infty} \frac{8}{n^3} \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{8}{n^{\cancel{3}}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{16n^2 + 24n + 8}{6n^2}$$

$$= \dots = \frac{16}{6} = \boxed{\frac{8}{3} = 2.6\overline{6}}$$

Ex: Use the definition of the definite integral to compute $\int_1^2 (1+4x)\,dx$.

Solution:

$$\int_1^2 (1+4x)\,dx = \lim_{x\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \quad \text{where} \quad f(x) = 1+4x,$$

$\Delta x = \frac{2-1}{n} = \frac{1}{n}$, and $x_i = 1 + i\Delta x = 1 + \frac{i}{n}$. Thus,

$$\int_1^2 (1+4x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + 4\left(1+\frac{i}{n}\right)\right) \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(5 + \frac{4}{n} i\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \left[\sum_{i=1}^{n} 5 + \frac{4}{n} \sum_{i=1}^{n} i \right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \left[5n + \frac{4}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)}{2} \right]$$

$$= \lim_{n\to\infty} 5 + \frac{2(n+1)}{n}$$

$$= 5 + 2$$

$$= \boxed{7}$$