# §3.2 – 3.4: Derivative Rules

We can use the definition of the derivative to establish some shortcuts for computing $f'(x)$.

## Derivative Rules

(i) If $f(x) = c$ (a constant), then $f'(x) = 0$.

(ii) If $f(x) = c \cdot g(x)$, then $f'(x) = c \cdot g'(x)$

(iii) Sum rule: $[f(x) + g(x)]' = f'(x) + g'(x)$

(iv) Product rule: $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

(v) Quotient rule: $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$

{low d-high minus high d-low, square the bottom and away we go!}

All can be proven with the definition of $f'(x)$.

We will prove (i) & (iv) and leave the rest as exercises.

<u>Proof:</u> (i) If $f(x) = c$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

(iv) If $y = f(x)g(x)$, then

$$y' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

<u>Trick:</u> Subtract and add $f(x)g(x+h)$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left[ \underbrace{\frac{f(x+h) - f(x)}{h}}_{\to f'(x)} \cdot \underbrace{g(x+h)}_{\to g(x)} + f(x) \cdot \underbrace{\frac{g(x+h) - g(x)}{h}}_{\to g'(x)} \right]$$

$$= f'(x)\cdot g(x) + f(x)\cdot g'(x)$$ ■

Ex: Find the derivative of each function below.

(a) $y = x^3 + x^{9/5}$

Solution: $y' = 3x^2 + \frac{9}{5}x^{9/5 - 1} = \boxed{3x^2 + \frac{9}{5}x^{4/5}}$

(b) $y = (x^2+3)(x-4)$

Product rule!

Solution: $y' = (x^2+3)'(x-4) + (x^2+3)(x-4)'$

$$= \boxed{2x(x-4) + (x^2+3)\cdot 1}$$

(c) $y = \dfrac{x^4+1}{x-7}$.

Quotient rule!

Solution: $y' = \dfrac{(x-7)(x^4+1)' - (x^4+1)(x-7)'}{(x-7)^2}$

$$= \boxed{\frac{4x^3(x-7) - (x^4+1)\cdot 1}{(x-7)^2}}$$

What about something like $y = (x^2+1)^{2000}$?

We need a rule to help with compositions!

**The Chain Rule**

If $y = f(g(x))$, then $y' = f'(g(x)) \cdot g'(x)$.

For $y = (x^2+1)^{2000}$, we have $y = f(g(x))$

where $f(x) = x^{2000}$ and $g(x) = x^2+1$. Hence

$$y' = f'(g(x)) \cdot g'(x) = 2000(x^2+1)^{1999} \cdot (2x)$$

Chain rule!

$$= \boxed{4000x(x^2+1)^{1999}}$$

Ex: Let $f(x) = x^3(1-2x)^5$. Find $f'(x)$.

Solution: $f'(x) = (x^3)' \cdot (1-2x)^5 + x^3 \cdot \left[(1-2x)^5\right]'$

(product rule!)

$$= 3x^2 \cdot (1-2x)^5 + x^3 \cdot 5(1-2x)^4 \cdot (1-2x)'$$

(chain rule!)

$$= \boxed{3x^2(1-2x)^5 + x^3 \cdot 5(1-2x)^4(-2)}$$

<u>Ex:</u> Let $f(x) = \dfrac{1}{(1+x+x^2)^9}$. Find $y'$.

<u>Solution:</u> To avoid quotient rule, write $f(x) = (1+x+x^2)^{-9}$

and use chain rule!

$$f'(x) = -9(1+x+x^2)^{-10} \cdot (1+x+x^2)' = \boxed{\dfrac{-9(1+2x)}{(1+x+x^2)^{10}}}$$