Even and Odd Functions

A function $y = f(x)$ is said to be

- even if $f(-x) = f(x)$ for all x in its domain
- odd if $f(-x) = -f(x)$ for all x in its domain.

Ex: $f(x) = \frac{x^3}{2}$

$$f(-x) = \frac{(-x)^3}{2} = \frac{(-1)^3 x^3}{2} = \frac{-x^3}{2} = -f(x)$$

$\therefore$ $f(x)$ is odd.

Ex: $g(x) = 1 + \frac{1}{x^2}$

$$g(-x) = 1 + \frac{1}{(-x)^2} = 1 + \frac{1}{x^2} = g(x) \qquad \therefore g \text{ is even.}$$

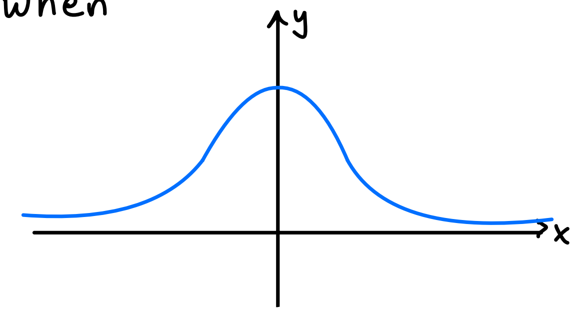
Ex: $h(x) = x^4 + 2x$

$\neq h(x), -h(x)$

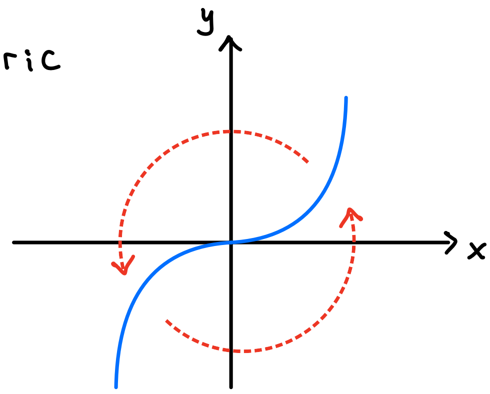
$$h(-x) = (-x)^4 + 2(-x) = x^4 - 2x$$

$\therefore$ Neither even nor odd!

Even functions are symmetric when reflected over the y-axis...



While odd functions are symmetric when reflected over <u>both</u> axes.



Similar to how a complex number can be broken up into real and imaginary parts, a function $f(x)$ can be broken up into <u>even and odd parts</u>!

$$\underbrace{f_e(x)}_{\text{The even part of } f.} = \frac{f(x) + f(-x)}{2}, \qquad \underbrace{f_o(x)}_{\text{The odd part of } f.} = \frac{f(x) - f(-x)}{2}$$

Ex: For $f(x) = x^4 + 2x$

$$f_e(x) = \frac{f(x) + f(-x)}{2} = \frac{(x^4 + 2x) + ((-x)^4 + 2(-x))}{2}$$

$$= \frac{(x^4 + \cancel{2x}) + (x^4 - \cancel{2x})}{2} = x^4$$

even!

$$f_o(x) = \frac{f(x) - f(-x)}{2} = \frac{(x^4 + 2x) - ((-x)^4 + 2(-x))}{2}$$

$$= \frac{\cancel{x^4} + 2x - (\cancel{x^4} - 2x)}{2} = 2x$$

odd!

Properties of f_e and f_o:

(i) f_e is an even function

(ii) f_o is an odd function

(iii) $f_e(x) + f_o(x) = f(x)$

Proof: (i) To see that f_e is even, note that

$$f_e(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = f_e(x).$$

(ii) To see that $f_o(x)$ is odd, note that

$$f_o(-x) = \frac{f(-x) - f(-(-x))}{2}$$

$$= \frac{f(-x) - f(x)}{2}$$

$$= \frac{-[f(x) - f(-x)]}{2} = -f_o(x).$$

(iii) Finally,

$$f_e(x) + f_o(x) = \left[\frac{f(x) + \cancel{f(-x)}}{2}\right] + \left[\frac{f(x) - \cancel{f(-x)}}{2}\right]$$

$$= \frac{f(x) + f(x)}{2} = f(x),$$

as claimed.

"End of proof." $\blacksquare$

<u>Ex:</u> Suppose f is an even function. Show that

$$f_e(x) = f(x) \text{ and } f_o(x) = 0.$$

Solution: We have

$=$ (since f is even)

$$f_e(x) = \frac{f(x)+f(-x)}{2} = \frac{f(x)+f(x)}{2} = \frac{2f(x)}{2} = f(x)$$

$=$ (again, since f is even)

$$f_o(x) = \frac{f(x)-f(-x)}{2} = \frac{f(x)-f(x)}{2} = \frac{0}{2} = 0$$

$\blacksquare$

A similar statement as above is true for odd functions — try proving it as an exercise!

Additional Exercises:

1. Determine $f_e(x)$ and $f_o(x)$ for $f(x) = \frac{1}{x+1}$.

2. Is there a function that is both even <u>and</u> odd? Explain.

Solutions:

1. With $f(x) = \frac{1}{1+x}$, we have

$$f_e(x) = \frac{f(x) + f(-x)}{2} = \frac{\frac{1}{1+x} + \frac{1}{1-x}}{2}$$

$$= \frac{\frac{(1-\cancel{x}) + (1+\cancel{x})}{(1+x)(1-x)}}{2}$$

$$= \frac{\cancel{2}}{\cancel{2}(1-x^2)} = \boxed{\frac{1}{1-x^2}}$$

$$f_o(x) = \frac{f(x) - f(-x)}{2} = \frac{\frac{1}{1+x} - \frac{1}{1-x}}{2}$$

$$= \frac{\frac{(\cancel{1}-x) - (\cancel{1}+x)}{(1+x)(1-x)}}{2}$$

$$= \frac{-\cancel{2}x}{\cancel{2}(1-x^2)} = \boxed{\frac{-x}{1-x^2}}$$

2. For f to be both even and odd, we would need

$$f(-x) = f(x) \quad (1)$$

and $$f(-x) = -f(x) \quad (2)$$

for all x in the domain. Hence, subtracting (2) from (1) we get $0 = 2f(x)$, or equivalently, $f(x) = 0$ for all x in the domain. Therefore, the only function that is both even and odd is the constant function, $\boxed{f(x) = 0.}$