Derivatives of Logarithmic & Exponential Functions

Recall: e is the unique base such that $y = e^x$ has slope 1 at $x = 0$. That is, $f(x) = e^x$ has derivative $f'(0) = 1$. Thus,

$$f'(0) = \lim_{h \to 0} \frac{e^{0+h} - e^0}{h} = \boxed{\lim_{h \to 0} \frac{e^h - 1}{h} = 1}$$

We can use this limit to find $(e^x)'$!

$$(e^x)' = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} e^x \left(\cancelto{1}{\frac{e^h - 1}{h}} \right) = e^x$$

$$\therefore \boxed{(e^x)' = e^x}$$

What about the derivative of a^x?

Trick: $a^x = (e^{\ln a})^x = e^{x \ln a}$!

We then have

$$(a^x)' = (e^{x\ln a})' = \underbrace{e^{x\ln a}}_{=a^x} \cdot \underbrace{(x\ln a)'}_{=\ln a} = a^x \ln(a)$$

$$\therefore \boxed{(a^x)' = a^x \cdot \ln a}$$

To find the derivative of $y = \log_a(x)$, we can use implicit differentiation:

$$y = \log_a x \Rightarrow a^y = x$$

Implicit Differentiation

$$\Rightarrow a^y \cdot \ln(a) \cdot y' = 1$$

$$\Rightarrow y' = \frac{1}{a^y \cdot \ln(a)} = \frac{1}{x \cdot \ln(a)} \quad (\text{since } a^y = x)$$

$$\therefore \boxed{(\log_a x)' = \frac{1}{x \cdot \ln(a)}}$$

When $a = e$, we get

$$\boxed{(\ln x)' = \frac{1}{x}}$$

Ex: Find the following derivatives.

(a) $y = \dfrac{\ln x}{x}$ (b) $y = 2^{e^x \cdot \tan(x)}$

Solution:

(a) $y' = \dfrac{x(\ln x)' - \ln x\,(x)'}{x^2}$

$= \dfrac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \boxed{\dfrac{1 - \ln x}{x^2}}$

(b) $y' = 2^{e^x \tan x} \cdot \ln(2) \cdot (e^x \tan x)'$

$= \boxed{2^{e^x \tan x} \cdot \ln(2)\left(e^x \tan x + e^x \sec^2 x\right)}$

When covering logarithmic differentiation, we'll often encounter functions of the form $y = \ln(f(x))$. From the chain rule, we have

$$\boxed{\left[\ln(f(x))\right]' = \frac{f'(x)}{f(x)}}$$