§6.4 – The Fundamental Theorem of Calculus – Part I

Recall:

The definite integral of $f(x)$ from $x=a$ to $x=b$ is

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x}$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

This definition is AWFUL to use, and so, in practice we instead compute integrals using the following theorem.

The Fundamental Theorem of Calculus (FTC) Part I

If $f(x)$ is continuous, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a).}$$

where F is any antiderivative of $f(x)$ (i.e., $F' = f$)

Ex: Use the FTC to evaluate $\int_0^2 x^2\,dx$.

Solution: An antiderivative of $f(x) = x^2$ is $F(x) = \frac{x^3}{3}$.

Thus, $$\int_0^2 x^2\,dx = F(2) - F(0) = \frac{2^3}{3} - \frac{0^3}{3} = \boxed{\frac{8}{3}}$$

Ex: Use the FTC to evaluate $\int_1^2 (1+4x)\,dx$.

Solution: An antiderivative of $f(x) = 1+4x$ is

$F(x) = x + 2x^2$, so

Notation for $F(b) - F(a)$

$$\int_1^2 (1+4x)\,dx = \left[x + 2x^2\right]_1^2$$

$$= \left(2 + 2(2)^2\right) - \left(1 + 2(1)^2\right) = 10 - 3 = \boxed{7}$$

Intuition for the FTC: If $y = F(x)$ then,

from our differentials lesson, $dy = F'(x)\,dx$. Thus,

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = \int_a^b dy$$

can be viewed as a sum of tiny changes in $y = F(x)$ as x varies from a to b. The FTC states that the sum of these changes is the same as the net change, $F(b) - F(a)$.

Ex: What is $\int_0^1 x^5\,dx$?

Solution: $\int_0^1 x^5\,dx = \left[\frac{x^6}{6}\right]_0^1 = \frac{1^6}{6} - \frac{0^6}{6} = \boxed{\frac{1}{6}}$

But wait a second... isn't $\frac{x^6}{6} + 1$ also an antiderivative of x^5? And $\frac{x^6}{6} - 2$? And $\frac{x^6}{6} + \pi$?

Yes! In fact, every antiderivative of x^5 has this form

If $F(x)$ is an antiderivative of $f(x)$, then every antiderivative of $f(x)$ has the form $F(x)+C$, where $C \in \mathbb{R}$ is a constant.

The collection of all antiderivatives of $f(x)$ is called the indefinite integral of $f(x)$, written

$$\boxed{\int f(x)\,dx = F(x) + C}$$

So for instance, $\int x^5\,dx = \frac{x^6}{6} + C$. More generally ...

The Power Rule for Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

Some Useful Antiderivatives

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C \qquad \int \sec x \cdot \tan x \, dx = \sec x + C$$

$$\int \frac{1}{1+x^2} \, dx = \arctan x + C \qquad \int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$$

$$\int e^x \, dx = e^x + C \qquad \int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

Why $|x|$? Well..., $\frac{1}{x}$ is defined for $x>0$ and $x<0$, so its antiderivative should be too!

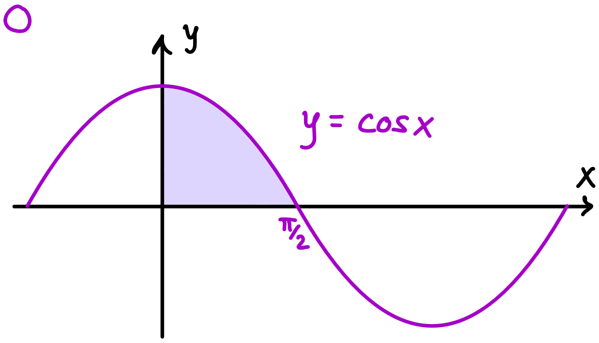
Ex: The area under $y = \cos x$ from $x = 0$ to $x = \pi/2$ is

$$\int_0^{\pi/2} \cos x \, dx = \Big[\sin x\Big]_0^{\pi/2}$$

$$= \sin \pi/2 - \sin 0$$

$$= 1 - 0$$

$$= \boxed{1}$$



Note: We don't write "+ C" when evaluating a definite integral $\int_a^b f(x)\,dx$ ... but if we <u>did</u>, the "+C" would just cancel out anyway! For instance,

$$\int_0^{\pi/2} \cos x \, dx = \Big[\sin x + C\Big]_0^{\pi/2}$$

$$= \left(\sin \tfrac{\pi}{2} + \cancel{C}\right) - \left(\sin 0 + \cancel{C}\right) = \boxed{1}$$

Antiderivatives of (slightly!) more complicated functions can be calculated using the following properties.

<u>Properties of Integrals</u>

(i) $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

(ii) $\int K \cdot f(x)\,dx = K \cdot \int f(x)\,dx \quad (K = \text{constant})$

<u>Ex:</u> Evaluate the following.

(a) $\int (11x^4 + 1)\,dx = 11\int x^4\,dx + \int 1\,dx = \boxed{\frac{11}{5}x^5 + x + C}$

(b) $\displaystyle\int \left(10^x + e^x + \sin x\right) dx = \boxed{\frac{10^x}{\ln 10} + e^x - \cos x + C}$

(c) $\displaystyle\int \left(\sqrt{x} - 3x^{4/3}\right) dx = \int \left(x^{1/2} - 3x^{4/3}\right) dx$

$$= \frac{x^{3/2}}{3/2} - 3\,\frac{x^{7/3}}{7/3} + C$$

$$= \boxed{\frac{2}{3}x^{3/2} - \frac{9}{7}x^{7/3} + C}$$

Sometimes, a bit of manipulation is needed.

(d) $\displaystyle\int (x^4+1)^2\, dx = \int \left(x^8 + 2x^4 + 1\right) dx = \boxed{\frac{x^9}{9} + \frac{2x^5}{5} + x + C}$

(e) $\displaystyle\int \frac{t^{1/3}+1}{t}\, dt = \int \left(t^{-2/3} + \frac{1}{t}\right) dt = \frac{t^{1/3}}{1/3} + \ln|t| + C$

$$= \boxed{3t^{1/3} + \ln|t| + C}$$

(f) $\displaystyle\int \frac{x^2}{1+x^2}\, dx = \int \frac{(x^2+1)-1}{1+x^2}\, dx = \int \left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right) dx$

$$= \int 1\, dx - \int \frac{1}{1+x^2}\, dx = \boxed{x - \arctan(x) + C}$$

The following properties can help with certain definite integrals:

Given real numbers a, b, and c,

(i) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(ii) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

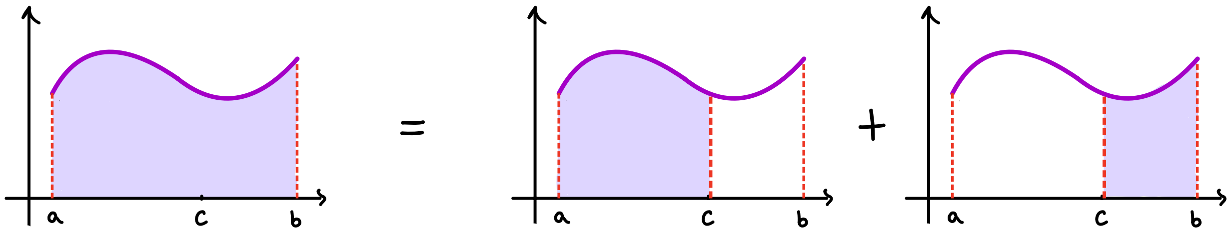
In the case that f is continuous, these properties quickly follow from the FTC. Indeed:

(i) $$\int_a^b f(x)\,dx = F(b) - F(a) = -\big(F(a) - F(b)\big) = -\int_b^a f(x)\,dx$$

(ii) $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \left[\cancel{F(c)} - F(a)\right] + \left[F(b) - \cancel{F(c)}\right]$$

$$= F(b) - F(a) = \int_a^b f(x)\,dx$$

Alternatively, you can think of (ii) in terms of areas:



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

<u>Ex:</u> $\displaystyle\int_{-2}^{3} |2x|\,dx$

<u>Solution:</u> Since $|2x| = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$, we have

$$\begin{aligned}
\int_{-2}^{3} |2x|\,dx &= \int_{-2}^{0} |2x|\,dx + \int_{0}^{3} |2x|\,dx \\
&= \int_{-2}^{0} -2x\,dx + \int_{0}^{3} 2x\,dx \\
&= -\left[x^2\right]_{-2}^{0} + \left[x^2\right]_{0}^{3} \\
&= -\left(0^2 - (-2)^2\right) + \left(3^2 - 0^2\right) = \boxed{13}
\end{aligned}$$

Ex: Evaluate $\int_0^3 f(x)\,dx$ if $f(x) = \begin{cases} 1-x^2 & \text{if } x \le 1 \\ 2 & \text{if } x > 1 \end{cases}$.

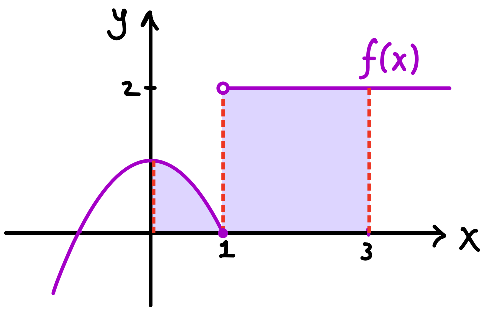
Solution:

$$\int_0^3 f(x)\,dx = \int_0^1 (1-x^2)\,dx + \int_1^3 2\,dx$$

$$= \left[x - \frac{x^3}{3}\right]_0^1 + \left[2x\right]_1^3$$

$$= \left[\left(1-\frac{1}{3}\right) - \left(0 - \frac{0}{3}\right)\right] + \left[2(3) - 2(1)\right]$$

$$= \frac{2}{3} + 4 = \boxed{\frac{14}{3}}$$



For some (not too complicated) functions, you can find an antiderivative by making an educated guess and checking with differentiation.

Ex: What is $\int e^{3x+1}\,dx$?

Solution: Is it $e^{3x+1} + C$? Not quite: $\left(e^{3x+1} + C\right)' = 3e^{3x+1}$

Fix: Divide by 3! We get

$$\boxed{\int e^{3x+1}\,dx = \frac{e^{3x+1}}{3} + C}$$

Ex: Evaluate $\displaystyle\int_0^{\pi} \cos(2x)\,dx$.

Solution: Maybe $\sin(2x)$ is an antiderivative? Not quite,

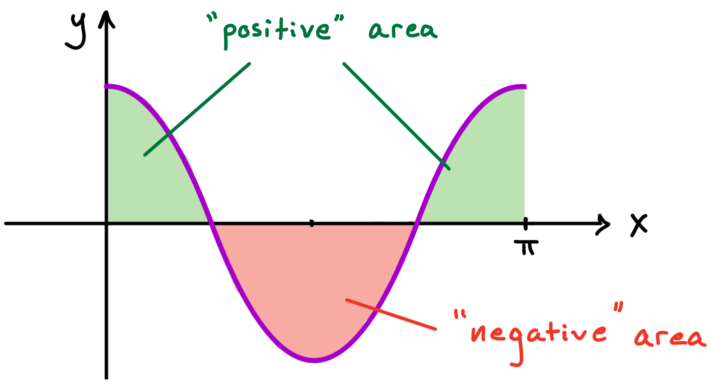
since $(\sin(2x))' = 2\cos(2x)$. Fix: Divide by 2!

$$\int_0^{\pi} \cos(2x)\,dx = \left[\frac{\sin(2x)}{2}\right]_0^{\pi}$$

$$= \frac{\sin(2\pi)}{2} - \frac{\sin(0)}{2} = \boxed{0}$$

Wait ... if $\displaystyle\int_0^{\pi} \cos(2x)\,dx$ represents an area, how can this integral possibly be 0??

It turns out that $\displaystyle\int_a^b f(x)\,dx$ actually represents a signed area, with area below the x-axis counted negatively:



In the above example, the area above the x-axis cancels perfectly with the area below, giving

$$\boxed{\int_0^{\pi} \cos(2x)\,dx = 0}$$