§6.5 – The Fundamental Theorem of Calculus – Part II

From Part I of the FTC, we know how useful antiderivatives are for evaluating $\int_a^b f(x)\,dx$. But does every continuous function even have an antiderivative?

Yes! This is Part II of the FTC!

The Fundamental Theorem of Calculus (FTC) – Part II

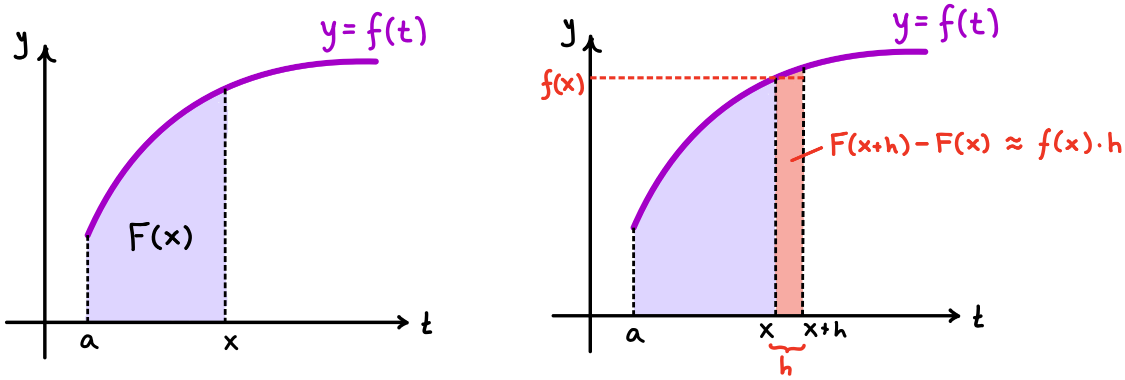
If f is continuous on $[a,b]$, then the function

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable and $\frac{d}{dx} F(x) = f(x)$. That is, $F(x)$ is an antiderivative of $f(x)$.

Okay... let's unpack this statement carefully...

Given x, the function $F(x) = \int_a^x f(t)\,dt$ outputs the area under $y = f(t)$ from $t = a$ to $t = x$.



Moving from x to $x+h$, the change in area $F(x+h) - F(x)$ is approximately $f(x) \cdot h$, hence $\dfrac{F(x+h) - F(x)}{h} \approx f(x)$.

When $h \to 0$, the approximation becomes exact:

$$\frac{d}{dx} F(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

<u>Takeaway</u>: FTC Part II tells us that

variable

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

constant

Ex: What is $\frac{d}{dx}\int_0^x t^2\,dt$?

Solution: By the FTC part II, $\frac{d}{dx}\int_1^x t^2\,dt = \boxed{x^2}$

Alternatively, we could have first evaluated the integral and then differentiated:

$$\frac{d}{dx}\int_1^x t^2\,dt = \frac{d}{dx}\left[\frac{t^3}{3}\right]_1^x = \frac{d}{dx}\left[\frac{x^3}{3} - \frac{1}{3}\right] = \boxed{x^2}$$

The advantage to using the FTC is that we don't need to first evaluate the integral — or even know how to!

Ex: What is $\frac{d}{dx}\underbrace{\int_2^x \tan(t^2\cdot e^t)\,dt}$?

No clue how to integrate this... but that's ok!

Solution: By FTC II,

$$\frac{d}{dx}\int_2^x \tan(t^2 e^t)\,dt = \boxed{\tan(x^2 e^x)}$$

A function, $h(x)$!

Ex: What is $\dfrac{d}{dx}\displaystyle\int_0^{8x+1} \sqrt{1+t^3}\, dt$?

It turns out that

$$\boxed{\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = f\big(h(x)\big)\cdot h'(x)}$$

Why? Well ... if $F(x) = \displaystyle\int_a^x f(t)\,dt$, then $F'(x) = f(x)$ by FTC II, hence

$$\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = \frac{d}{dx} F\big(h(x)\big)$$

$$= F'\big(h(x)\big)\cdot h'(x)$$

$$= f\big(h(x)\big)\cdot h'(x),$$

as claimed. Now let's revisit our example!

Solution:

$$\frac{d}{dx}\int_0^{8x+1} \sqrt{1+t^3}\, dt = \sqrt{1+(8x+1)^3}\cdot(8x+1)'$$

$$= \boxed{\sqrt{1+(8x+1)^3}\cdot 8}$$

Ex: What is $\frac{d}{dx}\int_{2x}^{\sin x} e^{t^2}\,dt$?

A function, h(x)!

Another function, g(x)!

In general

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = \frac{d}{dx}\left(\int_{g(x)}^{0} f(t)\,dt + \int_{0}^{h(x)} f(t)\,dt\right)$$

$$= \frac{d}{dx}\left(\int_{0}^{h(x)} f(t)\,dt - \int_{0}^{g(x)} f(t)\,dt\right)$$

$$= f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

This is the most general version of FTC II :

$$\boxed{\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)}$$

Solution:

$$\frac{d}{dx}\int_{2x}^{\sin x} e^{t^2}\,dt = e^{(\sin x)^2}\cdot(\sin x)' - e^{(2x)^2}\cdot(2x)'$$

$$= \boxed{e^{\sin^2 x}\cdot \cos x - e^{4x^2}\cdot 2}$$

FTC II tells us that every continuous function f has an antiderivative, $F(x) = \int_0^x f(t)\,dt$. Sometimes, we can express this antiderivative in simple terms...

e.g. x^2 has antiderivative $F(x) = \int_0^x t^2\,dt$, which we can write simply as

$$F(x) = \int_0^x t^2\,dt = \left[\frac{t^3}{3}\right]_0^x = \frac{x^3}{3} \quad \text{(nice!)}$$

... but sometimes it's not possible to express $F(x) = \int_a^x f(t)\,dt$ in terms of elementary (i.e., familiar) functions.

e.g. $F(x) = \int_0^x e^{t^2}\,dt$ is an antiderivative of e^{x^2}, but there is no way to express $F(x)$ in terms of polynomials, roots, trig functions, exponentials, logs, etc. !

So, while every continuous function has an antiderivative,

not all continuous functions have <u>nice</u> antiderivatives!