Generalized Sine Functions

Fact: Every function of the form

angular frequency

$$\boxed{B \cdot \sin(\omega x) + C \cdot \cos(\omega x)}$$

constants that are known to us

can be written in the form

$$\boxed{A \sin(\omega x + \varphi),}$$

constants to be determined

which we call a generalized sine function.

To determine A and φ, let's expand $\sin(\omega x + \varphi)$:

$$\boxed{B}\sin(\omega x) + \boxed{C}\cos(\omega x) = A\sin(\omega x + \varphi)$$

$$= A\left[\sin(\omega x)\cos\varphi + \cos(\omega x)\sin\varphi\right]$$

$$= \boxed{(A\cos\varphi)}\sin(\omega x) + \boxed{(A\sin\varphi)}\cos(\omega x)$$

By comparing coefficients on both sides, we get

$B = A\cos\varphi$ and $C = A\sin\varphi$. It then follows that

$$\frac{C}{B} = \frac{\cancel{A}\sin\varphi}{\cancel{A}\cos\varphi} = \tan\varphi,$$

hence $\underline{\varphi = \tan^{-1}\left(\frac{C}{B}\right)}$. We can then use $\underline{B = A\cos\varphi}$ (or $C = A\sin\varphi$) to get A.

<u>Summary:</u> To write $B\sin(\omega x) + C\cos(\omega x)$ as $A\sin(\omega x + \varphi)$...

① Calculate $\varphi = \tan^{-1}\left(\frac{C}{B}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

② Calculate $A = \dfrac{B}{\cos\varphi}$.

<u>Ex:</u> Write $2\sin(3x) - 2\cos(3x)$ in the form $A\sin(3x + \varphi)$.

<u>Solution:</u> Here, $B = 2$, $C = -2$. We have

① $\varphi = \tan^{-1}\left(\frac{C}{B}\right) = \tan^{-1}\left(\frac{-2}{2}\right) = \tan^{-1}(-1) = \underline{-\frac{\pi}{4}}$ and

② $A = \dfrac{B}{\cos\varphi} = \dfrac{2}{\cos\left(-\frac{\pi}{4}\right)} = \dfrac{2}{\left(\sqrt{2}/2\right)} = \underline{\dfrac{4}{\sqrt{2}}}$

Hence, $\boxed{2\sin(3x) - 2\cos(3x) = \frac{4}{\sqrt{2}}\sin\left(3x - \frac{\pi}{4}\right)}$

$\underline{\text{Ex:}}$ Let $f(x) = 3\sin\left(\frac{x}{2}\right) + 3\sqrt{3}\cos\left(\frac{x}{2}\right)$.

(a) Write $f(x)$ in the form $A\sin\left(\frac{x}{2} + \varphi\right)$.

(b) Find all x such that $f(x) = 6$.

<u>Solution:</u>

(a) Here, $B = 3$, $C = 3\sqrt{3}$. We have

① $\varphi = \tan^{-1}\left(\frac{C}{B}\right) = \tan^{-1}\left(\frac{3\sqrt{3}}{3}\right) = \tan^{-1}(\sqrt{3}) = \underline{\frac{\pi}{3}}$, and

② $A = \frac{B}{\cos\varphi} = \frac{3}{\cos(\pi/3)} = \frac{3}{(1/2)} = \underline{6}$

Hence, $\boxed{f(x) = 6\sin\left(\frac{x}{2} + \frac{\pi}{3}\right)}$

(b) It will be easiest to work with the description of $f(x)$ from (a). We have

$$f(x) = 6 \Rightarrow 6\sin\left(\frac{x}{2} + \frac{\pi}{3}\right) = 6$$

$$\Rightarrow \sin\left(\frac{x}{2} + \frac{\pi}{3}\right) = 1$$

$$\Rightarrow \frac{x}{2} + \frac{\pi}{3} = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$$

The set of all integers.

$$\Rightarrow \frac{x}{2} = \left(\frac{\pi}{2} - \frac{\pi}{3}\right) + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \frac{x}{2} = \frac{\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow \boxed{x = \frac{\pi}{3} + 4k\pi, \quad k \in \mathbb{Z}}$$

$$\left(= \frac{\pi}{3}, \ \frac{\pi}{3} \pm 4\pi, \ \frac{\pi}{3} \pm 8\pi, \ldots \right)$$