Hyperbolic Trigonometric Functions

The hyperbolic sine and cosine functions are

$$\boxed{\sinh(x) = \frac{e^x - e^{-x}}{2}}$$

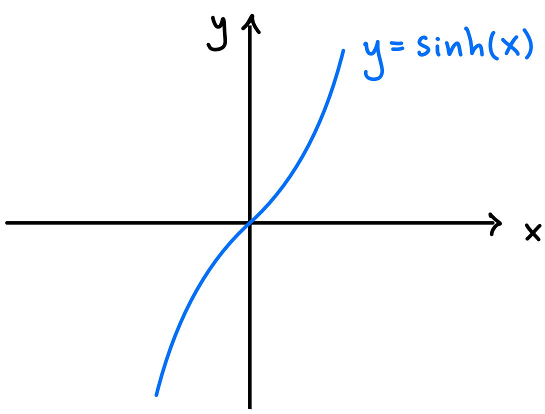
"sinch"

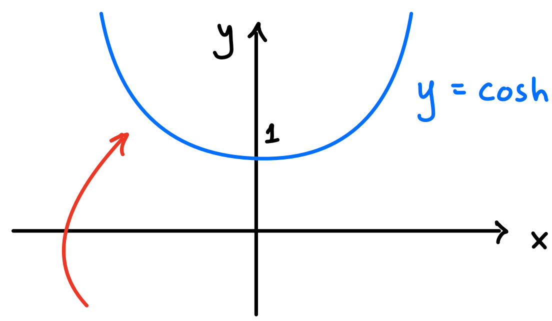
and

$$\boxed{\cosh(x) = \frac{e^x + e^{-x}}{2}}$$

"cosh"

These are the odd and even parts of e^x!





Not a parabola — this is the shape of a hanging cable!

These functions satisfy very similar identities to those for $\sin(x)$ and $\cos(x)$!

<u>Ex:</u> Show that $\sinh(2x) = 2\sinh(x)\cosh(x)$

Solution: From the right-hand side, we have

$$2\sinh(x)\cosh(x) = \cancel{2}\left(\frac{e^x - e^{-x}}{\cancel{2}}\right)\left(\frac{e^x + e^{-x}}{2}\right)$$

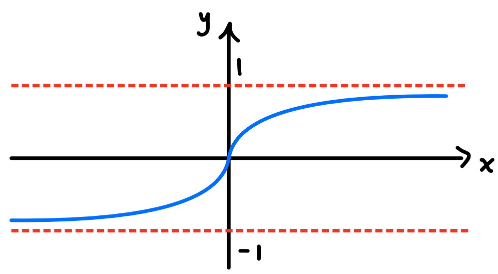
$$= \frac{e^x e^x + \cancel{e^x e^{-x}} - \cancel{e^{-x} e^x} - e^{-x} e^{-x}}{2}$$

$$= \frac{e^{2x} - e^{-2x}}{2},$$

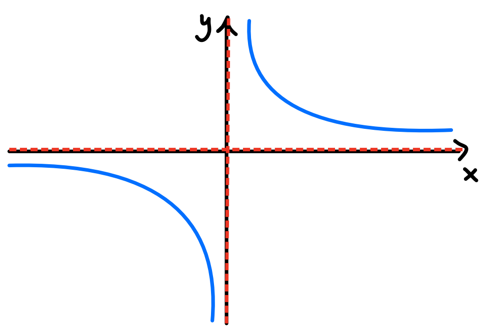
which we recognize as $\sinh(2x)$. ∎

The other hyperbolic trig functions can be defined in terms of sinh and cosh as we might expect:

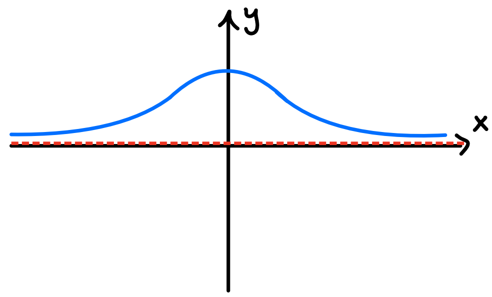
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



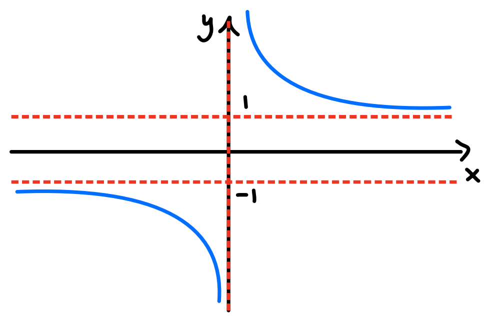
$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$$



$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$



$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



Additional Exercises:

1. Find simplified expressions for

   $\cosh(x) + \sinh(x)$ and $\cosh(x) - \sinh(x)$.

2. Show that $\cosh^2(x) - \sinh^2(x) = 1$.

Solutions:

1. We have

$$\cosh(x) + \sinh(x) = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \frac{2e^x}{2} = \boxed{e^x}$$

$$\cosh(x) - \sinh(x) = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = \frac{2e^{-x}}{2} = \boxed{e^{-x}}$$

2. To prove $\cosh^2(x) - \sinh^2(x) = 1$, we start with the LHS:

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{\cancel{(e^x)^2} + 2e^x e^{-x} + \cancel{(e^{-x})^2}}{4} - \frac{\cancel{(e^x)^2} - 2e^x e^{-x} + \cancel{(e^{-x})^2}}{4}$$

$$= \frac{\cancel{4} e^x e^{-x}}{\cancel{4}}$$

$$= \cancel{e^x} \cdot \frac{1}{\cancel{e^x}} = 1. \qquad \blacksquare$$

Alternatively, from 1.,

$$\cosh^2(x) - \sinh^2(x) = [\cosh(x) + \sinh(x)][\cosh(x) - \sinh(x)]$$

$$= e^x \cdot e^{-x} = 1. \qquad \blacksquare$$