§8.2 - Integration by Parts (IBP)

Recall: If u and v are functions of x, then by the product rule,

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

Integrating both sides, we have

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{\cancel{dx}}\,\cancel{dx} + \int v\frac{du}{\cancel{dx}}\,\cancel{dx}$$

$$\Longrightarrow \quad uv = \int u\,dv + \int v\,du$$

$$\overset{\text{Rearrange}}{\Longrightarrow} \quad \boxed{\int u\,dv = uv - \int v\,du.}$$

Integration by parts (IBP) formula.

Remarks:

1. IBP allows us to trade one integral, $\int u\,dv$ for another (hopefully simpler) integral, $\int v\,du$.

2. IBP can help when integrating a product of functions. One function, u, will be differentiated while the other, dv, will be integrated.

Ex: $\int 2x \cos x \, dx$

Strategy: Pick u to be a function that gets simpler when differentiated. Everything else is dv.

Solution: Let's define u and dv as follows:

$$u = 2x \qquad\qquad v = \sin x$$

$$\text{differentiate} \downarrow \qquad\qquad \uparrow \text{integrate}$$

$$du = 2\,dx \qquad\qquad dv = \cos x \, dx$$

Then

$$\int \underbrace{2x}_{u} \underbrace{\cos x \, dx}_{dv} = uv - \int v \, du$$

$$= 2x\sin x - \underbrace{\int 2\sin x\,dx}_{= -\cos x + C}$$

$$= \boxed{2x\sin x + 2\cos x + C}$$

A better strategy for picking u is the LIATE method.

Logs
Inverse trig
Algebraic (x^n, polynomials)
Trig
Exponential

Let u be the first function on this list that appears in your integral. Let everything else be dv.

Ex: Evaluate $\int x^2 \ln x\,dx$

Solution: Using IBP and the LIATE method,

$$u = \ln x \qquad v = x^3/3$$

$$du = \frac{1}{x}\,dx \qquad dv = x^2\,dx$$

$$\therefore \int x^2 \ln x\,dx = uv - \int v\,du$$

$$= (\ln x)\cdot \frac{x^3}{3} - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx$$

$$= \boxed{\frac{x^3}{3}\ln x - \frac{x^3}{9} + C}$$

Ex: Evaluate $\int x^2 e^x\,dx$

Solution: Let $u = x^2$ $\quad v = e^x$

$du = 2x\,dx$ $\quad dv = e^x\,dx$

$$\int x^2 e^x\,dx = x^2 e^x - \int e^x \cdot 2x\,dx$$

$$= x^2 e^x - 2\underbrace{\int x e^x\,dx}_{\text{IBP again!}}$$

$u = x$ $\quad v = e^x$

$du = dx$ $\quad dv = e^x dx$

$$= x^2 e^x - 2\left[x e^x - \int e^x\,dx\right]$$

$$= \boxed{x^2 e^x - 2xe^x + 2e^x + C}$$

Sometimes we may need to combine multiple methods.

Ex: Evaluate $\int \sin x \cos x\, e^{\sin x}\,dx$

Solution: Let's first clean up the integral using a u-substitution.

$$\int \sin x \cos x \, e^{\sin x} \, dx = \int \underbrace{\sin x \cdot e^{\sin x}}_{= ue^u} \cdot \underbrace{\cos x \, dx}_{= du}$$

Let $u = \sin x$, so

$du = \cos x \, dx$

$$= \int ue^u \, du$$

IBP

$= wv - \int v \, dw$ where

$w = u \qquad v = e^u$

$dw = du \qquad dv = e^u du$

$$= ue^u - \int e^u \, du$$

$$= ue^u - e^u + C$$

$$= \sin x \, e^{\sin x} - e^{\sin x} + C$$

Sometimes IBP can also help when the integrand doesn't look like a product!

Ex: Evaluate $\int \arctan x \, dx$

Solution: Let $u = \arctan x$ $\quad v = x$

$du = \frac{1}{1+x^2}\,dx \quad dv = 1\,dx$

Then

$$\int \arctan x\,dx = x \cdot \arctan x - \int \frac{x}{1+x^2}\,dx$$

u-substitution!
Let $u = 1+x^2$, so
$du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$

$$= x \cdot \arctan x - \int \frac{\cancel{x}}{u}\left(\frac{du}{2\cancel{x}}\right)$$

$$= x \cdot \arctan x - \frac{1}{2}\int \frac{1}{u}\,du$$

$$= x \cdot \arctan x - \frac{1}{2}\ln|u| + C$$

$$= \boxed{x \cdot \arctan x - \frac{1}{2}\ln|1+x^2| + C}$$

Here's one more interesting type of IBP problem...

Ex: Evaluate $\int e^x \sin x\,dx$

Solution: Use IBP with $u = \sin x$ $\quad v = e^x$

$du = \cos x\,dx \quad dv = e^x\,dx$

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

IBP again!

$u = \cos x \qquad v = e^x$

$du = -\sin x \, dx \qquad dv = e^x \, dx$

$$= e^x \sin x - \left[e^x \cos x - \int e^x (-\sin x) \, dx \right]$$

$$= e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x \, dx}_{\text{Aha! This is exactly the integral we started with!}}$$

If $I = \int e^x \sin x \, dx$, then we've just shown that

$$I = e^x \sin x - e^x \cos x - I,$$

hence $2I = e^x(\sin x - \cos x)$, and therefore

$$\boxed{I = \int e^x \sin x \, dx = \frac{e^x(\sin x - \cos x)}{2} + C}$$

Using integration by parts with definite integrals is no different — just don't forget the bounds!

$$\boxed{\int_a^b u \, dv = \left[uv\right]_a^b - \int_a^b v \, du}$$

Ex: Evaluate $\int_1^e \ln x \, dx$.

Solution: Let $u = \ln x$ $\qquad v = x$

$du = \frac{1}{x}\, dx \qquad dv = 1\, dx$

Then
$$\int_1^e \ln x\, dx = \Big[x \ln x\Big]_1^e - \int_1^e x \cdot \frac{1}{x}\, dx$$
$$= \Big[x \ln x\Big]_1^e - \Big[x\Big]_1^e$$
$$= \Big(e \underbrace{\ln(e)}_{=1} - 1\, \underbrace{\ln(1)}_{=0}\Big) - (e - 1)$$
$$= \boxed{1}$$