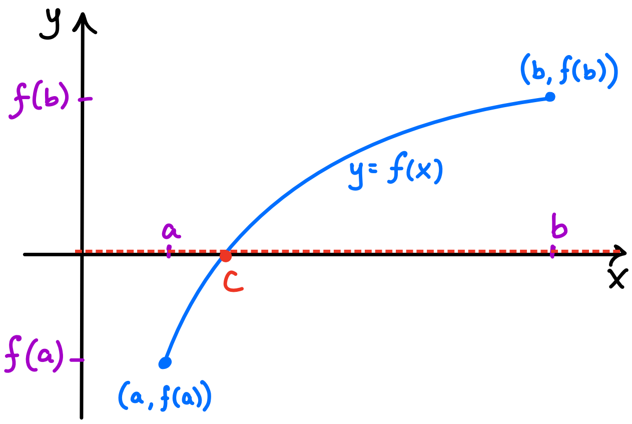
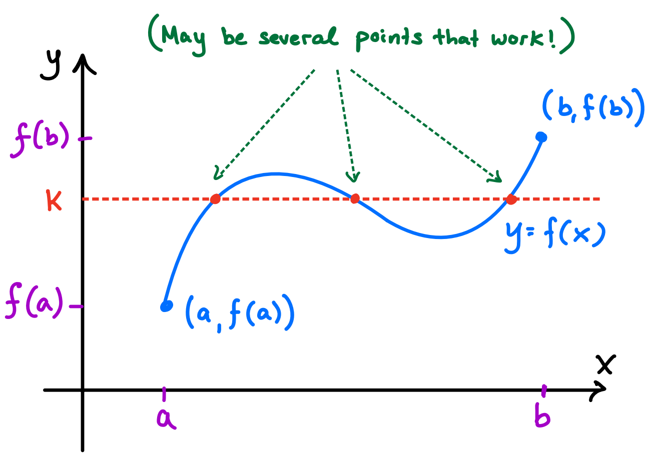
The Intermediate Value Theorem

Suppose f is continuous for all $x \in [a,b]$, so we can draw its graph from $(x,y) = (a, f(a))$ to $(x,y) = (b, f(b))$ without lifting our pen:



Note: If $f(a) < 0$ and $f(b) > 0$ (or vice versa) then there must exist $c \in (a,b)$ with $f(c) = 0$.



More generally: if $f(a) < K$ and $f(b) > K$ (or vice versa), then there exists $c \in (a,b)$ such that $f(c) = K$.

This leads us to the following theorem:

The Intermediate Value Theorem (IVT)

If f is continuous for all $x \in [a,b]$ and $f(a) < K < f(b)$ or $f(b) < K < f(a)$, then there exists at least one $c \in (a,b)$ such that $f(c) = K$.

Ex: Show that $f(x) = x^5 + x - 1$ has a root (i.e., a solution to $f(x) = 0$) in $[0,1]$.

Solution: $f(x) = x^5 + x - 1$ is a polynomial and hence is continuous everywhere. Furthermore,

$$f(0) = 0^5 + 0 - 1 = -1 \quad (<0) \text{ and}$$

$$f(1) = 1^5 + 1 - 1 = 1 \quad (>0)$$

Thus, by the IVT, there exists $c \in (0,1)$ such that $f(c) = 0$, as desired. ■

Ex: Show that there is a solution to the equation

$$\cos(x) = 2x$$

Solution: Finding a solution to $\cos(x) = 2x$ is equivalent to finding a solution to $\cos(x) - 2x = 0$. Note that $\cos(x) - 2x$ is a difference of two continuous functions and hence is continuous.

Furthermore,

$$\cos(0) - 2(0) = 1 \quad (>0)$$

$$\cos(\pi/2) - 2\left(\tfrac{\pi}{2}\right) = -\pi \quad (<0)$$

We have not been told what interval (a,b) to consider, so we need to use trial and error!

By the IVT, there exists some $c \in (0, \pi/2)$ such that $\cos(c) - 2c = 0$, or equivalently,

$$\cos(c) = 2c.$$

■

Additional Exercises

1. Prove that there exists a number c such that $2^c = c^4$.

2. Show that the equation $\sin(x) = \frac{1}{x}$ has infinitely many solutions

Solutions:

1. The function $f(x) = 2^x - x^4$ is continuous, as it is a difference of continuous functions. Furthermore, $f(0) = 2^0 - 0^4 = 1$ (>0) while $f(2) = 2^2 - 2^4 = -12$ (<0). By the IVT, there exists $c \in (0,2)$ such that $f(c) = 0$, or equivalently, $2^c = c^4$. ∎

2. Consider the function $f(x) = \sin x - \frac{1}{x}$ which is continuous for all $x \neq 0$, and, in particular, is continuous on each interval

$$\left[2k\pi, \frac{\pi}{2} + 2k\pi\right], \quad k = 1, 2, 3, \ldots$$

Furthermore,

$$f(2k\pi) = \underbrace{\sin(2k\pi)}_{=0} - \frac{1}{2k\pi} = \frac{-1}{2k\pi} \quad (<0)$$

$$f\left(\frac{\pi}{2} + 2k\pi\right) = \sin\left(\frac{\pi}{2} + 2k\pi\right) - \frac{1}{\frac{\pi}{2} + 2k\pi}$$

$$= \sin\left(\frac{\pi}{2}\right) - \frac{1}{\frac{\pi}{2} + 2k\pi}$$

$$= 1 - \underbrace{\frac{1}{\frac{\pi}{2} + 2k\pi}}_{<1} \quad (>0).$$

By the IVT, there exists a solution c_k to $f(x) = 0$ in each interval $\left[2k\pi, \frac{\pi}{2} + 2k\pi\right]$, $k = 1, 2, 3, \ldots$

Since none of the intervals $\left[2k\pi, \frac{\pi}{2} + 2k\pi\right]$ overlap, all of the solutions $c_1, c_2, c_3, \ldots$ must be distinct; hence, $\sin(x) = \frac{1}{x}$ has infinitely many solutions! $\blacksquare$