# §4.2 – Increasing and Decreasing Functions

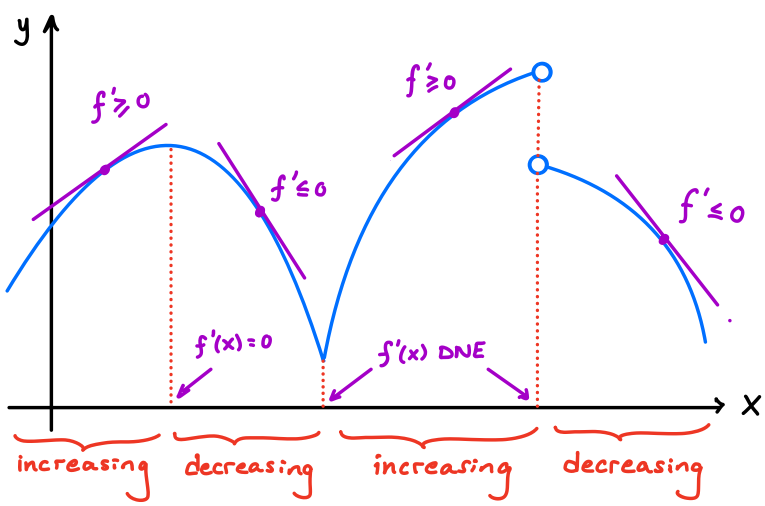
In these next few sections, we will learn how to use properties of $f'$ to study the behaviour of $f$.

In fact, last time we used the MVT to prove the following connection between $f$ and $f'$:

**Test for Increase / Decrease:**

If $f'(x) \geq 0$ for all $x \in (a,b)$, then $f$ is increasing on $(a,b)$.

If $f'(x) \leq 0$ for all $x \in (a,b)$, then $f$ is decreasing on $(a,b)$.




**Observation:**

Intervals of increase/decrease are separated by points where $f'(x) = 0$ or DNE.

Cool! To figure out where $f$ is increasing / decreasing, we can find any points where $f'(x) = 0$ or DNE, then check the sign of $f'$ in the surrounding intervals!

**Remarks:**

1. We will include an endpoint on an interval of increase / decrease if $f$ is continuous there.

2. An endpoint can belong to both an interval of increase and an interval of decrease.

3. Don't joint intervals of increase/decrease with unions ($\cup$), just write them out separately.

Ex: Where is $f$ increasing? Where is $f$ decreasing?

(a) $f(x) = x^2$

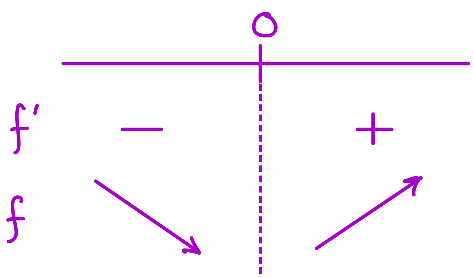
Solution:

[First, find any $x$'s where $f'(x) = 0$ or $f'(x)$ DNE.]

$f'(x) = 2x$, which exists everywhere.

$$f'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$$

[Next, check the sign of $f'$ around these points.]




$\therefore f$ increasing on $[0, \infty)$

decreasing on $(-\infty, 0]$

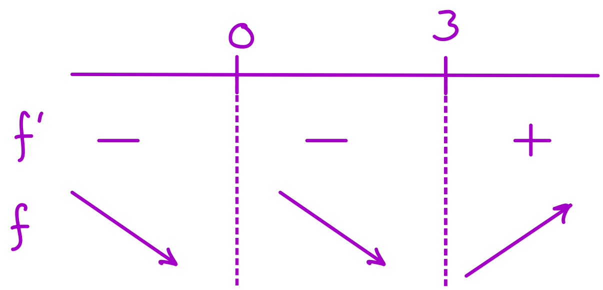
[$f(x) = x^2$ is continuous at $x = 0$, so include the endpoint.]

(b) $f(x) = x^4 - 4x^3 + 1$

Solution: $f'(x) = 4x^3 - 12x^2$, which exists everywhere.

$$f'(x) = 4x^3 - 12x^2 = 0 \Rightarrow 4x^2(x-3) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 3.$$




$\therefore$ $\boxed{f \text{ is decreasing on } (-\infty, 3]\,;\ \text{increasing on } [3, \infty).}$

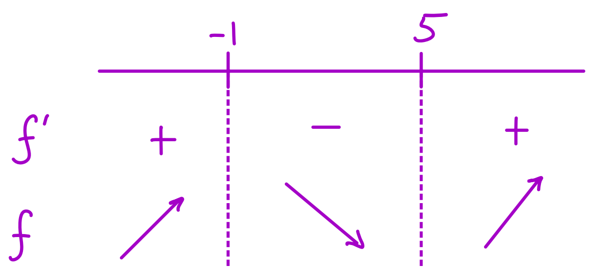
(c) $f(x) = \dfrac{2-x}{(x+1)^2}$

Solution: $$f'(x) = \frac{(x+1)^2(2-x)' - (2-x)\left[(x+1)^2\right]'}{(x+1)^4}$$

$$= \frac{-(x+1)^{\cancel{2}} - 2(2-x)\cancel{(x+1)}}{(x+1)^{\cancel{4}\,3}}$$

$$= \frac{-(x+1) - 2(2-x)}{(x+1)^3} = \frac{x-5}{(x+1)^3}$$

DNE when $x = -1$,
$f'(x) = 0$ when $x = 5$.



$f$ is increasing on $(-\infty,-1)$ and on $[5,\infty)$.

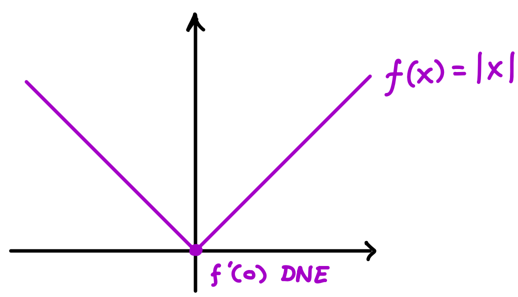
$f$ is decreasing on $(-1,5]$.

($x=-1$ not included since $f$ not continuous there!)

A point $x=c$ in the domain of $f$ is said to be a critical point (CP) of $f$ if $f'(c)=0$ or $f'(c)$ DNE.

These points will be very important when studying maxima and minima in the next section.

Ex: $f(x)=|x|$ has a CP at $x=0$ since $f'(0)$ DNE.




Ex: We saw that $f(x)=x^4-4x^3+1$ has critical points at $x=0$ and $x=3$, since $f'(0)=0$ and $f'(3)=0$.

Ex: We saw that $f(x) = \frac{2-x}{(x+1)^2}$ has a critical point at $x = 5$.

$\boxed{\text{Note: Even though } f'(-1) \text{ DNE, } x = -1 \text{ is NOT a critical point as it isn't in the domain of } f.}$