§ 4.11 - L'Hopital's Rule

It turns out that derivatives can help us evaluate limits, specifically limits of indeterminate form (where we can't just "plug in" $x = a$):

$$\text{“}0/0\text{”},\ \text{“}\infty/\infty\text{”},\ \text{“}0\cdot\infty\text{”},\ \text{“}\infty^0\text{”},\ \text{“}0^0\text{”},\ \text{“}1^\infty\text{”},\ \text{“}\infty-\infty\text{”}$$

L'Hopital's Rule

Suppose that near $x = a$, except possibly at $x = a$, f and g are differentiable and $g'(x) \neq 0$.

If $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ has the form "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$", and if $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ exists or is $\pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Examples: Evaluate the following limits.

(a) $\lim\limits_{x \to 3} \dfrac{x^3 - x - 24}{x-3} \quad \left(\dfrac{0}{0}\right)$

$$\overset{LH}{=} \lim_{x \to 3} \frac{(x^3 - x - 24)'}{(x-4)'}$$

$$= \lim_{x \to 3} \frac{3x^2 - 1}{1} = 3(3)^2 - 1 = \boxed{26}$$

(b) $\lim\limits_{x \to 2} \dfrac{x-2}{\sqrt{x} - \sqrt{2}} \quad \left(\dfrac{0}{0}\right)$

$$\overset{LH}{=} \lim_{x \to 2} \frac{1}{\frac{1}{2} x^{-1/2}} = \lim_{x \to 2} 2\sqrt{x} = \boxed{2\sqrt{2}}$$

Note: L'Hopital's rule also works on limits where

$x \longrightarrow a^+$, $\quad x \longrightarrow a^-$, $\quad$ or $\quad x \longrightarrow \pm\infty$

(c) $\lim\limits_{x \to \infty} \dfrac{x^3 + x - 7}{3x^3 + x + 1} \quad \left(\dfrac{\infty}{\infty}\right)$

$$\overset{LH}{=} \lim_{x \to \infty} \frac{3x^2 + 1}{9x^2 + 1} \quad \left(\frac{\infty}{\infty} \text{ again!}\right)$$

$$\overset{LH}{=} \lim_{x\to\infty} \frac{6\cancel{x}}{18\cancel{x}} = \frac{6}{18} = \boxed{3}$$

(d) $\displaystyle \lim_{x\to\infty} \frac{\ln(x)}{x} \quad \left(\frac{\infty}{\infty}\right)$

$$\overset{LH}{=} \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = \boxed{0}$$

(e) $\displaystyle \lim_{x\to\infty} \frac{e^x}{x^2+1} \quad \left(\frac{\infty}{\infty}\right)$

$$\overset{LH}{=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{LH}{=} \lim_{x\to\infty} \frac{e^x}{2} = \boxed{\infty}$$

(∞/∞)

How would we have evaluated these limits before learning L'Hopital's rule??

(f) $\displaystyle \lim_{x\to 1} \frac{x^2-1}{x^2+1} \quad \left(\text{“}\frac{0}{2}\text{”} \Rightarrow \text{Don't use L'Hopital!!}\right)$

$$= \frac{0}{2} = \boxed{0}$$

For other indeterminate forms, try to rewrite the limit in the form "0/0" or "∞/∞", then use L'Hopital.

For "$0 \cdot \infty$", move one function to the denominator as a reciprocal: "$0 \cdot \infty$" $\longrightarrow$ "$\frac{0}{1/\infty}$" $\longrightarrow$ "$\frac{0}{0}$" !

Ex: Evaluate the following limits.

(a) $\lim_{x \to 0^+} x \cdot \ln(x)$ $\quad (0 \cdot -\infty)$

$$= \lim_{x \to 0^+} \frac{\ln(x)}{1/x} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{LH}{=} \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} \frac{-x^2}{x} = \lim_{x \to 0^+} -x = \boxed{0}$$

Simplify before proceeding!

(b) $\lim_{x \to \infty} x^2 \cdot \sin\left(\frac{1}{x}\right)$ $\quad (\infty \cdot 0)$

$$= \lim_{x \to \infty} \frac{\sin(1/x)}{1/x^2} \quad \left(\frac{0}{0}\right)$$

$$\overset{LH}{=} \lim_{x \to \infty} \frac{-1/x^2 \cos(1/x)}{-2/x^3}$$

$$= \lim_{x \to \infty} \frac{x \cos(1/x)}{2} = \frac{\infty \cdot 1}{2} = \boxed{\infty}$$

For "0^0", "∞^0", or "1^∞", apply a logarithm to the limit to bring down the exponent!

Example: Evaluate the following limits.

(a) $\lim_{x \to 0^+} x^x$ (0^0, indeterminate!)

Let $L = \lim_{x \to 0^+} x^x$. We have

$$\ln L = \ln\left(\lim_{x \to 0^+} x^x\right) = \lim_{x \to 0^+} \ln(x^x)$$

$$= \lim_{x \to 0^+} x \ln x = \overbrace{\lim_{x \to 0^+} \frac{\ln(x)}{1/x} = \cdots = 0}^{\text{LH from an earlier example}}$$

Since $\ln L = 0$, we have $L = \lim_{x \to 0^+} x^x = e^0 = \boxed{1}$

(b) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ (1^∞, indeterminate)

Let $L = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$, so

$$\ln L = \lim_{x \to \infty} x \ln\left(1 + \frac{1}{x}\right) \qquad (\infty \cdot 0)$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\overset{LH}{=} \lim_{x\to\infty} \frac{\cancel{\frac{-1}{x^2}} \cdot \frac{1}{1+1/x}}{\cancel{\frac{-1}{x^2}}} = \frac{1}{1+0} = 1$$

Thus, $L = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e^1 = \boxed{e}$

(which matches our definition of e from §1.9 !)

(c) $\lim_{x\to \pi/2^-} (\sec x)^{\cos x}$ $\quad (\infty^0,$ indeterminate$)$

Let $L = \lim_{x\to \pi/2^-} (\sec x)^{\cos x}$. Then

$$\ln L = \lim_{x\to\pi/2^-} \cos(x) \cdot \ln(\sec x) \quad (0\cdot\infty)$$

$$= \lim_{x\to\pi/2^-} \frac{\ln(\sec x)}{\sec x} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{LH}{=} \lim_{x\to\pi/2^-} \frac{\cancel{(\sec x)'} \ \frac{1}{\sec x}}{\cancel{(\sec x)'}} = \lim_{x\to\pi/2^-} \cos x = 0.$$

$$\therefore L = \lim_{x\to\pi/2^-} (\sec x)^{\cos x} = e^0 = \boxed{1}$$

For "$\infty - \infty$" limits: Try putting everything over a common denominator or multiplying by a conjugate to get a "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" limit.

Ex: $\lim_{x \to \pi/2} \left(\sec x - \tan x \right)$ $(\infty - \infty)$

$$= \lim_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$= \lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x} \quad \left(\frac{0}{0}\right)$$

$$\overset{LH}{=} \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x}$$

$$= \frac{\cos(\pi/2)}{\sin(\pi/2)}$$

$$= \frac{0}{1} = \boxed{0}$$