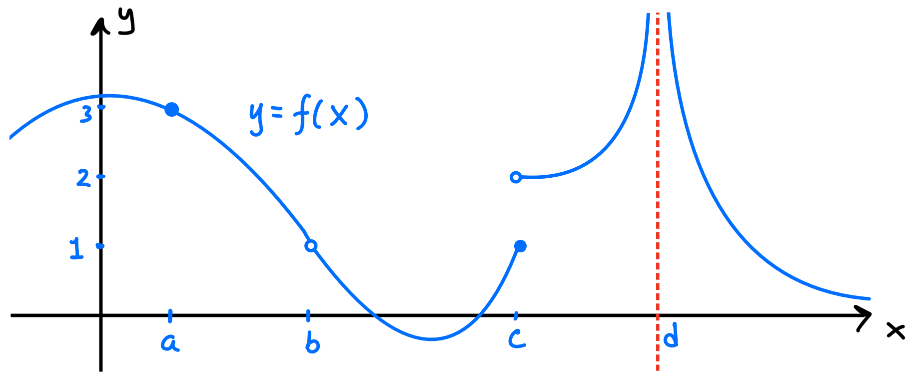
Chapter 2: Calculus Begins!

§ 2.1 – 2.3: Limits

We've studied how functions behave at a point, but often we'll be interested in how a function behaves near a point.

Intro Example: Consider the graph of $f(x)$ shown below.



As x approaches a, $f(x)$ approaches 3.

As x approaches b, $f(x)$ approaches 1.

As x approaches c, $f(x)$ approaches 1 if x comes from left
2 if x comes from right

As x approaches d, $f(x)$ approaches ∞.

If $f(x)$ approaches a finite number L as x gets infinitely close to a but not equal to a, we write $\boxed{\lim_{x \to a} f(x) = L}$

"The limit of $f(x)$ as x approaches a exists and is equal to L"

Note: The limit L must be the same if x comes from the left or from the right! These limits are denoted

$$\lim_{x \to a^-} f(x) \quad (x \to a \text{ from left})$$

$$\lim_{x \to a^+} f(x) \quad (x \to a \text{ from right})$$

If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ or if $f(x)$ approaches $\pm \infty$ or if $f(x)$ doesn't approach anything as $x \to a$, we say $\lim_{x \to a} f(x)$ does not exist (DNE).

In our intro example...

- $\lim_{x \to a} f(x) = 3$
- $\lim_{x \to b} f(x) = 1$ ← Don't care that f isn't defined <u>at</u> $x = b$.
- $\lim_{x \to c} f(x)$ DNE since $\lim_{x \to c^-} f(x) = 1$ while $\lim_{x \to c^+} f(x) = 2$

  Not equal
- $\lim_{x \to d} f(x) = \infty$ (so the limit <u>DNE</u>!)

<u>Limit Laws:</u> If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$,

then ... (finite) real numbers

(i) $\lim_{x \to a} c \cdot f(x) = c \cdot L \quad (c \in \mathbb{R})$

(ii) $\lim_{x \to a} f(x) \pm g(x) = L \pm M$

(iii) $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M$

(iv) $\lim_{x \to a} f(x) / g(x) = L / M$ (provided $M \neq 0$)

Using these laws, we can evaluate basic limits of polynomials and rational functions.

Ex: $\lim\limits_{x \to 1} x^2 + 3x = \left[\lim\limits_{x \to 1} x^2\right] + 3\left[\lim\limits_{x \to 1} x\right] = 1^2 + 3 \cdot 1 = \boxed{4}$

Ex: $\lim\limits_{x \to 3} \dfrac{x^2+3x+2}{x+1} = \dfrac{\lim\limits_{x \to 3} (x^2+3x+2)}{\lim\limits_{x \to 3} (x+1)} = \dfrac{3^2+3(3)+2}{3+1} = \boxed{5}$

If the limit results in an indeterminate form

(e.g., "$\frac{0}{0}$", "$\frac{\pm\infty}{\pm\infty}$", "$0 \cdot \pm\infty$", "$\infty - \infty$") more work must be done!

Ex: $\lim\limits_{x \to 2} \dfrac{x^2+2x-8}{x-2}$

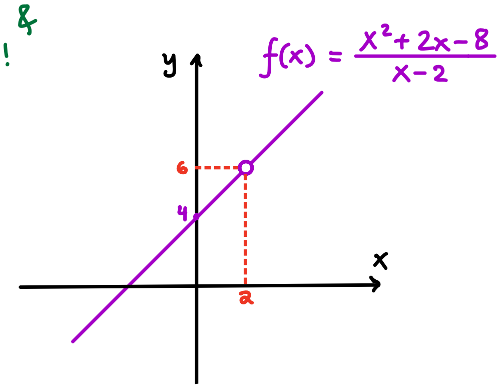
Plugging in $x = -1$ gives "$\frac{0}{0}$", which is indeterminate!

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+4)}{\cancel{x-2}}$$

Factor & Cancel!

$$= \lim_{x \to 2} (x+4)$$

$$= 2 + 4 = \boxed{6}$$



Ex: $\lim_{x \to \pi/2} \frac{\cos(x)}{\sin(2x)}$ ⟵ "$\frac{0}{0}$" type $\Rightarrow$ indeterminate!

$$= \lim_{x \to \pi/2} \frac{\cancel{\cos(x)}}{2\sin(x)\cancel{\cos(x)}} = \lim_{x \to \pi/2} \frac{1}{2\sin(x)} = \frac{1}{2\sin(\pi/2)} = \boxed{\frac{1}{2}}$$

Another Trick: Multiply and divide by the conjugate of the numerator or denominator.

Ex: $\lim_{x \to 1} \frac{\sqrt{2} - \sqrt{x+1}}{x-1}$ ⟵ "$\frac{0}{0}$" type $\Rightarrow$ indeterminate!

$$= \lim_{x \to 1} \frac{\sqrt{2} - \sqrt{x+1}}{x-1} \cdot \frac{\sqrt{2} + \sqrt{x+1}}{\sqrt{2} + \sqrt{x+1}}$$

"conjugate" of the numerator

$$= \lim_{x \to 1} \frac{2 + \cancel{\sqrt{2}\sqrt{x+1}} - \cancel{\sqrt{2}\sqrt{x+1}} - (x+1)}{(x-1)(\sqrt{2} + \sqrt{x+1})}$$

$$= \lim_{x \to 1} \frac{\cancel{(1-x)}}{-\cancel{(1-x)}(\sqrt{2} + \sqrt{x+1})}$$

$$= \lim_{x \to 1} \frac{-1}{\sqrt{2} + \sqrt{x+1}}$$

$$= \frac{-1}{\sqrt{2} + \sqrt{1+1}} = \boxed{\frac{-1}{2\sqrt{2}}}$$

$\boxed{\text{With absolute value and other piecewise functions, you'll often need to check left and right limits.}}$

<u>Ex:</u> $\displaystyle\lim_{x\to 0} \frac{|x|-x}{x}$ ← "$\frac{0}{0}$" type $\Rightarrow$ indeterminate!

The left- and right-sided limits are

$$\lim_{x\to 0^-} \frac{|x|-x}{x} = \lim_{x\to 0^-} \frac{-x-x}{x} = \lim_{x\to 0^-} \frac{-2\cancel{x}}{\cancel{x}} = \lim_{x\to 0^-} -2 = -2$$

$$\lim_{x\to 0^+} \frac{|x|-x}{x} = \lim_{x\to 0^+} \frac{x-x}{x} = \lim_{x\to 0^+} \frac{0}{x} = \lim_{x\to 0^+} 0 = 0$$

Since $\displaystyle\lim_{x\to 0^-} \frac{|x|-x}{x} \neq \lim_{x\to 0^+} \frac{|x|-x}{x}$, $\boxed{\text{the limit DNE!}}$

More complicated limits require more advanced methods!

<u>Ex:</u> $\displaystyle\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$ ← "$0 \cdot ???$" $\Rightarrow$ indeterminate!

Notice that $-1 \le \cos\left(\frac{1}{x}\right) \le 1$ for all x, hence

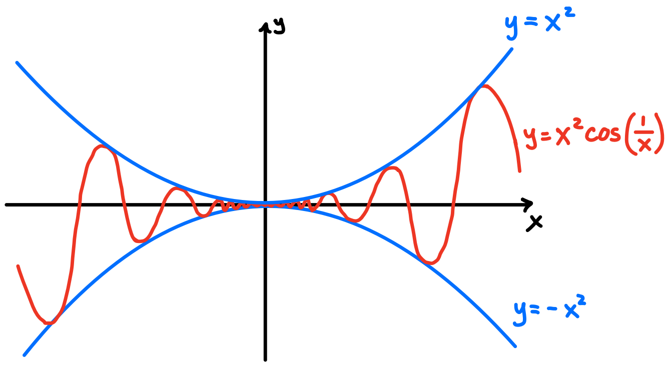
$$\xrightarrow{\cdot x^2} \quad -x^2 \le x^2\cos\left(\frac{1}{x}\right) \le x^2$$

Taking limits as $x \to 0$, we have

$$\underbrace{\lim_{x\to 0} -x^2}_{=0} \le \lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) \le \underbrace{\lim_{x\to 0} x^2}_{=0}$$

$$\Rightarrow \qquad 0 \le \lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) \le 0$$

So, $\lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right)$ must also be $\boxed{0.}$



We have just seen our first application of the Limit Squeeze Theorem!

The Squeeze Theorem

If $f(x) \le g(x) \le h(x)$ for all x near a and

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$ too!

Ex: $\lim\limits_{x\to\infty} \frac{\sin x}{x}$ $\longleftarrow$ "$\frac{???}{\infty}$" $\Rightarrow$ indeterminate

We have

$$-1 \le \sin x \le 1 \xRightarrow{\div x} \frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

$$\Longrightarrow \underbrace{\lim_{x\to\infty} \frac{-1}{x}}_{=0} \le \lim_{x\to\infty} \frac{\sin x}{x} \le \underbrace{\lim_{x\to\infty} \frac{1}{x}}_{=0}$$

$$\Longrightarrow 0 \le \lim_{x\to\infty} \frac{\sin x}{x} \le 0$$

Hence by the squeeze theorem, $\boxed{\lim\limits_{x\to\infty} \frac{\sin x}{x} = 0.}$

Additional Exercise: Evaluate the following.

(a) $\lim\limits_{x\to 1} \frac{x-1}{2-\sqrt{x+3}}$

(b) $\lim\limits_{x\to 5^-} \frac{x^2-25}{x^3-4x^2-5x}$

(c) $\lim\limits_{x\to \pi} \frac{\sin(x+\pi)}{\sin(x-\pi)}$

(d) $\lim\limits_{x\to 0} |x|\cdot\sin\left(\frac{1}{2x}\right)$

(e) $\lim\limits_{x\to 3} \frac{|x|+|x-3|-3}{x-3}$

(f) $\lim\limits_{x\to \frac{\pi}{2}^-} \cos x\cdot\cos(\tan x)$

Solutions

"$\frac{0}{0}$" $\Rightarrow$ indeterminate!

(a) $$\lim_{x\to 1}\frac{x-1}{2-\sqrt{x+3}} = \lim_{x\to 1}\frac{x-1}{2-\sqrt{x+3}}\cdot\frac{(2+\sqrt{x+3})}{(2+\sqrt{x+3})}$$

conjugate of denominator

$$= \lim_{x\to 1}\frac{(x-1)(2+\sqrt{x+3})}{(2-\sqrt{x+3})(2+\sqrt{x+3})}$$

$$= \lim_{x\to 1}\frac{(x-1)(2+\sqrt{x+3})}{4+\cancel{2\sqrt{x+3}}-\cancel{2\sqrt{x+3}}-(\sqrt{x+3})^2}$$

$$= \lim_{x\to 1}\frac{(x-1)(2+\sqrt{x+3})}{4-(x+3)}$$

$$= \lim_{x\to 1}\frac{\cancel{(x-1)}(2+\sqrt{x+3})}{-\cancel{(x-1)}} = -(2+\sqrt{1+3}) = \boxed{-4}$$

(b) $$\lim_{x\to 5^-}\frac{x^2-25}{x^3-4x^2-5x} = \lim_{x\to 5^-}\frac{\cancel{(x-5)}(x+5)}{x\cancel{(x-5)}(x+1)}$$

"$\frac{0}{0}$" $\Rightarrow$ indeterminate!

$$= \lim_{x\to 5^-}\frac{x+5}{x(x+1)} = \frac{10}{5\cdot 6} = \boxed{\frac{1}{3}}$$

(c) $$\lim_{x\to \pi}\frac{\sin(x+\pi)}{\sin(x-\pi)} = \lim_{x\to\pi}\frac{\sin x\cdot\overbrace{\cos\pi}^{=-1} + \cos x\cdot\cancel{\sin\pi}^{=0}}{\sin x\cdot\underbrace{\cos\pi}_{=-1} + \cos x\cdot\cancel{\sin\pi}_{=0}}$$

"$\frac{0}{0}$" $\Rightarrow$ indeterminate!

$$= \lim_{x\to\pi}\frac{\sin x\cdot(-1)}{\sin x\cdot(-1)} = \boxed{1}$$

(d) $\lim_{x \to 0} |x| \cdot \sin\left(\frac{1}{2x}\right) \longleftarrow$ "$0 \cdot ???$" $\Rightarrow$ indeterminate!

Note that $-1 \le \sin\left(\frac{1}{2x}\right) \le 1$ for all x, hence

$\cdot |x|$

$$-|x| \le |x| \sin\left(\frac{1}{2x}\right) \le |x|$$

$$\Rightarrow \underbrace{\lim_{x \to 0} -|x|}_{=0} \le \lim_{x \to 0} |x| \sin\left(\frac{1}{2x}\right) \le \underbrace{\lim_{x \to 0} |x|}_{=0}$$

$$\Rightarrow 0 \le \lim_{x \to 0} |x| \sin\left(\frac{1}{2x}\right) \le 0$$

Hence, by the Squeeze Theorem, $\lim_{x \to 0} |x| \sin\left(\frac{1}{2x}\right) = \boxed{0}$

(e) $\lim_{x \to 3} \frac{|x| + |x-3| - 3}{x-3} \longleftarrow$ "$\frac{0}{0}$" $\Rightarrow$ indeterminate!

Note that since $x \longrightarrow 3$, we have $x > 0$, hence $|x| = x$.

For $|x-3|$, let's look at the left- and right-sided limits!

$$\lim_{x \to 3^-} \frac{|x| + |x-3| - 3}{x-3} = \lim_{x \to 3^-} \frac{x - (x-3) - 3}{x-3}$$

$$= \lim_{x \to 3^-} \frac{0}{x-3} = \lim_{x \to 3^-} 0 = \underline{0}.$$

$$\lim_{x\to 3^+} \frac{|x|+|x-3|-3}{x-3} = \lim_{x\to 3^+} \frac{x+(x-3)-3}{x-3}$$

$$= \lim_{x\to 3^+} \frac{2x-6}{x-3} = \lim_{x\to 3^+} \frac{2\cancel{(x-3)}}{\cancel{x-3}} = \underline{2}$$

Since $\lim_{x\to 3^-} \frac{|x|+|x-3|-3}{x-3} \neq \lim_{x\to 3^+} \frac{|x|+|x-3|-3}{x-3}$, the limit $\boxed{\text{DNE}}$

(f) $\lim_{x\to \frac{\pi}{2}^-} \cos x \cdot \cos(\tan x)$ "$0 \cdot ???$" since $\tan x \to \infty$ as $x \to \frac{\pi}{2}^-$

Note that $-1 \leq \cos(\tan x) \leq 1$ for all x, hence

$$-\cos x \leq \cos x \cdot \cos(\tan x) \leq \cos x$$

$$\Rightarrow \underbrace{\lim_{x\to \pi/2^-} -\cos x}_{=0} \leq \lim_{x\to \pi/2^-} \cos x \cdot \cos(\tan x) \leq \underbrace{\lim_{x\to \pi/2^-} \cos x}_{=0}$$

$$\Rightarrow 0 \leq \lim_{x\to \pi/2^-} \cos x \cdot \cos(\tan x) \leq 0$$

By the Squeeze Theorem, $\lim_{x\to \pi/2^-} \cos x \cdot \cos(\tan x) = \boxed{0.}$