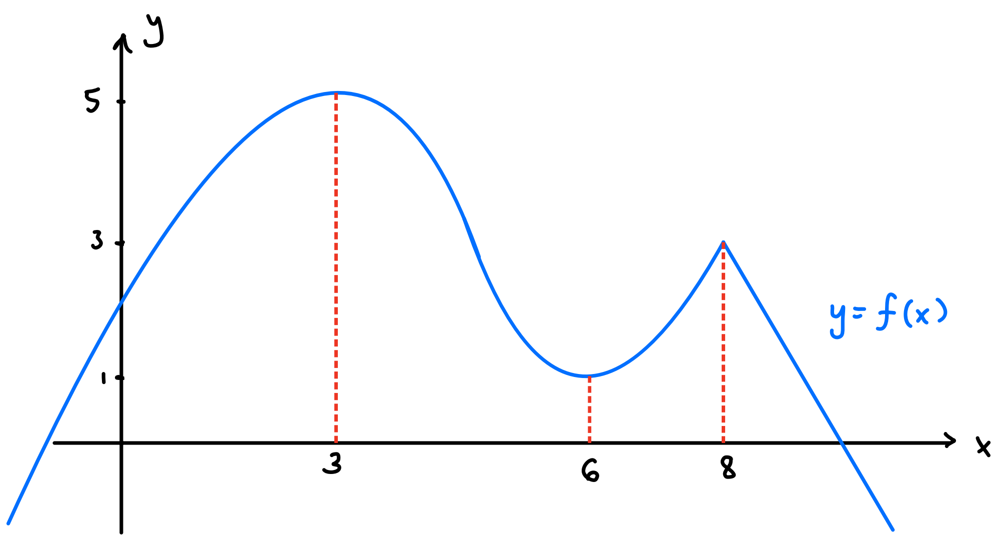
§4.3 – Local Maxima and Minima

f is said to have

(i) a **local maximum** at x_0 if there exists an open interval I around x_0 such that $f(x) \le f(x_0)$ for all $x \in I$.

(ii) a **local minimum** at x_0 if there exists an open interval I around x_0 such that $f(x) \ge f(x_0)$ for all $x \in I$.

Ex:



f has a local max at $x=3$ with value 5,

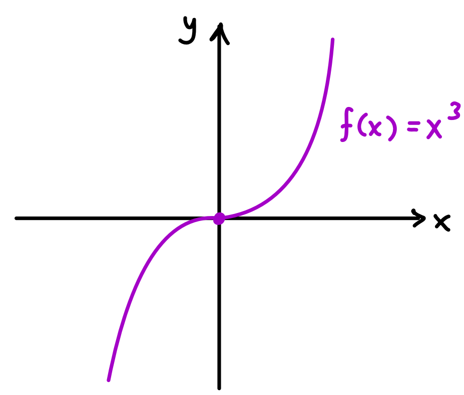
local min at $x=6$ with value 1,

local max at $x=8$ with value 3.

Fact: Local maxima and minima can only occur at critical points! However, not every critical point is necessarily a local max or min.

Ex: Let $f(x) = x^3$. Then $f'(x) = 3x^2$, which is 0 at $x=0$, hence $x=0$ is a C.P. of f.

However there is neither a local max nor min at $x=0$.



Given a critical point of f, how can we tell whether it corresponds to a local max, local min, or neither?

The First Derivative Test

Let $x=c$ be a critical point at which f is continuous

(i) If $f'(x)$ changes from $+$ to $-$ around $x=c$, then there is a local max at $x=c$.

(ii) If $f'(x)$ changes from $+$ to $-$ around $x=c$,

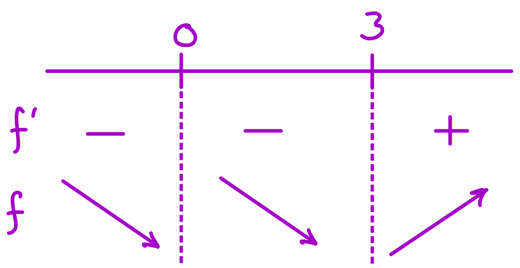
then there is a local max at $x=c$.

(iii) If $f'(x)$ doesn't change sign around $x=c$, then

there is neither a local max nor min at $x=c$.

<u>Ex:</u> We saw that $f(x)=x^4-4x^3+1$ has critical points

at $x=0$ and $x=3$.



$\searrow\ \searrow \Rightarrow$ Neither a local max nor min at $x=0$.

$\searrow\ \nearrow \Rightarrow$ Local min at $x=3$ with value $f(3)=-26$.

<u>Ex:</u> We saw that $f(x)=\dfrac{2-x}{(x+1)^2}$ has a critical point

at $x=5$ (and not at $x=-1$, since $f(-1)$ is not defined!)

$$\begin{array}{c|cc} & \multicolumn{2}{c}{5} \\ \hline f' & - & + \\ f & \searrow & \nearrow \end{array}$$

$\searrow \nearrow \implies$ Local min at $x = 5$ with value $f(5) = \frac{-1}{12}$

<u>Ex:</u> Find all local maxima and minima of $f(x) = e^{-x^2}$.

<u>Solution:</u> First find the critical points.

$f'(x) = -2xe^{-x^2}$, which exists everywhere.

$$f'(x) = -2x\underbrace{e^{-x^2}}_{>0 \text{ always}} = 0 \implies -2x = 0 \implies x = 0.$$

$$\begin{array}{c|cc} & \multicolumn{2}{c}{0} \\ \hline f' & + & - \\ f & \nearrow & \searrow \end{array}$$

$\nearrow \searrow \implies$ local max at $x = 0$ with value $f(0) = e^{-0^2} = 1$.