§3.14 – Rolle's Theorem and the Mean Value Theorem

These are two of the most important theorems in Calculus. They are the workhorses of the subject, appearing in the proofs of some of our most fundamental theorems!

Rolle's Theorem: Suppose that

(i) f is continuous for $x \in [a,b]$,

(ii) f is differentiable for $x \in (a,b)$, and

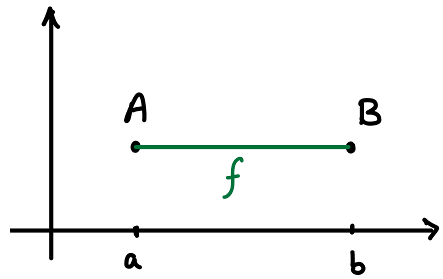
(iii) $f(a) = f(b)$.

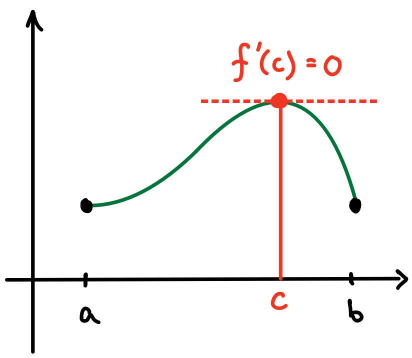
Then there exists at least one $c \in (a,b)$ with $f'(c) = 0$.

"Proof" (by picture!)

Connect A to B without lifting your pen (continuity) and without sharp corners (since f' exists).

Either the graph is flat, in which case $f'(x) = 0$ everywhere ...





... OR f increases then decreases (or vice-versa), in which case there is a peak (or valley) where $f'(c) = 0$. ∎

The Mean Value Theorem (MVT):

If f is (i) continuous for $x \in [a, b]$ and

(ii) differentiable for $x \in (a, b)$,

then there exists at least one $c \in (a, b)$ with

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

Proof: Consider the function

$$g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

Since f is continuous and differentiable, so is g.

Furthermore, $g(a) = g(b) = 0$. Thus, by Rolle's Theorem, there exists $c \in (a,b)$ with $g'(c) = 0$.

But

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

so $g'(c) = 0 \implies f'(c) - \dfrac{f(b)-f(a)}{b-a} = 0$

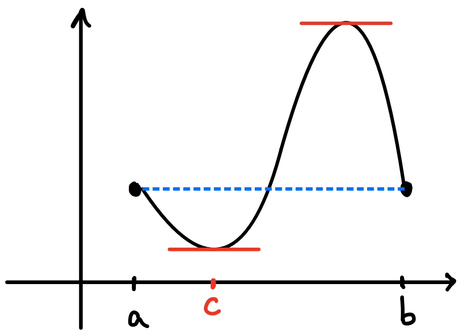
$$\implies f'(c) = \frac{f(b)-f(a)}{b-a}$$

■

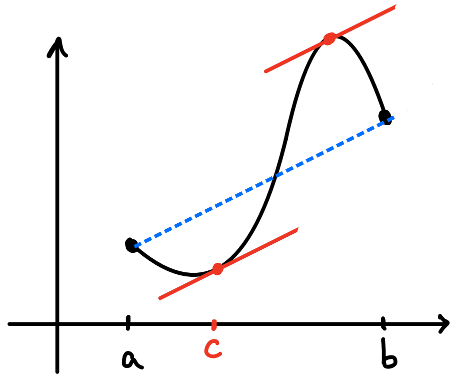
The MVT says that there is a point at which

$$\boxed{\text{instantaneous rate of change}} = \boxed{\text{average rate of change from } x=a \text{ to } x=b.}$$

$$\boxed{\text{slope of the tangent line}} = \boxed{\text{slope of the secant line through } (a, f(a)) \text{ and } (b, f(b))}$$



Rolle's Theorem



The Mean Value Theorem

The Mean Value Theorem is really a "tilted" version of Rolle's Theorem! In fact, in the special case of the MVT where $f(a) = f(b)$, we get

$$f'(c) = \frac{\overbrace{f(b) - f(a)}^{=0}}{b - a} = 0,$$

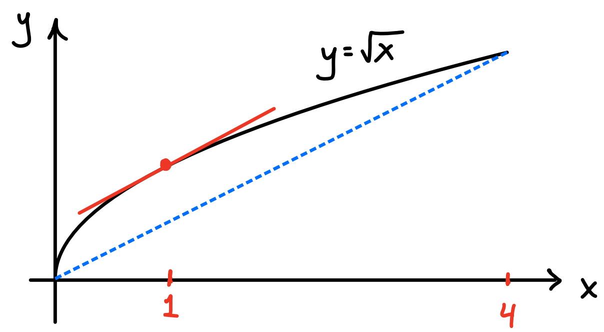
which is exactly Rolle's Theorem!

Numerical Example:

Let $f(x) = \sqrt{x}$, $x \in [0,4]$. Since f is continuous on $[0,4]$ and differentiable on $(0,4)$, by the MVT, there exists $c \in (0,4)$ such that

$$f'(c) = \frac{f(4) - f(0)}{4-0} = \frac{\sqrt{4} - \sqrt{0}}{4-0} = \frac{1}{2}$$

Indeed, $f'(x) = \dfrac{1}{2\sqrt{x}}$, which is $\frac{1}{2}$ when $x = 1$.



The MVT is super important because it provides a direct link between a function f and its derivative f'. We can use the MVT to translate properties of f' into properties of f!

Applications (but first, a definition!)

Definition: A function f is

(i) increasing on $[a,b]$ if, whenever $x_1, x_2 \in [a,b]$ with $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$

(ii) decreasing on $[a,b]$ if, whenever $x_1, x_2 \in [a,b]$ with $x_1 < x_2$, we have $f(x_1) \geq f(x_2)$

Ex: Use the MVT to show: If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is increasing on (a,b).

Proof: Suppose $f'(x) \geq 0$ for all $x \in (a,b)$. We will show that f is increasing on $[a,b]$. To this end, let $x_1, x_2 \in [a,b]$ with $x_1 < x_2$. By the MVT applied on the interval (x_1, x_2), there exists $c \in (x_1, x_2)$ with

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Thus,

$$f(x_2) - f(x_1) = \underbrace{f'(c)}_{\geqslant 0} \cdot \underbrace{(x_2 - x_1)}_{>0} \geqslant 0,$$

hence $f(x_2) \geqslant f(x_1)$. That is, f is increasing. ∎

Exercise: Show that if $f'(x) \leq 0$ for all $x \in (a,b)$, then f is decreasing on (a,b).

Ex: Use the MVT to show the equation $\cos x = 2x$ has at most one solution.

[Note: Earlier we used the IVT to show there was at least one solution, now we'll show there is at most one!]

Proof (by Contradiction): Suppose there were, in fact, multiple solutions to $\cos x = 2x$, or equivalently, multiple solutions to

$$f(x) = \cos x - 2x = 0.$$

If x_1 and x_2 are two solutions with $x_1 < x_2$, then $f(x_1) = 0$ and $f(x_2) = 0$. By the MVT applied on the interval (x_1, x_2), there exists $c \in (x_1, x_2)$ with

$$f'(c) = \frac{\overbrace{f(x_2)}^{=0} - \overbrace{f(x_1)}^{=0}}{x_2 - x_1} = 0.$$

But $f'(x) = -\sin(x) - 2$, so

$$f'(c) = 0 \Rightarrow -\sin(c) - 2 = 0 \Rightarrow \overbrace{\sin(c) = -2}^{\text{Impossible, since } \sin(c) \in [-1,1]\,!}$$

Thus, our assumption of multiple solutions must have been wrong! Hence, $\cos x = 2x$ has at most one solution. ∎