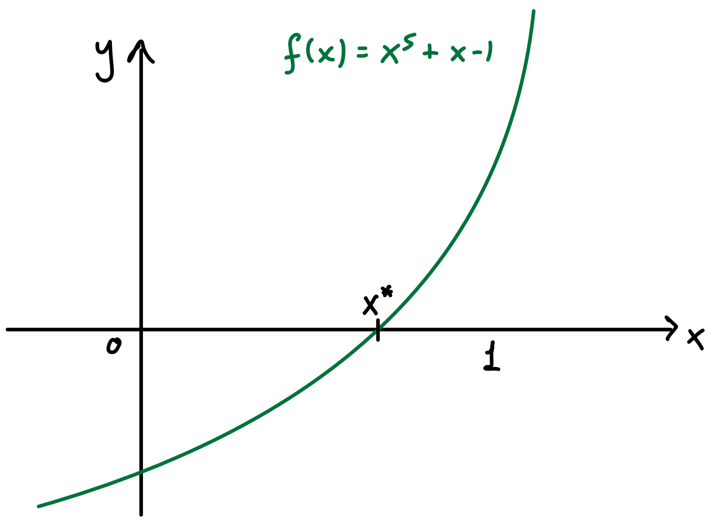
§4.1 – Newton's Method

Last time we used the IVT to show that

$$f(x) = x^5 + x - 1 = 0$$

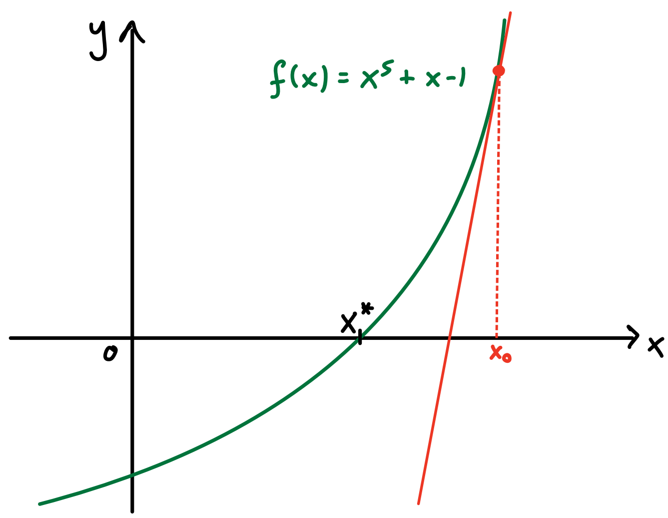
has a solution x^* in the interval $[0,1]$.



Great, we know a solution exists, but how do we find it? Unfortunately, there is no way to calculate x^* exaclty! Our best option:

$\boxed{\text{Approximate a Solution!}}$

We'll start with an inital guess, x_0. Then, rather than solving $f(x) = 0$ (too hard!), we'll instead find where the tangent line at x_0 is 0.



Question 1: What is the equation of the tangent line to f at x_0?

Answer: Slope $= f'(x_0)$, line passes through $(x_0, f(x_0))$.

Hence, if (x, y) is any point on the line, then

$$\frac{y - f(x_0)}{x - x_0} = \frac{\text{rise}}{\text{run}} = \text{slope} = f'(x_0)$$

$$\Rightarrow \quad y - f(x_0) = f'(x_0)(x - x_0)$$

$$\Rightarrow \quad \boxed{y = f(x_0) + f'(x_0)(x - x_0)}$$

equation of tangent line to f at x_0.

Question 2: At what point x is the tangent line equal to 0?

Answer:

$$y = f(x_0) + f'(x_0)(x - x_0) = 0 \quad \Rightarrow \quad f'(x_0)(x - x_0) = -f(x_0)$$

$$\Rightarrow \quad x - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

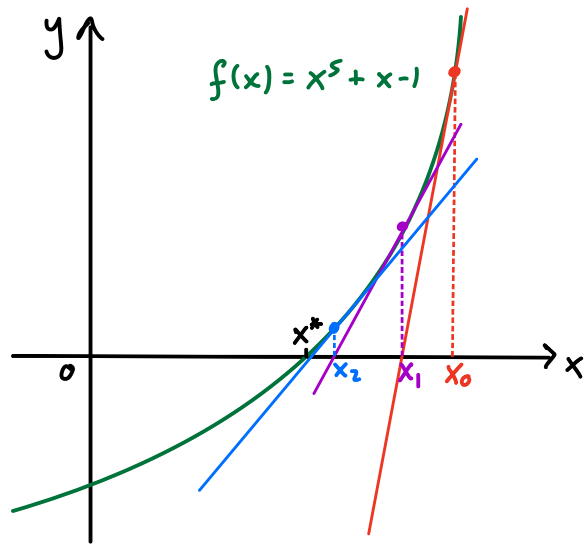
$$\Rightarrow \quad \boxed{x = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

We'll call this new approximation x_1:

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

Hopefully x_1 will be closer to x^* than x_0 was.

To get even closer, repeat with x_0 replaced by x_1! This process is known as...



Newton's Method

To approximate a solution x^* to an equation $f(x) = 0$,

1. Start with an initial guess x_0 (ideally, close to x^*).

2. For each $n = 1, 2, 3, \dots$, let

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Back to our example...

Suppose we wish to approximate a solution $x^* \in [0,1]$ to $f(x) = x^5 + x - 1 = 0$ correct to 6 decimal places.

Let's use Newton's method with initial guess $x_0 = 1$.

We have $f'(x) = 5x^4 + 1$, and hence

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{(x_0^5 + x_0 - 1)}{5x_0^4 + 1} = 0.8333\ldots$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.764382\ldots$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.755024\ldots$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.\underline{754877}\ldots$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0.\underline{754877}\ldots$$

First 6 decimal places have stabilized, so we can stop.

$$\boxed{x^* \approx 0.754877}$$

Ex: Previously, we used the IVT to show that $\cos x = 2x$ has a solution $x^* \in [0, \pi/2]$.

Approximate x^* correct to 7 decimal places.

Solution: We are attempting to approximate a root of $f(x) = \cos(x) - 2x = 0$. We have $f'(x) = -\sin(x) - 2$ and we'll use $x_0 = 0.5$ as an initial guess. We have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{\left(\cos(x_0) - 2x_0\right)}{-\sin(x_0) - 2}$$

$$= 0.450626693\ldots$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.\underline{4501836}47\ldots$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.\underline{4501836}11\ldots$$

Stop, since first 7 decimal places have stabilized

$$\therefore \boxed{x^* \approx 0.4501836}$$

Exercise: Use Newton's method to approximate $\sqrt{7}$ correct to 3 decimal places.

Solution: $\sqrt{7}$ is a root of $f(x) = x^2 - 7 = 0$ and we guess that $\sqrt{7}$ should be somewhat near $\sqrt{9} = 3$,

so $x_0 = 3$ will be our initial guess.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - 7}{2x_0} = 2.6\overline{6}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \underline{2.645}8\overline{3}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \underline{2.645}751312\ldots$$

Stop, since first 3 dec. points have stabilized.

$$\therefore \boxed{\sqrt{7} \approx 2.645}$$

WARNING: In some cases, Newton's method can fail!

Ex: $f(x) = \arctan(x)$ has a root at $x = 0$.

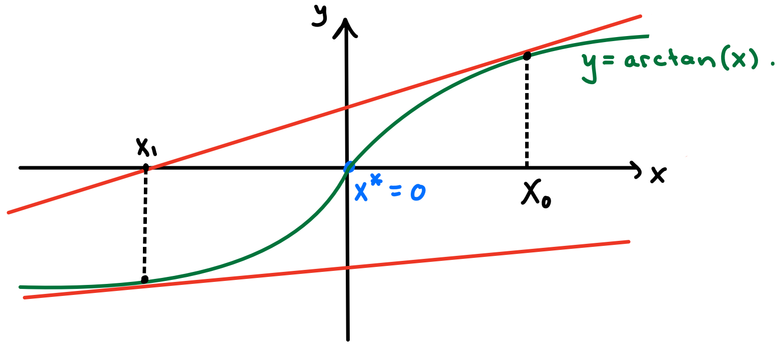
Let's attempt to approximate this root using Newton's method with $x_0 = 2$. We get...

$x_1 = -3.53\ldots$

$x_2 = 13.95\ldots$

$x_3 = -279.34\ldots$

$x_4 = 122016.99\ldots$



Not converging to anything!

Fix: Choose a new x_0 closer to x^*