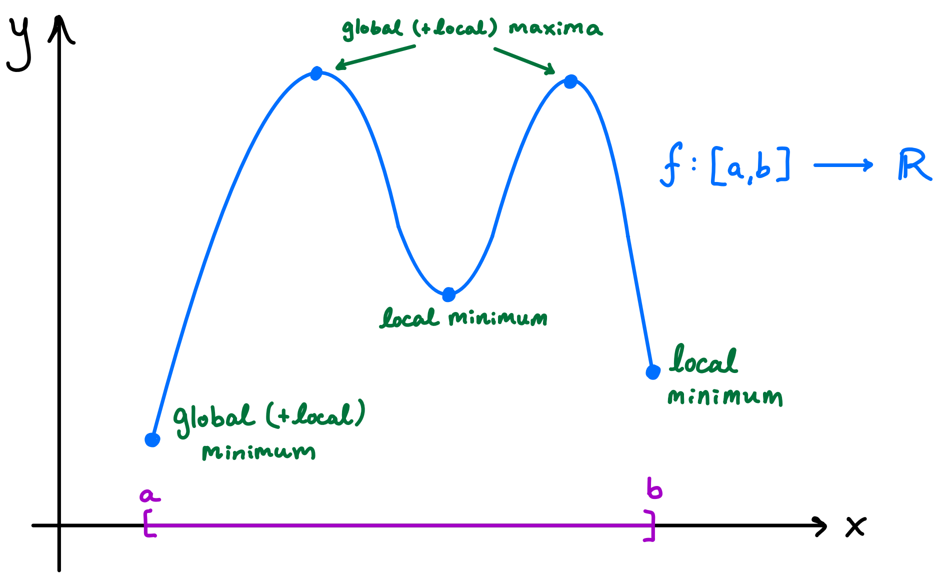
§4.7 – Global Extrema and Optimization

A function f has a

- global (or absolute) max on an interval I at x_0 if $f(x_0) \geq f(x)$ for all $x \in I$.
- global (or absolute) min on an interval I at x_0 if $f(x_0) \leq f(x)$ for all $x \in I$.



Fact: If $f : [a,b] \longrightarrow \mathbb{R}$ is continuous, then f will have global maxima and minima on $[a,b]$. They could occur at critical points in $[a,b]$ or at the endpoints.

The Process: To find global max/mins of a continuous function f on $[a,b]$,

(1) find all critical points of f in $[a,b]$,

(2) evaluate f(critical pts), $f(a)$ and $f(b)$,

(3) biggest = global max ; smallest = global min.

Ex: Find the global extrema of $f(x) = x^3 - 12x$ for $x \in [0,3]$.

Solution: Any critical points?

$$f'(x) = \underbrace{3x^2 - 12}_{\text{exists everywhere!}} = 0 \Rightarrow 3x^2 = 12 \Rightarrow x = -2 \text{ or } +2$$

Ignore — not in $[0,3]$.

CP!

Compare:

$f(0) = 0$ (Biggest!)

$f(2) = -16$ (Smallest!)

$f(3) = -9$

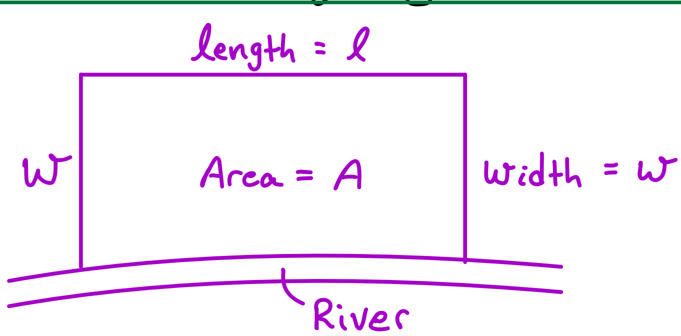
$\therefore$ Global max at $x=0$ with value $f(0)=0$; global min at $x=2$ with value $f(2) = -16$.

This technique can be used to solve all sorts of applied optimization problems!

<u>Ex</u>: A farmer has 800m of fencing to build a rectangular giraffe enclosure. One side of the enclosure lies along a river and does not need to be fenced. Find the dimensions that will enclose the largest area.

<u>Solution:</u>

① Draw a picture. Identify any variables.



② Find an expression for the quantity being maximized or minimized. Identify any constraints.

Want to maximize $A = \ell \cdot w$.

<u>Constraint:</u> $\ell + 2w = 800$ (so $\ell = 800 - 2w$)

③ Write the quantity being optimized as a function of one variable. State its domain.

$$A = \ell \cdot w = (800 - 2w)\cdot w = 800w - 2w^2.$$

We need $w \geq 0$ and $2w \leq 800$, so $w \leq 400$

$w = 0$ means all fencing is used for ℓ.

$2w = 800$ means all fencing is used for w.

④ Find the absolute max/min on this domain.

We maximize $A(w) = 800w - 2w^2$, $w \in [0, 400]$.

Critical points of $A(w)$?

$$A'(w) = \underbrace{800 - 4w}_{\text{exists everywhere}} = 0 \Rightarrow 4w = 800$$

$$\Rightarrow w = 200 \quad \text{(Critical point!)}$$

Compare: $A(0) = 0$

$A(200) = 800(200) - 2(200)^2 = 80000 \text{ m}^2$

$A(400) = 0$

Global max

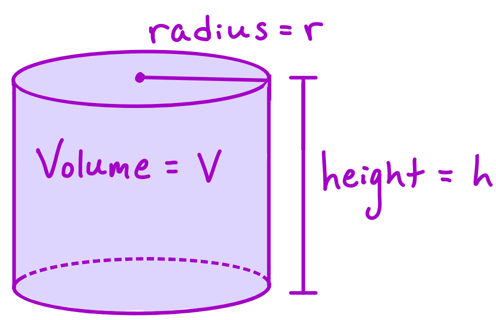
⑤ Write a concluding statement.

The maximum possible area is $80\,000 m^2$ and occurs when width $= 200m$ and length $= 800 - 2w = 400m$.

Ex: Suppose we have $300 cm^2$ of tin to build a cylindrical can (with top and bottom) with the largest possible volume. How much giraffe soup could such a can hold?

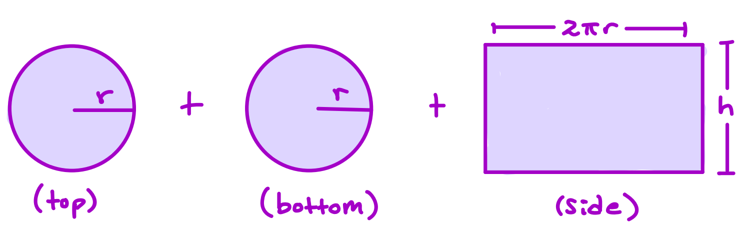
Solution:

Want to maximize $V = \pi r^2 h$.



Constraint:

Surface Area =



= 300

$$\Rightarrow \quad \pi r^2 + \pi r^2 + 2\pi r h = 300$$

$$\Rightarrow \quad 2\pi r^2 + 2\pi r h = 300.$$

From this equation, we have

$$2\pi r h = 300 - 2\pi r^2 \quad \Rightarrow \quad h = \frac{300 - 2\pi r^2}{2\pi r}$$

Thus, the volume function is

$$V = \pi r^2 h = \pi r^2 \left(\frac{300 - 2\pi r^2}{2\pi r} \right) = 150r - \pi r^3$$

We note that $r \geq 0$ and, if all tin is used for the base and top (i.e., no height), then

$$2\pi r^2 \leq 300 \quad \Rightarrow \quad r^2 \leq \frac{150}{\pi} \quad \Rightarrow \quad r \leq \sqrt{\frac{150}{\pi}}$$

Thus, we will maximize

$$V(r) = 150r - \pi r^3, \quad r \in \left[0, \sqrt{\frac{150}{\pi}} \right]$$

Any critical points?

$$V'(r) = \underbrace{150 - 3\pi r^2}_{\text{exists everywhere}} = 0 \Rightarrow 3\pi r^2 = 150$$

$$\Rightarrow r^2 = \frac{50}{\pi}$$

$$\Rightarrow r = \pm\sqrt{\frac{50}{\pi}}$$

Discard $r = -\sqrt{\frac{50}{\pi}}$ since not in domain.

Compare:

$$V(0) = 0$$

$$V\left(\sqrt{\frac{50}{\pi}}\right) \approx 398.9 \text{ cm}^3 \longleftarrow \text{max!}$$

$$V\left(\sqrt{\frac{150}{\pi}}\right) = 0$$

$$\boxed{\text{The largest can will hold} \approx 398.9 \text{ cm}^3 \text{ of giraffe soup.}}$$

Additional Exercises

1. A company produces two goods: apples and bananas. If the company produces A tons of apples and B tons of bananas, their profit is given by $A^2 + 2B^2$. Due to production constraints, $A + 3B$ cannot exceed 660 tons. How much of each good should be produced to maximize the company's profit?

Solution: The company will produce as much as possible to maximize profits, hence $A + 3B = 660$.

Thus, $A = 660 - 3B$.

We have to maximize $A^2 + 2B^2$, which can be written as

$$f(B) = (660 - 3B)^2 + 2B^2$$

We need $B \geq 0$ and $3B \leq 660$, so $B \leq 220$.

$B=0$ means we only produce A

$3B = 660$ means we only produce B

Thus, we maximize $f(B)$ for $B \in [0, 220]$.

Critical Points?

$$f'(B) = 2(660-3B)(-3) + 4B$$ (exists everywhere)

$$f'(B) = 0 \Rightarrow -3960 + 22B = 0$$

$$\Rightarrow B = \frac{3960}{22} = \underline{180}$$ (one critical point)

Compare:

$$f(0) = 435600$$ ← Max profit!

$$f(180) = 79200$$

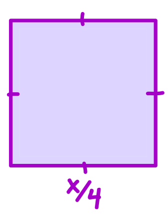
$$f(220) = 96800$$

Profits are maximized when the company produces $B=0$ tons of bananas and $A = 660-3B = 660$ tons of apples.

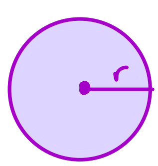
2. A wire 10 cm in length is cut into two pieces. One piece is bent into a square and the other is bent into a circle. How should the wire be cut if we wish to minimize the total area? What if we wish to maximize the total area?

<u>Solution:</u> Let x be the length used to form the square and y be the length used to form the circle.

We wish to optimize $A = A_{square} + A_{circle}$.



$$A_{square} = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$$



$$y = 2\pi r \Rightarrow r = \frac{y}{2\pi}$$

$$\therefore A_{circle} = \pi r^2 = \pi\left(\frac{y}{2\pi}\right)^2 = \frac{y^2}{4\pi}$$

Thus, $A = \dfrac{x^2}{16} + \dfrac{y^2}{4\pi}$.

We know that $x + y$ = length of wire = 10 cm,

So $y = 10 - x$. Thus, we optimize

$$A(x) = \frac{x^2}{16} + \frac{(10-x)^2}{4\pi} \quad \text{for } x \in [0, 10].$$

<u>Critical Points?</u>

$$A'(x) = \frac{x}{8} - \frac{(10-x)}{2\pi}$$ (exists everywhere)

$$A'(x) = 0 \Rightarrow \frac{\pi x - 4(10-x)}{8\pi} = 0$$

$$\Rightarrow (\pi + 4)x = 40$$

$$\Rightarrow x = \frac{40}{\pi+4}$$ (one critical point)

<u>Compare</u>:

$$A(0) = \frac{25}{\pi} \approx 7.96 \text{ cm}^2$$ (maximum!)

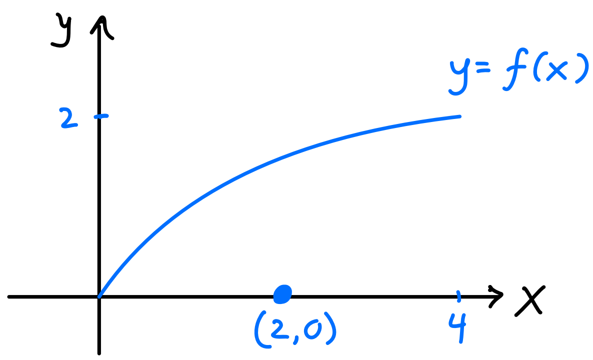
$$A\left(\frac{40}{\pi+4}\right) = \frac{25}{\pi+4} \approx 3.50 \text{ cm}^2$$ (minimum!)

$$A(10) = \frac{25}{4} = 6.25 \text{ cm}^2$$

Largest area will be $\approx 7.96\text{cm}^2$ and will occur when all 10cm of wire is used for the circle.

Smallest area will be $\approx 3.50\text{cm}^2$ and will occur when $x = \frac{40}{\pi+4}$ cm of wire is used for the square.

3. Consider the function $f(x) = \sqrt{x}$, $x \in [0,4]$.



Find the point (x,y) on the graph of $y = f(x)$ that is closest to $(2,0)$. What is this minimum distance?

<u>Hint</u>: Instead of minimizing the distance from (x,y) to $(2,0)$, it will be easier to minimize the <u>square</u> of this distance!

Solution: We wish to find the point (x,y) on graph of $y=\sqrt{x}$ that minimizes

$$d = \sqrt{(x-2)^2 + (y-0)^2} = \sqrt{x^2-4x+4+y^2},$$

the distance from (x,y) to $(2,0)$. Following the hint, we will instead (equivalently) find (x,y) that minimizes

$$d^2 = x^2-4x+4+y^2,$$

the square of this distance. Since $y=\sqrt{x}$, we can write this function as

$$g(x) = x^2-4x+4+(\sqrt{x})^2 = x^2-3x+4, \quad x \in [0,4].$$

Any critical points?

$$g'(x) = \underbrace{2x-3}_{\text{exists everywhere}} = 0 \implies x = 3/2 \quad \text{(one C.P.)}$$

Next, we compare the values of g at the critical points and the endpoints:

$$g(0) = 0^2 - 3(0) + 4 = 4$$

$$g(3/2) = (3/2)^2 - 3(3/2) + 4 = 7/4$$ ⟵ Minimum!

$$g(4) = 4^2 - 3(4) + 4 = 8$$

The closest point is $(x, y) = (x, \sqrt{x}) = \left(3/2, \sqrt{3/2}\right)$. The minimum distance is $\sqrt{g(3/2)} = \sqrt{7/4} = \sqrt{7}/2$.

Since g is the square of the distance