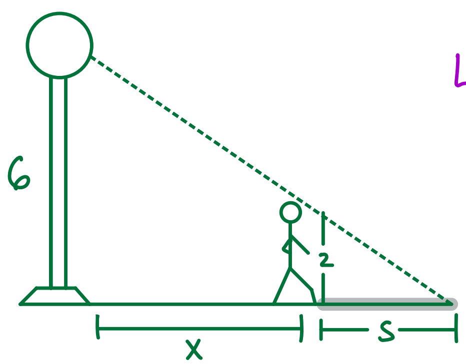
§ 4.9 – Related Rates

Idea: Suppose x and y change according to time, t. If x and y are related and we know dx/dt (i.e., how quickly x is changing), then we can figure out dy/dt.

Ex: A person is walking away from a street light at a rate of 4 m/s. If the person is 2m tall and the street light is 6m tall, how quickly is the length of the person's shadow changing?

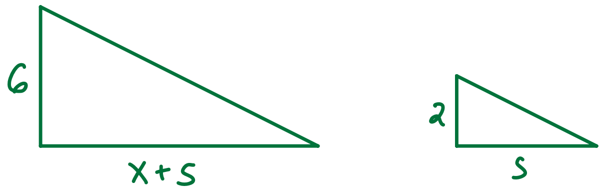
Solution: Start with a picture!



Let x = man's distance from the street light at time t

s = length of the shadow at time t.

These quantities are related by similar triangles!



$$\frac{6}{2} = \frac{x+s}{s} \Rightarrow 3s = x+s \Rightarrow \underline{s = x/2}$$

We know $\frac{dx}{dt} = 4$ and want to find $\frac{ds}{dt}$. Let's differentiate with respect to time.

$$s = \frac{x}{2} \Rightarrow \frac{ds}{dt} = \frac{1}{2}\frac{dx}{dt} = \frac{1}{2}(4) = 2.$$

$\therefore$ The shadow is growing at a rate of 2 m/s.

The Process

(1) Draw a picture and define your variables.

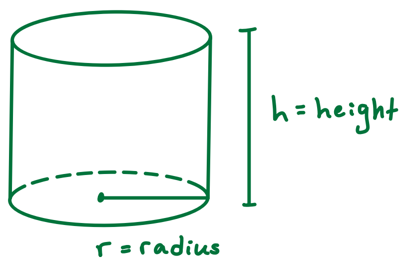
(2) Find a formula relating your variables.

(3) Differentiate (implicitly) with respect to t

(4) Solve for the desired quantity.

Ex: A steel cylinder is smooshed in a hydraulic press. During this process, the steel always remains cylindrical with a volume of $40 cm^3$. If the height is decreasing at a rate of 1 cm/s, find the rate at which the radius is increasing at the moment the radius is equal to 2cm.

Solution:



We know $V = 40 = \pi r^2 h$.

Want to find $\frac{dr}{dt}$ when $r = 2$.

$$40 = \pi r^2 h \overset{d/dt}{\Longrightarrow} \frac{d}{dt}(40) = \frac{d}{dt}(\pi r^2 h)$$

$$\Longrightarrow 0 = \pi \left[\frac{d(r^2)}{dt} \cdot h + r^2 \cdot \frac{dh}{dt} \right] \text{ (product rule)}$$

$$= \pi \left[2r \frac{dr}{dt} \cdot h + r^2 \cdot \frac{dh}{dt} \right]$$

what is h?

When $r=2$, $40 = \pi r^2 h = 4\pi h \implies h = \frac{40}{4\pi} = \frac{10}{\pi}$

Thus,

$$0 = \pi\left[2r\frac{dr}{dt}\cdot h + r^2\cdot\frac{dh}{dt}\right]$$

$$\overset{\div\pi}{\implies} \quad 0 = 2(2)\frac{dr}{dt}\cdot\frac{10}{\pi} + 2^2(-1)$$

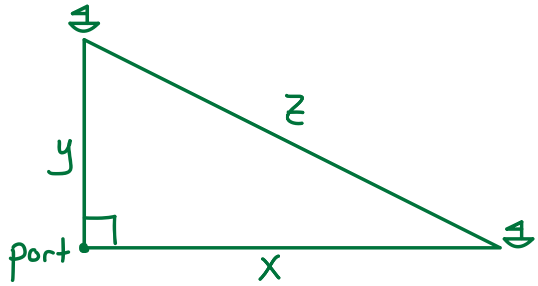
$$\implies \quad 4 = \frac{40}{\pi}\cdot\frac{dr}{dt}$$

$$\implies \quad \frac{dr}{dt} = \frac{\pi}{10} \text{ cm/s}$$

$\boxed{\text{When } r=2, \text{ the radius is increasing at } \pi/10 \text{ cm/s.}}$

Ex: A ship leaves port at noon, travelling 100 km/hr due east. At 2pm, another ship leaves the same port travelling 150 km/hr due north. At what rate is the distance between the ships increasing at 4pm?

Solution:



Let x = distance travelled by first ship at time t

y = distance travelled by second ship at time t.

z = distance between the ships at time t.

Since $x^2 + y^2 = z^2$, we have

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$$

We know $\frac{dx}{dt} = 100$ km/hr and $\frac{dy}{dt} = 150$ km/hr.

Furthermore, at 4pm:

$$x = 100 \text{ km/hr} \cdot 4 \text{ hrs} = 400 \text{ km}$$

$$y = 150 \text{ km/hr} \cdot 2 \text{ hrs} = 300 \text{ km}$$

and $z = \sqrt{x^2+y^2} = \sqrt{400^2 + 300^2} = 500$ km.

Therefore,

$$\cancel{2}x\,\frac{dx}{dt} + \cancel{2}y \cdot \frac{dy}{dt} = \cancel{2}z\,\frac{dz}{dt}$$

$$\Rightarrow \quad 400 \cdot 100 + 300 \cdot 150 = 500\,\frac{dz}{dt}$$

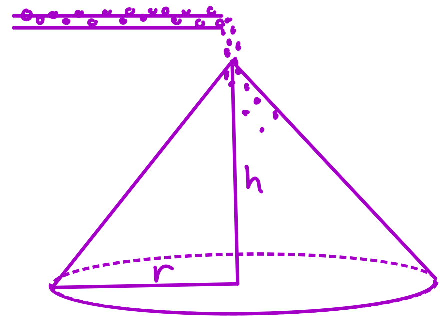
$$\Rightarrow \quad \frac{dz}{dt} = \frac{400 \cdot 100 + 300 \cdot 150}{500} = \underline{170 \text{ Km/hr}}.$$

$\boxed{\text{At 4pm, the distance between the ships is increasing at a rate of 170 Km/hr.}}$

Additional Exercise:

Gravel falls from a conveyor belt at a rate of $2\,m^3/s$. It forms a conical pile with height h always equal to $\sqrt{2}\,r$, where r is the radius of the base.

At what rate is the height of the pile increasing when it is 10m high? (The volume of a cone is $V = \frac{\pi r^2 h}{3}$.)



Solution: Since $h = \sqrt{2}r$

(or, equivalently, $r = \frac{h}{\sqrt{2}}$), we

have

$$V = \frac{\pi r^2 h}{3} = \frac{\pi \left(\frac{h}{\sqrt{2}}\right)^2 h}{3} = \frac{\pi h^3}{6}$$

Differentiating implicitly with respect to time, t:

$$\frac{dV}{dt} = \frac{\pi}{6} \cdot 3h^2 \cdot \frac{dh}{dt} = \frac{\pi h^2}{2} \frac{dh}{dt}$$

Substituting $\frac{dV}{dt} = 2 m^3/s$ and $h = 10m$, we have

$$\frac{dV}{dt} = \frac{\pi h^2}{2} \frac{dh}{dt} \Rightarrow 2 = \frac{\pi (10)^2}{2} \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{2 \cdot 2}{100\pi} = \frac{1}{25\pi} \text{ m/s.}$$

$$\boxed{\text{The height is increasing at a rate of } \frac{1}{25\pi} \text{ m/s.}}$$