§6.7 - The Substitution Rule (Reverse Chain Rule)

Ex: From the chain rule we have

$$\frac{d}{dx}(x^2+1)^7 = 7(x^2+1)^6 \cdot (x^2+1)' = 14x(x^2+1)^6$$

and hence $\displaystyle\int 14x(x^2+1)^6\,dx = (x^2+1)^7 + C$

How could we have evaluated $\displaystyle\int 14x(x^2+1)^6\,dx$ without knowing the answer in advance?

We can use a change of variable / substitution to make the integral nicer!

For $\displaystyle\int 14x(x^2+1)^6\,dx$, let $u = x^2+1 \implies \dfrac{du}{dx} = 2x$

$$\implies du = 2x\,dx$$

$$\implies dx = \frac{du}{2x}$$

We get $\displaystyle\int 14x\underbrace{(x^2+1)^6}_{=u}\underbrace{dx}_{=\frac{du}{2x}} = \int 14\cancel{x}\cdot u^6 \cdot \frac{du}{2\cancel{x}}$

$$= \int 7u^6 \, du$$

$$= u^7 + C$$ (now go back to x!)

$$= \boxed{(x^2+1)^7 + C}$$

Ex: Evaluate $\int x^2\sqrt{x^3+4}\,dx$.

Solution: Let $u = x^3+4$, so $du = 3x^2\,dx$

$$\Rightarrow dx = \frac{du}{3x^2}$$

Thus,

$$\int x^2\sqrt{x^3+4}\,dx = \int x^2\sqrt{u}\cdot\frac{du}{3x^2}$$

$$= \frac{1}{3}\int u^{1/2}\,du$$

$$= \frac{1}{3}\,\frac{u^{3/2}}{3/2} + C = \boxed{\frac{2}{9}(x^3+4)^{3/2} + C}$$

General Strategy

"Let $u =$ ______ , then $du =$ ______ dx".

Replace u and dx in the integral and evaluate.

Good Choices for u:

- u = function raised to an ugly power
- u = function inside sin, cos, ln, $e^{\text{—}}$, etc.
- u = function whose derivative is also in the integral.

Ex: Evaluate the following:

(a) $\displaystyle\int \sin^8(x)\cos(x)\,dx$

Solution: Let $u = \sin x$, so $du = \cos x\,dx$

$$\Rightarrow dx = \frac{du}{\cos x}$$

Thus, $\displaystyle\int \sin^8(x)\cos(x)\,dx = \int u^8\,du$

$$= \frac{u^9}{9} + C = \boxed{\frac{\sin^9 x}{9} + C}$$

(b) $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$

Solution: Let $u = x+1$, so $du = dx$.

$$\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{x}{\sqrt{u}}\,du$$

$u = x+1 \Rightarrow x = u-1$

$$= \int \frac{u-1}{\sqrt{u}}\,du$$

$$= \int \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right) du$$

$$= \int \left(u^{1/2} - u^{-1/2}\right) du$$

$$= \frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} + C = \boxed{\frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C}$$

(c) $\int x^3 (x^2+1)^{99}\,dx$

Solution: Let $u = x^2+1$, so $du = 2x\,dx$, hence $dx = \frac{du}{2x}$.

$$\int x^3 (x^2+1)^{99}\,dx = \int x^3 u^{99} \cdot \frac{du}{2x}$$

$$= \frac{1}{2}\int x^2 \cdot u^{99}\,du$$

$u = x^2+1 \Rightarrow x^2 = u-1$

$$= \frac{1}{2}\int (u-1)\cdot u^{99}\,du$$

$$= \frac{1}{2}\int (u^{100} - u^{99})\,du$$

$$= \frac{1}{2}\left[\frac{u^{101}}{101} - \frac{u^{100}}{100}\right] + C$$

$$= \boxed{\frac{(x^2+1)^{101}}{202} - \frac{(x^2+1)^{100}}{200} + C}$$

For definite integrals, $\int_a^b f(x)\,dx$, we must also deal with the bounds. There are two ways to do this.

Option 1: Don't change the bounds, but make sure to rewrite everything in terms of x before plugging them in.

Ex: $\int_0^1 e^x \cos(e^x)\,dx$

let $u = e^x$

$du = e^x\,dx$

$$= \int_{x=0}^{x=1} \cos(u)\,du$$

$$= \Big[\sin(u) \Big]_{x=0}^{x=1}$$

$$= \Big[\sin(e^x) \Big]_0^1 = \boxed{\sin(e) - \sin(1)}$$

Option 2: Change the bounds and don't worry about rewriting everything in terms of x.

Ex: $\displaystyle\int_0^1 e^x \cos(e^x)\, dx$

Let $u = e^x$

$du = e^x\, dx$

$$= \int_1^e \cos(u)\, du$$

Bounds:

When $x = 0$, $u = e^0 = 1$

When $x = 1$, $u = e^1 = e$

$$= \Big[\sin(u) \Big]_1^e$$

$$= \boxed{\sin(e) - \sin(1)}$$

Ex: Evaluate the following:

(a) $\displaystyle\int_1^e \frac{\ln x}{x}\, dx$

Solution: Let $u = \ln x$, so $du = \frac{1}{x}\,dx$

When $x = 1$, $u = \ln 1 = 0$.

When $x = e$, $u = \ln e = 1$.

Thus, $\displaystyle\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \boxed{\frac{1}{2}}$

(b) $\displaystyle\int_0^{\pi/3} \tan x\,dx$

Solution: $\displaystyle\int_0^{\pi/3} \tan x\,dx = \int_0^{\pi/3} \frac{\sin x}{\cos x}\,dx$

Let $u = \cos x$,

$du = -\sin x\,dx$

$\displaystyle = \int_1^{1/2} \frac{\sin x}{u} \cdot \frac{-du}{\sin x}$

$x = 0 \Rightarrow u = \cos(0) = 1$

$x = \pi/3 \Rightarrow u = \cos\frac{\pi}{3} = 1/2$

$\displaystyle = -\int_1^{1/2} \frac{1}{u}\,du$

$= -\Big[\ln|u|\Big]_1^{1/2}$

$= -\Big[\ln(1/2) - \ln(1)\Big]$

$= \boxed{-\ln(1/2)}$ (or $\ln 2$)

Additional Exercises

1. $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

2. $\displaystyle\int_0^{\pi/2} \sin^3 x \cos^3 x\,dx$

Solutions:

1. $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

Let $u = \frac{1}{x}$, so $du = \frac{-1}{x^2}\,dx$

$\Rightarrow dx = -x^2\,du$

$$= \int \frac{e^u}{\cancel{x^2}}\left(-\cancel{x^2}\,du\right)$$

$$= -\int e^u\,du = -e^u + C = \boxed{-e^{1/x} + C}$$

2. $\displaystyle\int_0^{\pi/2} \sin^3 x \cos^3 x\,dx$

Let $u = \sin x$

$du = \cos x\,dx \Rightarrow dx = \frac{du}{\cos x}$

When $x = 0$, $u = \sin(0) = 0$

When $x = \pi/2$, $u = \sin(\pi/2) = 1$

$$= \int_0^1 u^3 \cos^3 x \left(\frac{du}{\cos x}\right)$$

$$= \int_0^1 u^3 \cos^2 x\,du$$

$$= \int_0^1 u^3 (1 - \sin^2 x)\, dx$$

$$= \int_0^1 u^3 (1 - u^2)\, du$$

$$= \int_0^1 (u^3 - u^5)\, du$$

$$= \left[\frac{u^4}{4} - \frac{u^6}{6} \right]_0^1 = \frac{1}{4} - \frac{1}{6} = \boxed{\frac{1}{12}}$$