§3.9 – Derivatives of Trig Functions

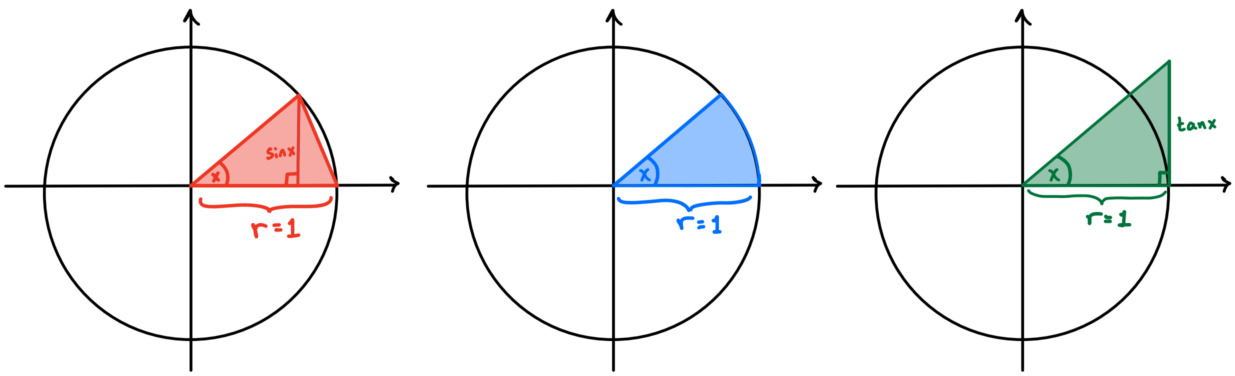
To find the derivative of $f(x) = \sin(x)$, we'll need two very special limits!

See below why these are true!

$$\boxed{\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{\cos(x)-1}{x} = 0.}$$

Let's start with $\lim_{x\to 0} \frac{\sin x}{x}$!

Suppose x is a positive number near 0.



Area $= \frac{1}{2}bh = \frac{1}{2}\sin x$ Area $= \pi r^2\left(\frac{x}{2\pi}\right) = \frac{1}{2}x$ Area $= \frac{1}{2}bh = \frac{1}{2}\tan x$

Comparing the areas above, we see that

$$\frac{1}{2}\sin x \le \frac{1}{2}x \le \frac{1}{2}\tan x$$

Dividing by $\frac{1}{2}\sin x$, we have

$$1 \leq \frac{x}{\sin x} \leq \frac{\tan x}{\sin x}.$$

Taking reciprocals,

$$1 \geq \frac{\sin x}{x} \geq \underbrace{\frac{\sin x}{\tan x}}_{=\cos x}.$$

Taking limits as $x \to 0^+$ (since we assumed $x > 0$ earlier!),

$$\underbrace{\lim_{x\to 0^+} 1}_{=1} \geq \lim_{x\to 0^+} \frac{\sin x}{x} \geq \underbrace{\lim_{x\to 0^+} \cos x}_{=1}$$

Therefore, by the Squeeze Theorem,

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

For $x < 0$, we have

Since $\sin x$ is odd!

$$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\sin(-x)}{-x} = \lim_{x\to 0^+} \frac{-\sin x}{-x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

Thus,

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

What about $\lim_{x\to 0} \dfrac{\cos x - 1}{x}$?

$$\lim_{x\to 0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} = \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x(\cos x + 1)}$$

$$= \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\to 1} \cdot \underbrace{\frac{\sin x}{\cos x + 1}}_{\to 0/2 = 0.}$$

$$= 1 \cdot 0 = \boxed{0.}$$

We may now calculate f' for $f(x) = \sin(x)$!

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\left[\cos(h) - 1\right] + \cos(x)\cdot\sin(h)}{h}$$

$$= \lim_{h\to 0} \sin(x)\cdot \left[\cancelto{0}{\frac{\cos(h)-1}{h}}\right] + \cos(x)\,\cancelto{1}{\frac{\sin(h)}{h}}$$

$$= \boxed{\cos(x)}$$

Therefore,

$$\boxed{(\sin x)' = \cos x}$$

Similarly,

$$\boxed{(\cos x)' = -\sin x}$$

We can now find the derivatives of tan x, sec x, cot x and csc x using derivative rules!

Ex: Find $(\tan x)'$ and $(\sec x)'$.

Solution:

Quotient rule!

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \overbrace{(\sin x)'}^{\cos x} - \sin x \cdot \overbrace{(\cos x)'}^{-\sin x}}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{\cos x \cdot \overbrace{1'}^{0} - 1 \cdot \overbrace{(\cos x)'}^{-\sin x}}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \boxed{\sec x \cdot \tan x}$$

<u>Summary:</u>

$(\sin x)' = \cos x$ $\quad$ $(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$ $\quad$ $(\sec x)' = \sec x \cdot \tan x$

$(\cot x)' = -\csc^2 x$ $\quad$ $(\csc x)' = -\csc x \cdot \cot x$

Exercises!

<u>Ex</u>: Find the derivative of each function below.

(a) $y = \sec(1+3x)$

(b) $y = \sin^2(6x)$

Solution:

(a) $y' = \sec(1+3x)\tan(1+3x) \cdot (1+3x)'$ (chain rule)

$= \boxed{\sec(1+3x)\tan(1+3x) \cdot 3}$

(b) $y' = 2\sin(6x) \cdot [\sin(6x)]'$ (chain rule)

$= 2\sin(6x) \cdot \cos(6x) \cdot (6x)'$ (chain rule... again!)

$= \boxed{2\sin(6x)\cos(6x) \cdot 6}$