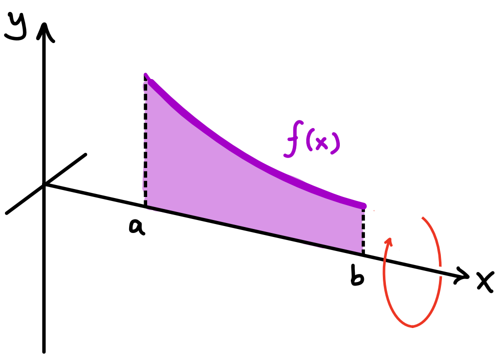
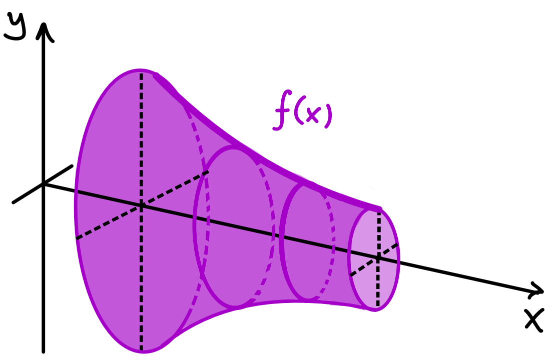
§7.2 – Volumes by Disks / Washers

Integrals don't just tell us about areas, we can also use them for calculating volumes of 3D Solids!

The solids we'll consider in MATH 116 are called solids of revolution, and are obtained by revolving a 2D region about an axis:

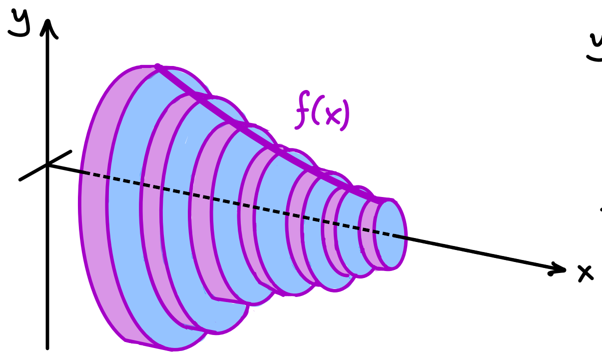


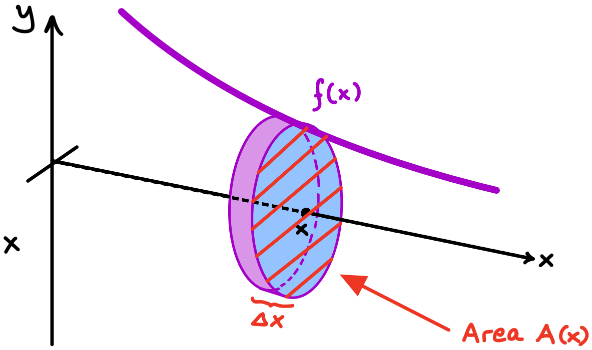


There are two ways to find the volume of such a solid.

1. The Disk / Washer Method

Start by slicing the solid into thin disks.





Each disk has width Δx. If $A(x)$ denotes the area of the disk at each point x, then the volume of a typical disk is $A(x)\Delta x$. Adding these volumes:

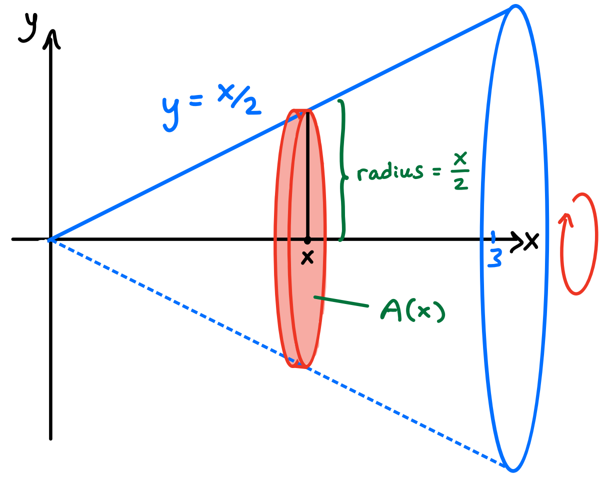
$$\boxed{\text{Volume of the Solid} = \int_a^b A(x)\,dx}$$

Ex: Consider the region between the x-axis and $y = \frac{x}{2}$ from $x = 0$ to $x = 3$. Find the volume of the solid obtained by rotating this region about the x-axis.

Solution: Start with a sketch showing the region and a typical disk. The area of a disk is

$A(x) = \pi \cdot \text{radius}^2 = \pi \cdot \left(\frac{x}{2}\right)^2,$

hence



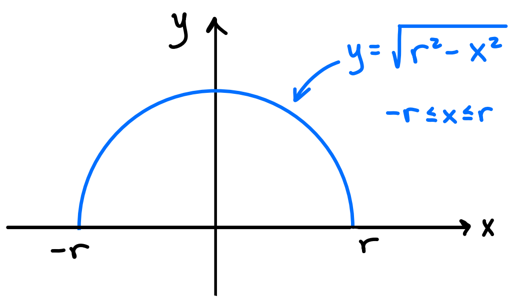
$$\text{Volume} = \int_0^3 A(x)\,dx = \int_0^3 \pi\left(\frac{x}{2}\right)^2 dx$$

$$= \frac{\pi}{4}\int_0^3 x^2\,dx$$

$$= \frac{\pi}{4}\left[\frac{x^3}{3}\right]_0^3 = \boxed{\frac{9\pi}{4}}$$

Ex: Find the volume of a sphere of radius r.

Solution: A sphere can be obtained by rotating the top half of the circle $x^2+y^2=r^2$ about the x-axis!



$$\text{Volume} = \int_{-r}^{r} A(x)\,dx$$

$$= \int_{-r}^{r} \pi \cdot \text{radius}^2\,dx$$

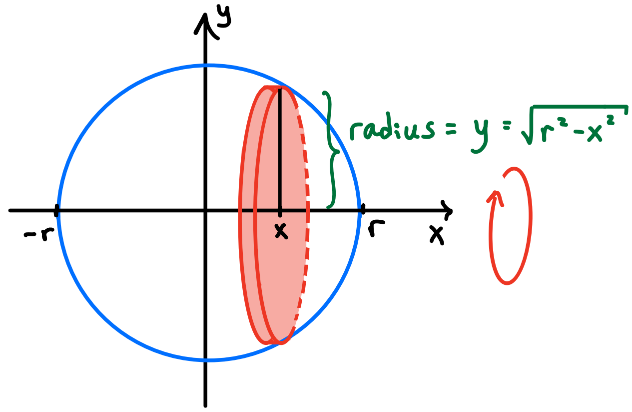
$$= \int_{-r}^{r} \pi\left(\sqrt{r^2-x^2}\right)^2 dx$$

$$= \int_{-r}^{r} \pi\left(r^2-x^2\right) dx$$

$$= \left[\pi r^2 x - \frac{\pi x^3}{3}\right]_{-r}^{r}$$

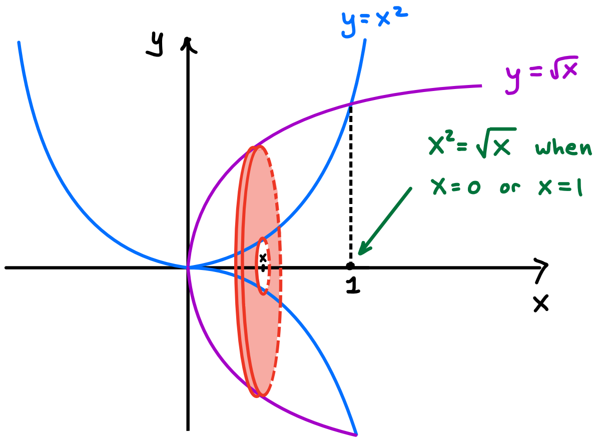
$$= \left(\pi r^3 - \frac{\pi r^3}{3}\right) - \left(-\pi r^3 + \frac{\pi r^3}{3}\right)$$

$$= \boxed{\frac{4\pi r^3}{3}}$$



<u>Ex:</u> Set up the integral that gives the volume of the solid obtained by rotating the region bounded between $y = x^2$ and $y = \sqrt{x}$ about the x-axis.

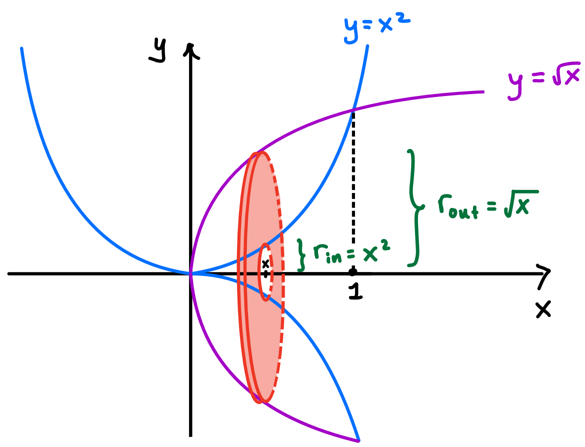
Solution: Start with a sketch!



This time our cross-section isn't a disk... it's a washer!

In this case,

$$\boxed{\text{Area} = A(x) = \pi \cdot (\text{outer radius})^2 - \pi (\text{inner radius})^2}$$



Outer radius $= \sqrt{x}$

Inner radius $= x^2$

Bounds: $0 \leq x \leq 1$

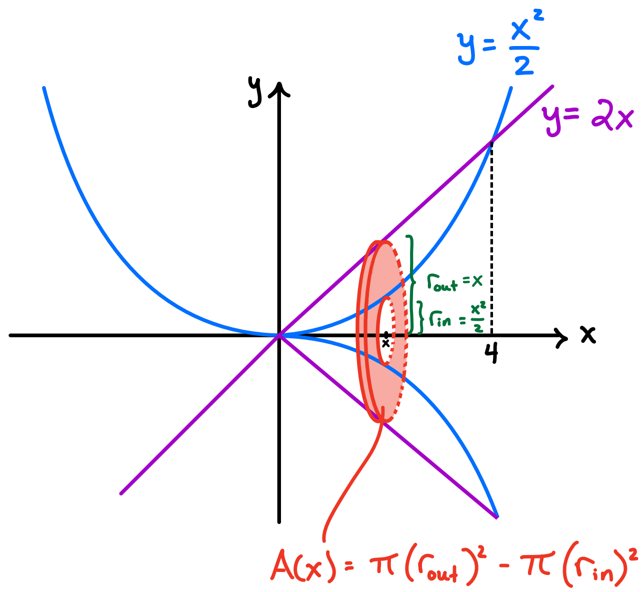
$$\therefore \text{Volume} = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi (r_{out})^2 - \pi (r_{in})^2\right] dx$$

$$= \boxed{\int_0^1 \left[\pi (\sqrt{x})^2 - \pi (x^2)^2\right] dx}$$

Ex: Consider the region between $y = 2x$ and $y = x^2/2$ for $x \in [0,4]$. Set up the integral for the volume of the solid obtained by rotating the region about...

(a) the x-axis

Solution: Sketch!



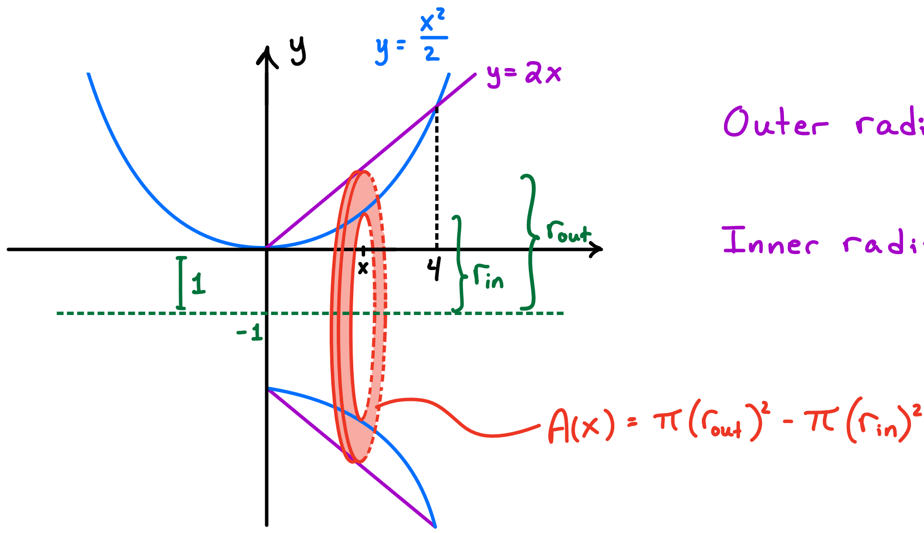
Outer radius: $2x$

Inner radius: $\frac{x^2}{2}$

$$\therefore \text{Volume} = \int_0^4 \underbrace{A(x)} \, dx = \boxed{\int_0^4 \left[\pi(2x)^2 - \pi\left(\frac{x^2}{2}\right)^2\right] dx}$$

(b) the line $y = -1$

Solution:



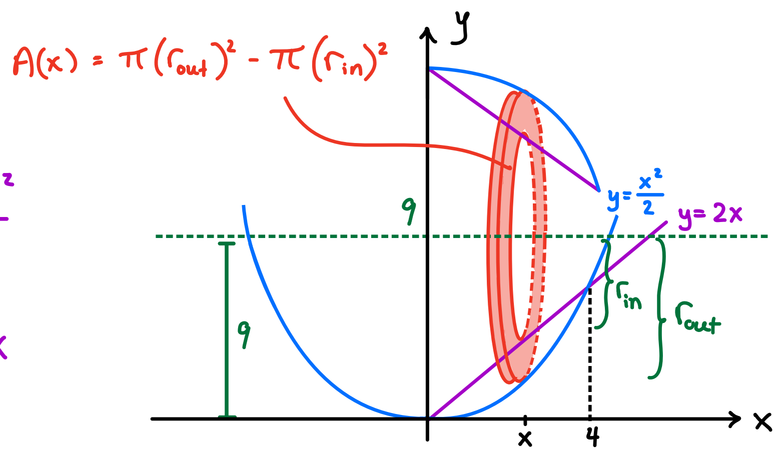
Outer radius $= 1 + 2x$

Inner radius $= 1 + \frac{x^2}{2}$

$$\text{Volume} = \int_0^4 \underbrace{A(x)}\,dx = \boxed{\int_0^4 \left[\pi(1+2x)^2 - \pi\left(1+\frac{x^2}{2}\right)^2\right]dx}$$

(c) the line $y = 9$

Solution:



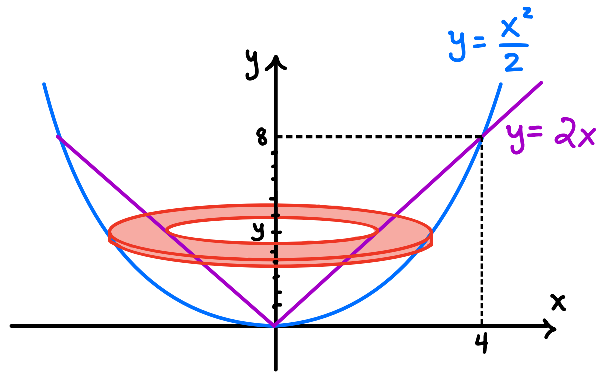
Outer radius: $9 - \frac{x^2}{2}$

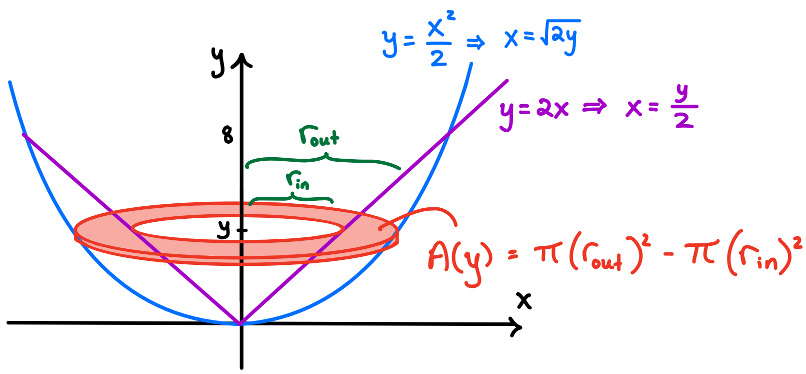
Inner radius: $9 - 2x$

$$\therefore \text{ Volume } = \int_0^4 A(x)\,dx = \boxed{\int_0^4 \left[\pi\left(9-\frac{x^2}{2}\right)^2 - \pi(9-2x)^2\right] dx}$$

(d) <u>the y-axis</u>

<u>Solution</u>: We have a washer at each $y \in [0, 8]$, so we'll be integrating with respect to y.





Outer radius: $\sqrt{2y}$

Inner radius: $\frac{y}{2}$

<u>Bounds</u>: $0 \le y \le 8$.

$$\therefore \text{ Volume } = \int_0^8 A(y)\,dy = \boxed{\int_0^8 \left[\pi\left(\sqrt{2y}\right)^2 - \pi\left(\frac{y}{2}\right)^2\right] dy}$$

Summary for disks / washers

Revolving around horizontal axis? Use functions of x.

Revolving around vertical axis? Use functions of y.