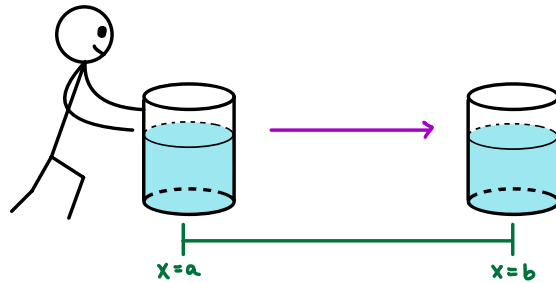


§ 7.4 - Work

Suppose a constant force F is applied to an object, moving it from $x=a$ to $x=b$.

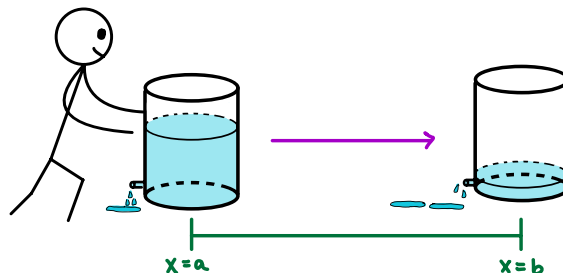


In this case, the work done on the object is

$$\text{Work} = \text{Force} \cdot \text{Distance} = F \cdot (b-a)$$

What if the force is not constant, and instead changes throughout the process (so force = $F(x)$)?

For example, if the container above were leaking...



... then it would get lighter as x increases, meaning less and less force would be needed to move it!

If force = $F(x)$, then the work done in moving the object a small distance $\Delta x \approx dx$ is roughly $F(x)\Delta x$.

Adding these work amounts from $x=a$ to $x=b$, we get

$$\text{Work} = \int_a^b F(x) dx$$

Ex: A 10m chain of mass 40Kg hangs from the top of a building. How much work is done in lifting the chain to the top?

Solution: First, let's figure out how much mass is being moved at each point of the process.

The chain has density $\frac{40 \text{ kg}}{10 \text{ m}} = 4 \text{ kg/m}$ so
once x metres have been lifted (and $10-x$ metres
remain), the remaining chain has mass

$$4 \frac{\text{kg}}{\text{m}} \cdot (10-x) \text{ m} = 40 - 4x \text{ kg}$$

Hence,

$$\text{Force} = \text{Mass} \cdot \text{Acceleration}$$

Due to gravity
 $= 9.81 \text{ m/s}^2$

$$\Rightarrow F(x) = (40 - 4x) \cdot 9.81 \text{ N (Newtons)}$$

Thus,

$$\begin{aligned} \text{Work} &= \int_0^{10} F(x) dx = \int_0^{10} (40 - 4x) \cdot 9.81 dx \\ &= 9.81 \left[40x - 2x^2 \right]_0^{10} \\ &= 9.81 (400 - 2 \cdot 100) \\ &= \boxed{1962 \text{ J}} \text{ (joules)} \end{aligned}$$

Ex: Rapunzel's hair has a constant density of 3 kg/m and hangs to the bottom of her 100m tower. Set up the integrals that give each of the following.

(a) The work done in lifting the hair to the top of the tower.

Solution: Once x metres have been lifted (and $100 - x$ metres remain), we have

$$M_{\text{hair}} = 3 \frac{\text{kg}}{\text{m}} \cdot (100 - x) \text{ m} = 300 - 3x \text{ kg.}$$

Hence,

$$\text{Force} = \text{Mass} \cdot \text{Acceleration}$$

$$\Rightarrow F(x) = (300 - 3x) \cdot 9.81 \text{ N}$$

Thus,

$$\text{Work} = \int_0^{100} F(x) dx = \int_0^{100} (300 - 3x) \cdot 9.81 dx$$

(b) The work done in lifting the hair the first half of the way up.

Solution: We need only modify the bounds from (a).

$$\text{Work} = \int_0^{50} F(x) dx = \int_0^{50} (300 - 3x) \cdot 9.81 dx$$

(c) The work done in lifting the hair to the top, given that her 10kg "Skip the Dishes" order is tied to the end.

Solution: Once x metres of hair have been lifted,

$$m_{\text{hair}} = 300 - 3x \text{ kg}, \quad m_{\text{food}} = 10 \text{ kg (constant)}$$

Thus,

$$\text{Force} = \text{Mass} \cdot \text{Acceleration}$$

$$\Rightarrow F(x) = \left[\underbrace{(300 - 3x)}_{m_{\text{hair}}} + \underbrace{10}_{m_{\text{food}}} \right] \cdot 9.81 \text{ N}$$


Hence, we get

$$\text{Work} = \int_0^{100} F(x) dx = \int_0^{100} (310 - 3x) \cdot 9.81 dx$$

(d) Same as (c), but the "Skip the Dishes" order falls off after being lifted 75 m.


Solution: In the first 75 m, we have

$$\text{Work} = \int_0^{75} (310 - 3x) \cdot 9.81 dx$$


 $m_{\text{hair}} + m_{\text{food}}$

In the last 25 m, we have

$$\text{Work} = \int_{75}^{100} (300 - 3x) \cdot 9.81 dx$$


 m_{hair}

Hence, the total work is

$$\int_0^{75} (310 - 3x) \cdot 9.81 dx + \int_{75}^{100} (300 - 3x) \cdot 9.81 dx$$

Ex: A 50m rope of density 2kg/m hangs into a well.

The rope is attached to a 10kg bucket initially filled

with 6kg of water. If water leaks from the bucket

at a constant rate and only 1kg of water remains

when the bucket reaches the top of the well, determine

the work needed to lift the bucket to the top.

Assume acceleration due to gravity is 10m/s^2 .

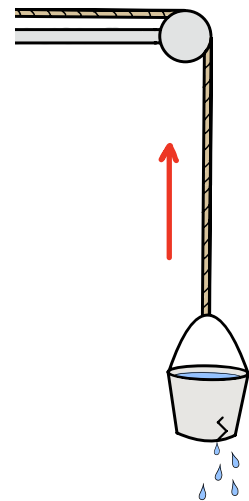
Solution: Over the 50m of distance, $6-1 = 5\text{kg}$ of water

are lost, so the water is leaking at a

rate of $\frac{5\text{kg}}{50\text{m}} = \frac{1}{10} \frac{\text{kg}}{\text{m}}$.

Thus, once x metres of rope have been

lifted and $(50-x)$ metres remain, we have



$$M_{\text{rope}} = 2 \frac{\text{kg}}{\text{m}} \cdot (50-x) \text{ m} = 100 - 2x \text{ kg} ,$$

$$M_{\text{bucket}} = 10 \text{ kg} \text{ (constant)} ,$$

$$m_{\text{water}} = 6 \text{ kg} - \left(\frac{1}{10} \frac{\text{kg}}{\text{m}} \right) (x \text{ m}) = 6 - \frac{x}{10} \text{ kg}$$

Therefore ,

$$\begin{aligned} \text{Work} &= \int_0^{50} F(x) dx \\ &= \int_0^{50} \left[\underbrace{\left((100 - 2x) + 10 + \left(6 - \frac{x}{10} \right) \right)}_{\substack{\text{Mass} \\ m_{\text{hair}} \quad m_{\text{bucket}} \quad m_{\text{water}}}} \right] \cdot \underbrace{10}_{\substack{\text{Acceleration} \\ \text{due to gravity}}} dx \\ &= \int_0^{50} (1160 - 21x) dx \\ &= \boxed{31750 \text{ J}} \end{aligned}$$